163 Discrete Mathematics

Review 2

Use the following to answer question 1:

In the question below suppose that a “word” is any string of seven letters of the alphabet, with repeated letters allowed.

1. How many words begin with A or B or end with A or B?

Solution. Number of words beginning with A or B is $2 \cdot 26^6$ - two possibilities for the first letter and 26 for each of the other 6 letters. In the same way number of words ending with A or B is $2 \cdot 26^6$. Number of words both starting with A or B and ending with A or B is $4 \cdot 26^5$ - 2 possibilities for the first letter, two for the last and 26 for each of 5 letters in the middle. Applying the formula

$$n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

we see that the answer to the problem is

$$2 \cdot 26^6 + 2 \cdot 26^6 - 4 \cdot 26^5 = 4 \cdot 26^5 (26-1) = 100 \cdot 26^5 = 1188137600$$

Use the following to answer question 2:

In the question below let $A$ be the set of all bit strings of length 10.

2. How many bit strings of length 10 have exactly six 0s?

Solution. We have to choose 6 places for 0s among ten. The number is

$$C(10,6) = C(10,4) = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210$$

Use the following to answer question 3:

In the question below a club with 20 women and 17 men needs to form a committee of size six.

3. How many committees are possible if the committee must consist of all women or all men?

Solution. Number of committees of all women is $C(20,6)$, of all men - $C(17,6)$. The answer is $C(20,6) + C(17,6) = 38760 + 12376 = 51136$. 

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4. Find the number of subsets of \( S = \{1,2,3,\ldots,10\} \) that contain exactly five elements, the sum of which is even.

**Solution.** If the sum of the elements of a subset is even then the subset contains 0, or 2, or 4 odd numbers and respectively 5, or 3, or 1 even numbers. Applying the multiplication principle we see that the total number of such subsets is

\[
C(5, 0) \cdot C(5, 5) + C(5, 2) \cdot C(5, 3) + C(5, 4) \cdot C(5, 1) = 1 \cdot 1 + 10 \cdot 10 + 5 \cdot 5 = 126.
\]

5. Find the coefficient of \( x^5 y^6 \) in the expansion of \( (2x - y)^{11} \).

**Solution.** The corresponding term in the expansion is \( C(11,5)(2x)^5(-y)^6 \) whence the coefficient of \( x^5 y^6 \) is \( C(11,5) \cdot 2^5 = 462 \cdot 32 = 14784 \).

Use the following to answer question 6:

In the questions below assume that you have 50 pennies and three jars, labeled \( A, B, \) and \( C. \)

6. In how many ways can you put the pennies in the jars, assuming that the pennies are identical and each jar must have at least two pennies put into it?

**Solution.** If we put two pennies in each box we will have 44 identical (indistinguishable) pennies to be distributed into 3 labeled (distinguishable) jars. The number of ways to distribute \( n \) indistinguishable objects into \( k \) distinguishable boxes is (see page 377 and Theorem 2 on page 373)

\[
C(n + k - 1, k - 1)
\]

In our case the number is \( C(46, 2) = 1035 \).

Use the following to answer question 7:

In the questions below you pick a bit string from the set of all bit strings of length ten.

7. What is the probability that the bit string has more 0s than 1s?

**Solution.** Let \( p_1, p_2, p_3 \) be the probabilities to have more 0s than 1s, more 1s than 0s, and the same number of 0s and 1s, respectively. Then \( p_1 = p_2 \), \( p_3 = C(10,5)(1/2)^5(1/2)^5 \), and \( p_1 + p_2 + p_3 = 1 \). Therefore

\[
p_1 = \frac{1 - C(10,5)/2^{10}}{2} = 0.376953125
\]
Use the following to answer question 8:

In the questions below an experiment consists of picking at random a bit string of length five. Consider the following events:

\( E_1 \): the bit string chosen begins with 1;
\( E_2 \): the bit string chosen ends with 1;
\( E_3 \): the bit string chosen has exactly three 1s.

8. Determine whether \( E_2 \) and \( E_3 \) are independent.

**Solution.** Let us first find the probability of \( E_2 \). The total number of bit strings of length 5 is \( 2^5 = 32 \). The number of strings ending with 1 is \( 2^4 = 16 \). Therefore,

\[
p(E_2) = \frac{16}{32} = \frac{1}{2}.
\]

In a similar way probability of \( E_3 \) is

\[
P(E_3) = \frac{C(5,3)}{2^5} = \frac{5}{16}.
\]

Finally let us compute the probability of \( E_2 \cup E_3 \). The number of strings ending with 1 and having exactly three 1s is \( C(4,2) = 6 \) whence

\[
P(E_2 \cup E_3) = \frac{6}{32} = \frac{3}{16}.
\]

It follows that

\[
p(E_2)p(E_3) = \frac{5}{32} \neq \frac{3}{16}
\]

and \( E_2 \) and \( E_3 \) are not independent.

