Mathematics 017: Introductory Algebra

Joanne Darken

Community College of Philadelphia

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Preface

Although this text is tagged as the Abstract in the Front Matter, it will appear in the typeset document as the Preface. Replace this text with your preface.

Basics

0.1 Introduction

Algebra is based on patterns that occur in the operations on numbers: addition, subtraction, multiplication and exponentiation. It is useful in solving various practical problems that arise in science, business and everyday life, and you need it to advance in mathematics and science. Also, and very importantly, algebra trains you in ways of thinking needed to work effectively with complex technology and a complex society.

A major goal of this introductory course is to learn to solve equations; that is, find a number that satisfies certain conditions. An example: suppose you need to come up with 350 dollars in 3 days, and all you have right now is \$23.75. You wonder if you can get the \$350 if you work very hard. Suppose you're can work 16 hours a day for the three days. How much would you have to clear per hour to reach your goal? Well, you need \$350 - \$23.75 = \$326.25, and you have 48 hours of work. You divide \$326.25 by 48 and find you need to make \$6.80 an hour after taxes.

You don't see any algebra in that little chain of reasoning, but the problem can be set up as an algebra problem: $23.75 + 48 \times P = 350$. Here the letter P stands for the pay rate per hour, and when we put in \$6.80 for P we get $23.75 + 48 \times 6.80 = 350.15$. It's a few cents over the goal because the 6.80 was rounded up to the nearest cent.

Far more complicated problems can be solved by the methods of algebra, and we will spend time later in the course on methods that take you from beginning to end step by step so that even if the problem is too hard to see all at once, you can to the answer by stages, the same way you can get to a new place by following directions of the type "Go three blocks, turn right, go to the second stop since and turn left, etc., etc." But as with getting to a place, it's easier if you know the territory (for example, if you know the area, you can take an alternate route if your first plan is blocked). So it is with algebra. Besides learning the routines to use, you should try to understand the territory—why things work the way they do. Then you can get an overview.

Part of the effort of learning algebra goes into becoming familiar with the terminology and notation, and explaining things to yourself and others In a way it's like learning a new language—you have to use it. And we have to be meticulous about what words mean and what is allowed and what isn't. So we will spend considerable time looking carefully at details.

But before we get technical about details, here is a set of problems to give you a sense of the sort of endeavor you are embarking on. Each equation below has a letter in it, and the job is to find a number to replace the letter so that the equation is true. The letter is called a *variable.*

Definition 1 *A variable is a symbol used to stand for a number.*

For each equation, the same number must be used each time the letter appears, if it appears more than once.

Definition 2 *A number that makes an equation true when substituted for the variable is called a solution of the equation.*

Each of the equations has at least one solution among the whole numbers (0, 1, 2, 3, 4, etc.), and in each case a solution can be found by inspection or persistent trial and error. Some equations have more than one solution.

Exercise 3 *Find at least one solution for each equation:*

1. $x + 1 = 13$

2. $2 \times r - 1 = 19$

3. $x + 10 = 10 + x$

4. $(A - 1) \times (A - 1) = A \times A - 2 \times A + 1$

5. $x - 13 = 29$

6. $B \times B = 25$

7. $S \times S = 169$

8. $a \times a - a = 6$

9. $T \times (T - 1) = 0$

10. $x^2 - 3 = 46$

11. $\frac{1}{2} \times C = 12$

12. $4 \times (P - 1) = 20$

13. $y + 3 = 2 \times y$

14. $L - (L - 1) = 1$

15. $2 \times (x + 1) = x + 5$

The equations above are all of a type you will learn how to solve in a methodical way in this course. When you learn the methods you will no longer have to use trial and error, but it is a good idea when you come to a new equation to give a quick look to see if you can find the answer by inspection. Even if you can't the effort will be good practice in reading equations.

0.2 Operations on Whole Numbers

Now we review the basic operations on numbers. For now we will work only with whole numbers (0, 1, 2, 3, 4, etc). . As you work the exercises, look for patterns and shortcuts.

Addition

Addition is denoted by +. In arithmetic, it may be written vertically.

