Math 270/Linear Algebra

Exam 1 (100 points)

1. Let $G_{10} := \mathbb{R} \setminus \{10\}$. Define a binary operation on G_{10} as follows:

$$a \otimes b := ab - 10a - 10b + 110,$$

where juxtaposition denotes ordinary real number multiplication and "+" denotes ordinary real number addition. Prove that $\langle G_{10}, \otimes \rangle$ is a group:

(a) (**Closure**) Show that $(a \in G_{10}) \wedge (b \in G_{10}) \implies a \otimes b \in G_{10}$.

(b) (**Associativity**) Show that $\forall a, b, c \in G_{10}$, one has $a \otimes (b \otimes c) = (a \otimes b) \otimes c$.

(c) (**Identity**) Identify that element $e \in G_{10}$ for which $a \otimes e = e \otimes a = a$ holds $\forall a \in G_{10}$.

(d) (**Existence of Inverse**) For each $a \in G_{10}$, identify that element $a^{-1} \in G_{10}$ for which $a \otimes a^{-1} = a^{-1} \otimes a = e$ holds.

2. Given that the multiplication for $\mathbb{Z}_{13}$ is

$\cdot$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{8}$	$\overline{9}$	$\overline{10}$	$\overline{11}$	$\overline{12}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{8}$	$\overline{9}$	$\overline{10}$	$\overline{11}$	$\overline{12}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{8}$	$\overline{1}$	$\overline{12}$	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{7}$	$\overline{9}$	$\overline{11}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{9}$	$\overline{12}$	$\overline{2}$	$\overline{5}$	$\overline{8}$	$\overline{11}$	$\overline{1}$	$\overline{4}$	$\overline{7}$	$\overline{9}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{8}$	$\overline{12}$	$\overline{3}$	$\overline{7}$	$\overline{11}$	$\overline{2}$	$\overline{6}$	$\overline{10}$	$\overline{1}$	$\overline{5}$	$\overline{9}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{10}$	$\overline{2}$	$\overline{7}$	$\overline{12}$	$\overline{4}$	$\overline{9}$	$\overline{1}$	$\overline{6}$	$\overline{11}$	$\overline{3}$	$\overline{8}$
$\overline{6}$	$\overline{0}$	$\overline{6}$	$\overline{12}$	$\overline{5}$	$\overline{11}$	$\overline{4}$	$\overline{10}$	$\overline{3}$	$\overline{9}$	$\overline{2}$	$\overline{8}$	$\overline{1}$	$\overline{7}$
$\overline{7}$	$\overline{0}$	$\overline{7}$	$\overline{1}$	$\overline{8}$	$\overline{2}$	$\overline{9}$	$\overline{3}$	$\overline{10}$	$\overline{4}$	$\overline{11}$	$\overline{5}$	$\overline{12}$	$\overline{6}$
$\overline{8}$	$\overline{0}$	$\overline{8}$	$\overline{3}$	$\overline{11}$	$\overline{6}$	$\overline{1}$	$\overline{9}$	$\overline{4}$	$\overline{12}$	$\overline{7}$	$\overline{2}$	$\overline{10}$	$\overline{5}$
$\overline{9}$	$\overline{0}$	$\overline{9}$	$\overline{5}$	$\overline{1}$	$\overline{10}$	$\overline{6}$	$\overline{2}$	$\overline{11}$	$\overline{7}$	$\overline{3}$	$\overline{12}$	$\overline{8}$	$\overline{4}$
$\overline{10}$	$\overline{0}$	$\overline{10}$	$\overline{7}$	$\overline{4}$	$\overline{1}$	$\overline{11}$	$\overline{8}$	$\overline{5}$	$\overline{2}$	$\overline{12}$	$\overline{9}$	$\overline{6}$	$\overline{3}$
$\overline{11}$	$\overline{0}$	$\overline{11}$	$\overline{9}$	$\overline{7}$	$\overline{5}$	$\overline{3}$	$\overline{1}$	$\overline{12}$	$\overline{10}$	$\overline{8}$	$\overline{6}$	$\overline{4}$	$\overline{2}$
$\overline{12}$	$\overline{0}$	$\overline{12}$	$\overline{11}$	$\overline{10}$	$\overline{9}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

(a) solve the equation $\overline{5}x = \overline{3}$ in $\mathbb{Z}_{13}$;

(b) solve the equation $\overline{6}x^2 = \overline{7}$ in $\mathbb{Z}_{13}$.

3. Compute the matrix products

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{bmatrix} a & b & c & d \end{bmatrix}$$

and

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

4. Write out the 3×3 matrix whose entries are given by $x_{ij} = (-1)^{i-j}$.

5. (a) Show that if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

then the sum of the diagonal elements of $AB - BA$ is zero.

(b) If E is a 2×2 matrix and the sum of the diagonal elements of E is zero ($AB - BA$, for instance), show that $E^2 = \lambda I_2$ for some scalar λ.

(c) Deduce from the above that if A, B and C are 2×2 matrices, then

$$(AB - BA)^2 C = C(AB - BA)^2.$$

6. Let $X = \begin{bmatrix} a & b & c \end{bmatrix}$ and

$$A = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

where $a^2 + b^2 + c^2 = 1$.

(a) Show that $A^2 = X'X - I_2$.

(b) Prove that $A^3 = -A$. (Hint: use part (a).)

(c) Find A^4 in terms of X. (Hint: use part (b)).

7. Let A be the matrix

$$\begin{bmatrix} 0 & a & a^2 & a^3 \\ 0 & 0 & a & a^2 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Define the matrix B by

$$B := A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \frac{1}{4}A^4 + \ldots$$

Show that this series has only finitely many terms different from zero and calculate B.

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