Lesson 1

Topics: Variables and algebraic expressions; Evaluation of algebraic expressions.

Variables and algebraic expressions as symbolical representation of number

Suppose that you thought of a number but you did not tell me what it was. I can think about your number as a number x. Symbol x is an example of a variable.

Variable **A variable is a symbol that represents an unknown number.**

The choice of the name of a variable is arbitrary: one can as well call it n, m or Ψ. **We treat variables as they were numbers we could use.** We can, for example, add numbers to variables: $m+3$, or subtract other variables from them: $m-\Psi$. We can multiply them: $4\cdot m$, divide: $\frac{\Psi}{m}$, raise to any given power: m^2 and then, if we wish, add all expressions together: $4\cdot m+\frac{\Psi}{m}+m^2$. The resulting expressions are called algebraic expressions.

Algebraic Expression **An algebraic expression is a number, variable or combination of the two connected by some mathematical operations like addition, subtraction, multiplication, division, or exponentiation.**

Notice that numbers and variables are also examples of algebraic expressions. We can refer to 3, x, or y as algebraic expressions.
Just like $4\cdot 5$, $2-5$, or 3^2-1 are numbers (written in a 'complicated' way, but numbers), algebraic expressions $4\cdot m$, $x-y$ or a^n-b are symbolic representation of numbers. Both **variables and algebraic expressions could be thought of as unknown numbers.**

Correct language and conventions used when forming algebraic expressions

Algebraic expressions are read using the same terminology as in arithmetic. For example, $A+5$ can be read as "A plus 5" or "the sum of A and 5"; y^2 can be read as "y raised to the second power" or "y squared"; $-x$ is read "minus x" or "the opposite of x". The following convention is commonly adopted to indicate multiplication of a number and a variable, or multiplication of variables.

To denote the operation of multiplication, the sign of multiplication between a number and a variable or between two variables or expressions does not have to be explicitly displayed, so for example:

2*A* means 2 times *A*

***xy* means *x* times *y*,**

y*(*a*+*b*) means *y* times the quantity *a*+*b

According to the above convention the following is true.

$$\boldsymbol{x = 1\cdot x = 1x}$$

Although $x = 1x$, it is **customary to write *x* instead of 1*x*** (just like any time we want to write 4, we just write 4 not $1 \cdot 4$). The following is also true.

$$\boldsymbol{-x = -1\cdot x = -1x}$$

Again, it is **customary to write – *x* instead of –1*x*.**

When forming algebraic expressions, we place parentheses according to the notational convention adopted in arithmetic.

Any time two operation signs are next to each other, parentheses are needed. For example,

we write:	we *do not* write:
$2-(-1)$	$2--1$,
$2 \div (-1)$	$2 \div -1$
$2 \times (-1)$	2×-1

Likewise, when multiplying a and $-b$, we write $a(-b)$**.** The parentheses are needed even if the multiplication sign is not explicitly displayed Notice that if the parentheses are omitted, the expression changes its meaning from multiplication of a and $-b$ to subtraction: $a-b$.

Exponents pertain only to 'the closest' expression. You might, for example, recall that to indicate that the entire fraction is raised to a given power, we use parentheses. We write $\left(\frac{2}{5}\right)^2$. Similarly, **whenever you exponentiate any algebraic expression that is not represented by a single symbol, you must use parentheses**: $\left(\frac{x}{y}\right)^2$, $(-x)^2$ $(y-x)^2$. On the other hand, 5^2 x^2, y^2 do not require parentheses**.**

Example 1.1 How are the following expressions read?

a) x^2 b) xz c) x^3

d) x^m e) $-x$ f) $\frac{x}{y}$

Solution:

a) ‘ x raised to the second power’ or ‘ x squared’
b) ‘ x times z ’, or ‘ the product of x and z ’, or ‘ x multiplied by z ’
c) ‘ x raised to the third power’ or ‘ x cubed’
d) ‘ x raised to the m-th power’
e) ‘minus x ’ or ‘the opposite of x ’
f) ‘ x divided by y’ or ‘the quotient of x and y’

Example 1.2 In the following expressions parentheses are needed. Explain why they are needed.

a) $a \div (-c)$ b) $(-a)^n$

Solution:

a) Any time two operation signs are next to each other, parentheses are needed. In this case the “÷ sign is followed by “–”sign.

b) Without the parentheses, only a would be raised to the $n-$th power. With parentheses, we raise $-a$ to the n-th power. The two statements have different meaning, thus parentheses are needed.

Algebraic expressions allow us to express mathematical ideas in a general form

Algebraic expressions allow us to write mathematical ideas in symbols, without using specific numbers. For example, the area of a square is equal to the square of the length of its side. Not every square is going to have the same size, so we use a variable to represent the length of a side. If we denote s to be a side of a square, then the area of the square can be expressed as $s \cdot s = s^2$

Example 1.3 Let x and y denote two different numbers. Express the following statements using algebraic symbols.

a) The sum of x and y
b) The difference between x and y
c) The product of x and y
d) The quotient of x and y

Solution:

a) $x + y$
b) $x - y$
c) xy

d) $x \div y$ or equivalently $\frac{x}{y}$

Example 1.4 Find the algebraic expressions representing the opposite of the following expressions. Do not simplify.

a) x b) $-x$

Solution:
Recall that to find the opposite of a number, we must multiply the number by -1, thus

a) The opposite of x is $-1 \cdot x = -x$

b) The opposite of $-x$ is $-1 \cdot (-x) = -(-x)$ Please, notice that since the minus sign is directly followed by another minus sign, the parentheses are needed.