9. Each of 26 cards has a different letter of the alphabet on it. You pick one card at random. A vowel is worth 3 points and a consonant is worth 0 points. Let \( X \) = the value of the card picked. Find \( E(X) \), \( V(X) \), and the standard deviation of \( X \).

**Solution.** \( E(X) = 3 \cdot \frac{5}{26} + 0 \cdot \frac{21}{26} = \frac{15}{26} \).

\[
V(X) = \left(3 - \frac{15}{26}\right)^2 \cdot \frac{5}{26} + \left(0 - \frac{15}{26}\right)^2 \cdot \frac{21}{26} \approx 1.40.
\]

\[
\sigma(X) = \sqrt{V(X)} \approx 1.18
\]

Use the following to answer question 10:

In the questions below, describe each sequence recursively. Include initial conditions and assume that the sequences begin with \( a_1 \).
10. $1^2, 2^2, 3^2, 4^2, \ldots$

**Solution.** We see that $a_n = n^2, n = 1, 2, \ldots$ Therefore $a_{n+1} - a_n = (n+1)^2 - n^2 = 2n + 1$ and the sequence can be described recursively as

$$a_1 = 1, a_{n+1} = a_n + 2n + 1.$$ 

Use the following to answer question 11:

In the questions below solve the recurrence relation either by using the characteristic equation or by discovering a pattern formed by the terms.

11. $a_n = -10a_{n-1} - 21a_{n-2}, \quad a_0 = 2, a_1 = 1.$

**Solution.** The characteristic equation of the above recurrence relation is

$$r^2 + 10r + 21 = 0.$$ 

The left part can be factored as $(r + 3)(r + 7)$ whence the solutions of the characteristic equation are $r_1 = -3, r_2 = -7$, and the general solution of the recurrence relation is

$$a_n = \alpha_1(-3)^n + \alpha_2(-7)^n.$$ 

From the initial conditions we find

$$\alpha_1 + \alpha_2 = 2$$

$$3\alpha_1 + 7\alpha_2 = -1.$$ 

By multiplying the first equation by 7 and subtracting from it the second equation we get $4\alpha_1 = 15$ whence $\alpha_1 = \frac{15}{4}$ and $\alpha_2 = 2 - \alpha_1 = 2 - \frac{15}{4} = -\frac{7}{4}$. Finally

$$a_n = \frac{15}{4}(-3)^n - \frac{7}{4}(-7)^n.$$
12. What form does a particular solution of the linear nonhomogeneous recurrence relation 
\[ a_n = 4a_{n-1} - 4a_{n-2} + F(n) \] 
have when \( F(n) = (n^2 + 1)2^n \)?

**Solution.** The associated homogeneous recurrence relation is 
\[ a_n = 4a_{n-1} - 4a_{n-2} \]. Its characteristic equation \( r^2 - 4r + 4 = 0 \) has solution 2 of multiplicity 2. By theorem 6 on page 469 there is a particular solution of our nonhomogeneous recurrence relation of the form 
\[ a_n = n^2 2^n (An^2 + Bn + C) = 2^n (An^4 + Bn^3 + Cn^2) \]. To find the values of the coefficients \( A, B, \) and \( C \) let us introduce the polynomial 
\[ G(n) = An^4 + Bn^3 + Cn^2 \]. Then

\[ a_n = 2^n G(n), \quad a_{n+1} = 2 \cdot 2^n G(n+1), \quad \text{and} \quad a_{n+2} = 4 \cdot 2^n G(n+2). \]

Plugging in these expressions into the relation 
\[ a_{n+2} = 4a_{n+1} - 4a_n + [(n+2)^2 + 1] \cdot 2^n \cdot 4 \]
and dividing both parts by \( 4 \cdot 2^n \) we obtain the relation

\[ G(n+2) = 2G(n+1) - G(n) + n^2 + 4n + 5. \]

Or

\[ G(n+2) - 2G(n+1) + G(n) - n^2 - 4n - 5 = 0. \]

By expanding the expressions in the left part and combining the like terms we get

\[ (-1+12A)n^2 + (6B+24A-4)n - 5 + 6B + 2C + 14A = 0, \]

From here we find \( A = \frac{1}{12}, B = \frac{1}{3}, \) and \( C = \frac{11}{12} \).
13. A market sells ten kinds of soda. You want to buy 12 bottles. How many possibilities are there if you want at most three bottles of any kind?