Example

$$\begin{array}{r} 2 \\ \underline{+3} \\ 5 \end{array}$$

or horizontally:

$$2 + 3 = 5$$

In algebra, the horizontal layout is more common.

Exercise 4 *In all the exercise sets, * after the number of an exercise means the problem is harder.*

1. Compute the answer. If you see any shortcuts or notice that an answer will be the same as one you got before, use the fact to save yourself work. One of the problems is not written correctly. When you see such a problem, do not try to guess what the writer meant. Just say that it does not make sense (or abbreviate DNMS).

(**a**) $2 + 2 + 2 + 2 + 2 + 2$

(**b**) $2 + 2 + 2 + 3 +$

(**c**) $2 + 3 + 2 + 2 + 2$

(**d**) $3 + 3 + 3 + 2 + 2$

(**e**) $2 + 3 + 3 + 3 + 3$

(**f**) $2 + 2 + 2 + 5 + 5 + 5$

(**g**) $2 + 5 + 2 + 5 + 2 + 5$

(**h**) $1 + 1 + 1 + 1 + 1 + 1 + 1$

(**i**) $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

(**j**) $1 + 0 + 2 + 0 + 3 + 0$

(**k**) $1 + 0 + 0 + 17$

(**l**) $0 + 0 + 0 + 0$

(**m**) $1 + 9 + 2 + 8 + 3 + 7 + 4 + 6 + 5$

(**n**) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$

(**o**) You should have gotten the same answer for both **1m** and **1n**. Compare them and explain why.

(**p**) Find the sum of all the integers from 1 to 19. You can use what you noticed in **1o** to do this more easily.

(**q**) Find the sum of all the integers from 1 to 29. You can use what you noticed in **1o** to do this more easily.

(**r**) * If you see the pattern that makes it easier to do **1p** and **1q**, use it to find the sum of all the integers from 1 to 99.

2. There's a quicker way of computing $2 + 2{+}2 + 2{+}2{+}2$ than adding. What is it?

3. Use the better method for the previous problem to compute $2 + 2{+}2 + 2{+}2{+}2 + 2{+}2 + 2{+}2{+}2 + 2{+}2 + 2{+}2{+}2$ with a minimum of work.

4. Fill in the blanks.

(**a**) $1+$ _____ $= 9$

(**b**) $10 +$ _____ $= 17$

(**c**) $1 +$ _____ $+ 3 = 7$

(**d**) $1 + 1 + 1 +$ _____ $+ 1 = 7$

(**e**) $12 +$ _____ $= 29$

(**f**) $3 + 3 + 3 +$ _____ $+ 3 = 31$

(**g**) $1 + 2 + 3 = 3 +$ _____ $+ 1$

(**h**) $9 + 0 =$ _____ $+ 9$

(**i**) $0 + 0 + 0 + 0 =$ _____

(**j**) $1 + 2 + 4 + 8 + 16 = 16 + 8 + 4 +$ _____ $+ 1$

5. In the exercise above, you use another operation besides addition. What is it?

6. Some of the equations are true and some not, and it's possible to tell which without actually doing the additions. Which are true and which are not?

(**a**) $12 + 59 = 59 + 12$

(**b**) $12,345 + 91 = 91 + 12,345$

(**c**) $457 + 11 = 457 + 12$

(**d**) $27 + 2 + 3 = 2 + 3 + 27$

(**e**) $11 + 22 + 33 + 44 = 44 + 33 + 22 + 11$

(**f**) $1 + 2 + 3 + 4 + 5 + 6 = 1 + 6 + 2 + 5 + 3 + 4$

(**g**) $897,215 + 17 = 897,215 + 18$

7. Solve for x:

(**a**) $x + 5 = 42$

(**b**) $x - 13 = 84$

Subtraction

Subtraction is the opposite of addition; that is, it undoes addition. For example, if you add 2 to 6 to get 8 and want to get 6 back, you subtract 2 from 8.

Remember that you check a subtraction problem by addition.