Example 1.5 Use the letter x to represent a number and write the following statements as algebraic expressions.

a) Double a number
b) Two thirds of a number
c) A quantity increased by 3

Solution:

a) $2x$
b) $\frac{2}{3}x$
c) $x+3$

Example 1.6 Write the following statements as algebraic expressions.

a) x subtracted from A
b) $-x$ added to $-A$
c) $-x$ multiplied by $-y$
d) $\frac{a}{b}$ raised to sixteenth power

Solution:

a) $A-x$ Notice 'the reversed order' of variables (if you were asked to subtract 3 from 6, you would write $6-3$, similarly we write $A-x$).

b) $-x+(-A)$ Notice the use of parentheses: a plus sign, followed by a minus sign, requires parentheses.

c) $(-x)(-y)$ or $-x(-y)$ Notice the use of parentheses: multiplication sign (even if not explicitly displayed), followed by a minus sign, requires parentheses.

d) $\left(\frac{a}{b}\right)^{16}$ Parentheses are needed to indicate that entire $\frac{a}{b}$ is raised to the sixteenth power.

As mentioned, we can use variables and algebraic expressions to describe certain quantitative relationships without information about their specific values. In a certain store, a cake costs 5 dollars. Let x be a variable that represents the number of cakes we plan to buy in that store. To calculate how much we will pay for such a purchase, we multiply the price of one cake, 5 dollars, by the number of cakes we buy, x. Thus, we can express the cost as $5 \cdot x = 5x$. The algebraic expression $5x$ represents the cost of x cakes bought at 5 dollars each. From now on, any time we know the number of cakes we wish to buy, i.e. we know the value of x, we can find the price by **evaluating** the expression $5x$.

To Evaluate an Expression **To evaluate an algebraic expression means to find its value once we know the values of its variables. Each variable has to be replaced by its value and the resulting numerical expression has to be calculated.**

For example:
Evaluate $5x$ when $x = 10$ (in the above example it would mean finding the price of 10 cakes at 5 dollars each).

$$5x = 5 \times 10 = 50$$

The price of 10 cakes is 50 dollars.

Notice, that what we did was to substitute 10 for x in the expression $5x$. We could do that because x is equal to 10. The following fundamental principle underlies the process of evaluation.

> If two quantities are equal, you can always substitute one for the other.
> **"Equals can be substituted for equals"**

Can any algebraic expression be evaluated? Let us consider evaluation of $\frac{1}{a}$ when $a = 0$. If we replace a by its value 0, the operation we would have to perform is division by 0, but **division by zero is not defined**, so $\frac{1}{a}$ cannot be evaluated if $a = 0$. Expressions like $\frac{x}{0}$, or $x \div 0$ are undefined.

Example 1.7 Rewrite each expression replacing variables with their values, and then evaluate, if possible. If evaluation is not possible, explain why it is not possible.

a) $x - 6$, if $x = 5$ b) $-C$, if $C = -2$

c) $2x$, if $x = -3$ d) 2^n, if $n = 3$

e) $y - x$, if $x = -1$ and $y = \frac{1}{3}$ f) $\frac{1}{x+2}$, if $x = -2$

Solution:
Each variable should be replaced by its assumed value and the obtained numerical sentence has to be evaluated. Please, pay attention to the way parentheses are used.

a) $x-6=5-6=-1$
b) $-C=-(-2)=2$
c) $2x=2(-3)=-6$ Remember that $2x$ means multiplication of 2 and x.
d) $2^n=2^3=8$
e) $y-x=\frac{1}{3}-(-1)=\frac{1}{3}+1=1\frac{1}{3}$
f) $\frac{1}{x+2}=\frac{1}{-2+2}=\frac{1}{0}$ The expression cannot be evaluated, since division by zero is not defined.

Common mistakes and misconceptions

Mistake 1.1

When evaluating $2\times 10+1$, it is INCORRECT to write

$$2\times 10+1=20=21 \qquad (20\neq 21)$$

Instead, one should write:

$$2\times 10+1=20+1=21$$

Numbers or expressions not involved in the operation that is being carried out *must always be rewritten*. Equal sign means that the quantities on either side are equal.

Mistake 1.2

When $x=-1$, and you are asked to evaluate $-x$, you must be careful to write $-x=-(-1)$. Do not forget to recopy the minus sign before -1. It is incorrect to evaluate $-x$ by simply writing $-x=-1$.

Mistake 1.3

One should NOT think that $-x$ always represents a negative quantity. It depends on the value of x. If, for example, $x=-1$, then $-x=-(-1)=1$. SO we see that if x is a negative value, $-x$ actually represents a positive quantity

Mistake 1.4

When writing a^m, do NOT place m at the same level as a but slightly higher. Otherwise, a^m becomes am. These two expressions have different meanings.

Mistake 1.5

Expression x^2 is NOT read as 'x two' or 'two x'. Instead, read it as 'x raised to the second power' or 'x squared'