

# Math 152 Homework Assignment 1

February 13, 2007

**Exercise 1.** Let  $\Omega = \{H, T, E\}$  (a set with 3 elements, called  $H$ ,  $T$  and  $E$ . You can think of “heads”, “tails” and “edge” if you like, but this interpretation is not necessary).

1. List the elements of the *powerset*  $P(\Omega)$  of  $\Omega$ . (Recall that the powerset of a set is the set of all subsets of that set, including  $\emptyset$  (the empty set) and the whole set.)
2. Suppose  $P$  is a set function on  $\Omega$  (that is, a rule assigning real numbers to subsets of  $\Omega$ ). Suppose that

$$P(\{H, T\}) = \frac{3}{4}, P(\{H, E\}) = \frac{3}{4}, P(\{H\}) = \frac{1}{4}$$

Why is it impossible that  $P$  is a probability measure? (That is, no matter how  $P$  is defined on the remaining subsets of  $\Omega$ , it can't be a probability measure.)

3. Can you find a probability measure  $P$  on  $\Omega$  with  $P(\{H, T\}) = \frac{3}{4}$ ,  $P(\{H, E\}) = \frac{3}{4}$ ? What constraints are there on  $P(\{H\})$ ? That is, what values of  $P(\{H\})$  are possible and what values are not possible? (Hint: we saw in class that a probability measure is determined by its values on the one-element subsets. Using this you can determine that  $P(\{E\}) = P(\{T\})$ . What else do you know about the numbers  $P(\{H\})$ ,  $P(\{T\})$ ,  $P(\{E\})$ ? Use the properties of a probability measure and the information given.)

**Exercise 2.** Again let  $\Omega = \{H, T, E\}$  and regard  $\Omega$  as the sample space for the experiment of flipping a fair coin once, where we count “landing on edge” as a conceivable outcome (called  $E$ ).

1. What probability would you assign to the event  $\{E\}$ ? (If you have no idea, just pick a small positive number, like 0.001).
2. Given your assignment of  $P(\{E\})$  and given that the coin is fair (heads as likely as tails, but remember there's a third possibility) what are  $P(\{H\})$  and  $P(\{T\})$ ?

- Write all the subsets of  $\Omega$  (you did this already in the previous exercise) and their probabilities under the probability measure  $P$  you defined as a table (one column with the subset  $E$ , the next column with  $P(E)$ ).

**Exercise 3.** Using Venn diagrams, explain why the following identities or facts are true for any sets  $A, B, C$  (subsets of some set  $\Omega$ ).

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \setminus B = A \cap (\Omega \setminus B)$
- if  $A \subset B$  then  $\Omega \setminus A \supset \Omega \setminus B$
- $\Omega \setminus (A \cup B) = (\Omega \setminus A) \cap (\Omega \setminus B)$

**Note.** Properties 1 and 2 are distributive properties. Unlike addition and multiplication, either operation distributes over the other, so we have two distributive laws instead of one.

**Exercise 4.** If  $\Omega$  is any finite set, we can define a set function  $P$  on  $\Omega$  by the rule

$$P(E) = \frac{|E|}{|\Omega|}$$

here the bars  $|E|$  mean “number of elements in” the set  $E$ , also called the cardinality of  $E$  (the latter term is used even when  $E$  is infinite). The same thing is also written  $n(E)$  or  $\#E$  or  $\text{card}E$ .

Explain why this definition satisfies the properties of a probability measure. You can use Venn diagrams if that’s helpful. (We talked about this in class, so you may want to consult your notes.)

**Notes.** This measure  $P$  is called the *uniform measure* on  $\Omega$ , also called the *equiprobable measure*.

Note that for the set  $\Omega = (0, 1)$  of all real numbers strictly between 0 and 1, we can define a uniform measure as length. In this case all the one-element subsets have measure 0 (a point has 0 length). We can still do probability, but we have to rely on the measure rather than the probabilities of individual outcomes.

**Exercise 5.** Let  $\Omega = \{\text{TTT}, \text{TTH}, \dots, \text{HHH}\}$  be the sample space for the experiment of flipping a coin 3 times (only counting heads or tails as outcomes).

- What probability measure is appropriate if the coin is fair?
- What are the probabilities of the events  $\{\text{HHH}\}$  (three heads) and  $\{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}\}$  (at least two heads)?

**Exercise 6.** The *complement* of a set  $E \subset \Omega$  (relative to  $\Omega$ ) is the set difference  $\Omega \setminus E$ . The complement of  $E$  is often written in any of the following ways:  $E^c, \overline{E}, \widetilde{E}, E', -E$ .

1. Why is  $P(E^c) = 1 - P(E)$  for any subset  $E$  of  $\Omega$  and probability measure  $P$  on  $\Omega$ ?
2. Show that  $(E \cup F)^c = E^c \cap F^c$  via Venn diagrams. (What is  $E^c$  in terms of Venn diagrams?)

**Note.** Some of the facts in Exercise 3 can be expressed in terms of complements. You might want to rewrite them that way as it may be easier to interpret. In fact, you may notice that one of them is part 2 here.