Example

$$6 - 2 = 4$$

because

$$2 + 4 = 6$$

Exercise 5 *Subtraction*

1. For each subtraction write the corresponding addition.
(Example: For $9 - 2 = 7$, the corresponding addition is $7 + 2 = 9$.)

(**a**) $7 - 3 = 4$

(**b**) $13 - 6 = 7$

(**c**) $5 - 1 = 4$

(**d**) $5 - 2 = 3$

(**e**) $5 - 3 = 2$

(**f**) $5 - 4 = 1$

(**g**) $5 - 5 = 0$

(**h**) * $5 - 6 = -1$

(**i**) $123 - 57 = 66$

2. Fill in each blank with a number that makes the equation true. Check by writing the corresponding addition problem.

(**a**) _____ $-5 = 3$

(**b**) $\_\_\_\_\_ - 3 = 5$

(**c**) $7 - \_\_\_\_\_ = 3$

(**d**) $\_\_\_\_\_ - 0 = 9$

(**e**) $\_\_\_\_\_ - 0 = 23$

(**f**) $32 - \_\_\_\_\_ = 32$

(**g**) $158 - \_\_\_\_\_ = 0$

(**h**) $19 - \_\_\_\_\_ = 0$

(**i**) $14 - \_\_\_\_\_ = 14$

(**j**) $8 - 5 - \_\_\_\_\_ = 0$

3. Some of the equations are true and some not, and it's possible to tell which without actually doing the subtractions. Which are true and which are not?

(**a**) $9 - 5 = 5 - 9$

(**b**) $7 - 1 - 2 = 7 - 2 - 1$

(**c**) $457 - 93 = 457 - 90 - 3$

(**d**) $27 - 2 + 3 = 2 = 27 + 3 - 2$

(**e**) $11 - 0 = 0 - 11$

4. Solve for x:

(**a**) $x - 7 = 5$

(**b**) $12 - x = x$

Multiplication

Multiplication is denoted by $\times$ or $\cdot$ or parentheses:

Examples

$$\begin{aligned} 2 \times 3 &= 2 \cdot 3 = 2(3) = (2)3 = 6 \\ 2(3 \times 5) &= 2 \cdot (3 \cdot 5) = 2 \times (3 \cdot 5) \end{aligned}$$

Definition 6 *The answer to a multiplication is called the product.*

Definition 7 *The numbers being multiplied are called factors.*

In the first example just above, the factors are 2 and 3, and the product is 6, and in the second, the factors are 2, 3, and 5, and the product is 30. When we speak of the factors of a particular counting number we mean the counting numbers that go into it evenly; that is, without a remainder. This is the same as saying there is another counting number you can multiply the factor by to get the particular number under discussion.

(This gives us two different but related meaning for the word *factor:* we say 1, 2, 3, 5, 6, 10, 15 and 30 are the factors of 30—those are the numbers that go into 30 evenly. It is also true that $30 = \frac{1}{2} \times 60$ and we can speak of factoring $\frac{1}{2}$ out of 60 to get 30, and say $\frac{1}{2}$ is one of the factors in the product $\frac{1}{2} \times 60$. By we do not say $\frac{1}{2}$ is a factor of 30)

You need to get used to the terminology and the various ways of representing multiplication. In algebra, $\times$ is not often used, since it looks too much like the letter x, which we often use as a variable; the dot $(\cdot)$ and parentheses are much more common. If two variables are multiplied or a number is multiplied by a variable, then it is common to have no symbol between them; for example, ab means a times b, and $3x$ means 3 times x. Of course this doesn't work between two numbers: 23 means twenty three, not 2 times 3.

