Math 270/Linear Algebra

Assignment 1. Due Monday, 20 May 2002, at the beginning of class.

1. Let $r \in \mathbb{R}$ be an arbitrary real number and let $G_r := \mathbb{R} \setminus \{r\}$. Define a binary operation on G_r as follows:

$$a \otimes b := ab - ra - rb + r^2 + r,$$

where juxtaposition denotes ordinary real number multiplication and "+" denotes ordinary real number addition. Prove that $\langle G_r, \otimes \rangle$ is a group.

2. (a) Write the addition and multiplication tables of $\mathbb{Z}_{13}$ under natural addition and multiplication modulo 13.

   (b) Solve the equation $\overline{5}x = \overline{3}$ in $\mathbb{Z}_{13}$.

   (c) Solve the equation $\overline{6}x^2 = \overline{7}$ in $\mathbb{Z}_{13}$.

3. Let $F := \mathbb{Z} \times \mathbb{Z}^*$ and define a relation on F as follows:

$$(a, b) \sim (c, d) \iff ad - bc = 0.$$

   (a) Show that $\sim$ is an equivalence relation on F.

   (b) Let $\mathcal{F}$ denote the partition of F determined by the equivalence relation $\sim$ and consider $[(a, b)] \in \mathcal{F}$, $[(c, d)] \in \mathcal{F}$. Thus, for example, $[(a, b)] = \{(x, y) : ay - bx = 0\}$. Define addition on $\mathcal{F}$ as follows:

$$[(a, b)] + [(c, d)] := [(ad + bc, bd)].$$

   Show that $\langle \mathcal{F}, + \rangle$ is a group.

4. Let $\sigma \in S_n$. Prove that $\sigma^{n!} = \mathbf{Id}$ as follows:

   (a) Consider a k-cycle $\alpha \in S_n$. For what values of $t \in \mathbb{Z}_+$ will $\alpha^t = \mathbf{Id}$ hold?

   (b) Show that disjoint cycles $\alpha \in S_n$ and $\beta \in S_n$ commute; i.e., $\alpha\beta = \beta\alpha$.

   (c) Given a product of disjoint cycles, say $\alpha\beta$, show that $(\alpha\beta)^t = \alpha^t\beta^t$ for all $t \in \mathbb{Z}_+$.

   (d) Use (a), (b) and (c) to conclude that for an arbitrary $\sigma \in S_n$, one has $\sigma^{n!} = \mathbf{Id}$.

   Consider an arbitrary shuffle σ of an ordinary deck of 52 playing cards. Since $\sigma \in S_{52}$, one is guaranteed that the same shuffle repeated 52! times will return the deck of cards to its original order.

   John E. Majewicz: 1 May 2002