Solution. We will use the formula on page 506 and follow the pattern of Example 1 on the same page. Our problem can be reformulated as follows.

How many solutions does the equation

\[ x_1 + \ldots + x_{10} = 12 \]

have, where \( x_1, \ldots, x_{10} \) are nonnegative integers and \( x_i \leq 3, i = 1, \ldots, 10 \). Let \( P_i, i = 1, \ldots, 10 \), be the property that \( x_i > 3 \). Notice that for any four distinct indices \( 1 \leq i_1 < i_2 < i_3 < i_4 \leq 10 \) we have \( N(P_{i_1} P_{i_2} P_{i_3} P_{i_4}) = 0 \). Indeed, otherwise the number of bottles would be at least 16. Therefore in our case the formula on page 506 takes the form

\[
N(P_1 \ldots P_{10}) = N - \sum_{1 \leq i \leq 10} N(P_i) + \sum_{1 \leq i < j \leq 10} N(P_i P_j) - \sum_{1 \leq i < j < k \leq 10} N(P_i P_j P_k).
\]

Let us now compute each term in the right part of the formula above. We can find the total number \( N \) of nonnegative integer solutions of the equation \( x_1 + \ldots + x_{10} = 12 \) by following the pattern in Example 5 on page 373.

\( N = C(10 + 12 - 1, 12) = C(21, 12) = C(21, 9). \)

Next, again by following Example 5 on page 373, we see that for any index \( i, 1 \leq i \leq 10 \),

\( N(P_i) = C(10 + 8 - 1, 8) = C(17, 8) \)

whence

\[
\sum_{1 \leq i \leq 10} N(P_i) = 10 \cdot C(17, 8).
\]

Similarly for any \( i, j, 1 \leq i < j \leq 10 \) we have

\( N(P_i P_j) = C(10 + 4 - 1, 4) = C(13, 4) \)

and therefore,

\[
\sum_{1 \leq i < j \leq 10} N(P_i P_j) = C(10, 2)C(13, 4).
\]

Finally, for any \( i, j, k, 1 \leq i < j < k \leq 10 \)

\( N(P_i P_j P_k) = 1 \)

and
\[
\sum_{1 \leq i < j < k \leq 10} N(P_i P_j P_k) = C(10, 3).
\]

The answer is
\[
C(21, 9) - 10 \cdot C(17, 8) + 10 \cdot 2 \cdot C(13, 4) - C(10, 3) = 82885.
\]

14. How many permutations of all 26 letters of the alphabet are there, that contain at least one of the words SWORD, PLANT, CARTS?

**Solution.** Let \( A \), \( B \), and \( C \) be the sets of all permutations which contain SWORD, PLANT, or CARTS, respectively. Notice that \( A \cap C = B \cap C = \emptyset \). Therefore by inclusion – exclusion principle for three sets we have
\[
N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B).
\]

Let us compute \( N(A) \). SWORD can start from position 1 up to position 22; in each case the remaining 21 letters can be arranged in \( 21! \) ways. Therefore \( N(A) = 22 \cdot 21! = 22! \).

Similarly \( N(B) = N(C) = 22! \). Next we compute \( N(A \cap B) \). Let us first look at the case when SWORD goes before PLANT. SWORD can start from any position from 1 to 17, the respective numbers of ways to place PLANT will be 17, 16, …, 1. The remaining 16 letters can be arranged in \( 16! \) ways. Thus we obtain
\[
N(A \cap B) = 2(1 + 2 + \ldots + 17)16! = 2 \frac{17 \cdot 18}{2} 16! = 18!
\]

The answer is
\[
N(A \cup B \cup C) = 3 \cdot 22! - 18! = 3371995780959117312000.
\]

15. Find the number of ways to climb a 12-step staircase, if you go up either one or three steps at a time.

**Solution.** Let \( a_n \) be the number of ways to climb \( n \) steps by going one or three steps at a time. Then we have a recurrent relation
\[
a_n = a_{n-1} + a_{n-3}
\]

with the initial conditions \( a_1 = a_2 = 1, a_3 = 2 \). From here it is easy to compute directly that \( a_{12} = 60 \).