Exercise 8 *Multiplication*

1. Compute:

(**a**) $2 \cdot 4$	(**e**) $2 \cdot 3 \cdot 4 \cdot 2$
(**b**) $2(3 \times 4)$	(**f**) $3(3 \times 2)$
(**c**) $2 \times (3 \cdot 6)$	(**g**) $(2 \cdot 5)3$
(**d**) $4(4 \times 3)$	(**h**) $4 \cdot 5 \cdot 3$

2. List all the (counting number) factors of:
a) 6 b) 8 c) 12 d) 30 e) 96 f) 140

3. List all the (counting number) factors of each of the following:
a) 7 b) 9 c) 14 d) 24 e) 17 f) 1 g) 16 h) 50
i) 96 j) 37 k) 47

Multiplication is first presented as repeated addition; for example, $2+2+2=3\cdot 2$. Order doesn't matter in multiplication; for example $3\cdot 2=2\cdot 3$. This isn't obvious from the definition in terms of repeated addition, but it's always true. One way of looking at this is to consider an array of six dots in two rows of three:

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

You can think of this as two rows of 3 (that is, as $3+3$, or $2\cdot 3$) or three columns of 2 (that is, $2+2+2$, or $3\cdot 2$). The same two views can be taken for any multiplication of any two natural numbers, and the total numbers of dots is the same either way. When multiplying fractions or decimals the idea of repeated addition is not always applicable—multiplication has a life of its own separate from addition—but the patterns and rules that hold for the natural numbers still hold; for example,

$$\frac{1}{2}\times\frac{3}{4}=\frac{3}{4}\times\frac{1}{2}$$

You don't need to know what the answer is to know this is true; you can always switch the order in multiplication.

Exercise 9 *More multiplication*

1. Write the multiplication corresponding to each addition and find the answer both by adding and by multiplying. Which is quicker?

(**a**) $2+2+2+2$

(**b**) $7+7+7+7+7$

(**c**) $1+1+1+1+1+1+1+1+1+1$

(**d**) $23+23+23+23+23+23+23+23$

2. Some of the equations are true and some not, and it's possible to tell which without actually doing the multiplications. Which are true and which are not?

(**a**) $9\cdot 5=5\cdot 9$

(**b**) $7 \cdot 311 = 311 \cdot 7$

(**c**) $8 \cdot 7 \cdot 9 = 9 \cdot 8 \cdot 7$

(**d**) $34,217 \cdot 98,765 = 98,765 \cdot 34,217$

(**e**) $222 \cdot 333 = 222 \cdot 332$

(**f**) $897,654 \cdot 1 = 897,654 \cdot 0$

(**g**) $7865 \cdot 0 = 8542 \cdot 0$

3. Suppose you had fifteen quarters. What is the easiest way to compute how much money that is?

4. Suppose you buy a book priced at $36, but with a 25% discount, and 7% sales tax. How much difference does it make to what you pay if the discount is applied after the sales tax instead of before?

5. Solve for x:

(**a**) $4x = 80$

(**b**) $4x = 90$

(**c**) $4x = x$

Multiplication is the most interesting of the basic operations on numbers, because of special features of factorization and primes, which we now consider.

If you wanted to write a counting number as the sum of the simplest building blocks possible, you would write it as the sum of however many ones; *e.g.*, $3 = 1 + 1 + 1$. This is pretty dull. But if you want to write a counting number as a *product* of the simplest building blocks possible, things get more interesting. You write the number as a product of numbers that can't be broken down any further by factoring; *e.g.*, $12 = 2 \cdot 2 \cdot 3$. The factors (*i.e.*, the numbers being multiplied) cannot be factored any further, so they are the simplest building blocks for the original number when it's written as a product. We can write 12 in different ways as a product; *e.g.*, $12 = 2 \cdot 6$ or $12 = 3 \cdot 4$, but in each case one of the factors can be factored ($6 = 2 \cdot 3$, $4 = 2 \cdot 2$), so we haven't used the simplest possible building blocks. We need a name for these simplest factors.

Definition 10 *A prime number is a natural number which has exactly two divisors: itself and one.*

This definition is worded carefully so as to exclude one as a prime. If 1 were called prime, then it would always be possible to get another prime factorization but throwing in another factor of 1; *e.g.*, $12 = 2 \cdot 2 \cdot 3 \cdot 1$, and we don't want this. Leaving out 1 as a prime, there is only one way of factoring a number into primes. This is good to know. (If the original number is prime, like 3, then its prime factorization is just itself.)
ExampleThe numbers 2, 3, 5, 7,11, 13, 17, 19 and 23 are prime numbers.

Definition 11 *To factor a number is to write it as a product, which is called a factorization of the number. A factor of a natural number is a natural number that goes into it evenly; i.e., without a remainder.*

Example

$$\text{Factorizations of 12 are } 1 \cdot 12,\ 2 \cdot 6, 3 \cdot 4 \text{ and } 2 \cdot 2 \cdot 3, \text{ since}$$
$$12 = 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4 = 2 \cdot 2 \cdot 3$$

The numbers 1, 2, 3, 4, 6, and 12 are all factors of 12. The numbers 5, 7, 8, 9, 10, 11 and any number bigger than 12 are not.. More generally, a natural number a is a factor of a natural number b if a goes in b evenly, or equivalently, if b can be written as a product $b = ac$, where c is also a natural number.
Example

The factors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, 100.

Definition 12 *The prime factorization of a natural number is its unique factorization into primes.*

There is only one way to factor a number into primes (if order doesn't matter), but if 1 were counted as a prime there would be infinitely many; for example, $2 = 1 \cdot 2 = 1 \cdot 1 \cdot 2 = 1 \cdot 1 \cdot 1 \cdot 2$, *etc.* This is why we don't call 1 a prime number.

The prime factorization of 12 is $2^2 3$.

Example

The prime factors of 100 are 2 and 5.
The prime factorization of 100 is $2^2 5^2$.

There is no end to the primes, and their patterns and properties have been studied for many centuries and are still studied—there are still unanswered questions about them. Besides the interest that the mathematically inclined take in them for their own sake, they have practical uses. For example, if you know the product of two very large primes, you can use it to encode a message in such a way that would take a very long time for anyone to crack who doesn't know those prime factors. This method has been used by governments and industries to send secret messages.

In this course, we use primes in finding the smallest common denominator of fractions to be added or subtracted. In the case of a negative number, factor its opposite into primes and multiply by negative one to get a useful factorization of the negative number. (We usually arrange matters so that denominators are positive, as you will see.)

$$\begin{aligned} &\text{Factor } -28; \\ -28 &= (-1)2^2 7 \end{aligned}$$

This is the most complete factorization of -28: -1 times the prime factorization of 28. The term prime factorization is reserved for natural numbers, however.

Another reason to study factorizations of numbers is that we will have reason to factor algebraic expressions, and factoring numbers is a good refresher for the concept.

Exercise 13 .

1. Find all the primes from 1 to 100.

2. Find *all* factors of each of the following:

(**a**) 24 (**c**) 50 (**e**) 200
(**b**) 30 (**d**) 98 (**f**) 420

3. Write all possible factorizations for each number below into a product of two numbers.

(**a**) 24 (**c**) 50 (**e**) 200
(**b**) 30 (**d**) 98 (**f**) 420

4. Write the prime factorization of each number in Exercise **3**.

Division

Division is the opposite of multiplication in the sense that it undoes multiplication; for example, $5 \times 7 = 35$, so $35 \div 7 = 5$. The check of a division is a multiplication' for example, the check of $12 \div 3 = 4$ is $3 \times 4 = 12$.

Division is denoted by the symbol $\div$ or $/$ or by a fraction bar. (Any fraction can be thought of as the numerator divided by the denominator—whether it's useful to think of it this way depends on the context.)

Example

$$8 \div 2 = 8/2 = \frac{8}{2} = 4$$

Exercise 14 *Division*

1. Compute.

(**a**) $9 \div 3$ (**c**) $\frac{9}{3}$ (**e**) $10/2$ (**g**) $11/6$
(**b**) $9/3$ (**d**) $10 \div 2$ (**f**) $132 \div 3$ (**h**) $15/4$

2. Some of the equations are true and some not, and it's possible to tell which without actually doing the divisions. Which are true and which are not?

(**a**) $9 \div 5 = 5 \div 9$
(**b**) $7 \div 1 = 1 \div 7$
(**c**) $807 \div 1 = 807$

(**d**) $0 \div 95 = 0$

(**e**) $95 \div 0 = 0$

(**f**) $95 \div 0 = 0 \div 95$

3. Solve for x:

(**a**) $x \div 3 = 5$

(**b**) $10 \div x = 2$

(**c**) $12 \div x = 5$

A few comments on the four basic operations of arithmetic

Any two whole numbers can be added or multiplied to get another whole number. But when we subtract one whole number from another we get a whole number answer only if the smaller is subtracted from the larger (for example, $7 - 4$). If we do the subtraction the other way around $(4 - 7)$we don't get a whole number answer. Since people like answers, and answers do come in handy, new numbers have been invented so as to have answers for such problems. These are the negative numbers, which we go over in the next chapter and use throughout the course thereafter.

The same sort of issue arises with division: if we divide one whole number by another we may get a whole number answer or we may not. For example, $12 \div 3 = 4$ and 4 is a whole number, but $3 \div 12$ is not a whole number. Again, new numbers were invented to provide answers, because it is useful to have answers. BUT THERE IS ONE SITUATION IN WHICH WE CANNOT GET AN ANSWER WITHOUT DESTROYING ALL THE WORK AND ALL THE RULES SO FAR INVENTED: WE CANNOT DIVIDE BY ZERO. It turns out that if we try to divide by zero the check with multiplication cannot be made to work: it produces ridiculous results like $0 = 6$. You will see more about this in the section on properties of 0 and 1.

Exponentiation

To raise a number to a power is to multiply copies of it together, and this is called *exponentiation*. It is done often enough that there is special notation for it*;* for example,$2 \times 2 \times 2$ is written 2^3 $(= 8$, of course). The *base* is the number being multiplied (in this case 2), and the *exponent* is the number above it to the right, which tells how many copies of it are to be multiplied (in this case 3). The result (in this case 2^3, or 8) is called a *power* of the base. Speaking, we say 2^3 as "Two to the third power."

Examples

$$3^2 = 3 \times 3 = 9$$
$$5^4 = 5 \times 5 \times 5 \times 5 = 625$$

Exponentiation is related to multiplication the way multiplication is to addition: we start thinking about multiplication in terms of addition, and we start thinking about exponentiation in terms of multiplication. But like multiplication, exponentiation has a life of its own. However, that life is left for later mathematics courses, not this one.

Exercise 15 *Exponentiation*

1. Write in exponential notation and compute the answer.

(**a**) $2 \times 2 \times 2 \times 2$
(**b**) $3 \times 3 \times 3$
(**c**) 5×5
(**d**) $7 \times 7 \times 7$
(**e**) $2 \times 2 \times 2 \times 2 \times 2 \times 2$
(**f**) $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

2. Write as a multiplication and compute the value of:

(**a**) 6^2
(**b**) 9^2
(**c**) 10^3
(**d**) 10^5
(**e**) 3^4
(**f**) 2^6

3. Solve for x.

(**a**) $2^x = 8$
(**b**) $x^2 = 25$
(**c**) $3^x = 27$
(**d**) $x^3 = 125$
(**e**) $2^x = 32$
(**f**) $x^4 = 16$

4. "As I was walking to St. Ives I met a man with seven wives. Each wife had seven sacks, each sack had seven cats, each cat had seven kits." How many kits were there? Write this as a power and compute.

0.3 Order of operations

Often there are several operations to do in one problem, and they are not necessarily done left to right. By convention they are done in the following order:

First, any operations that can be done inside parentheses
Second, exponentiation
Third, multiplication and division, left to right
Fourth, addition and subtraction, left to right
Otherwise work left to right.

You may have heard of a trick for remembering the order of operations: "Please excuse my dear Aunt Sally." Here the first letter of each word stands for an operation: P(arentheses), E(xponentiation), M(ultiplication) & D(ivision), A(ddition) & S(ubtraction), and the they are given in the order in which they are to be done: Do any operations possible inside parentheses first, then exponentiations, then multiplications and divisions, then additions and subtractions. But BE CAREFUL: multiplication and division are on an equal footing, and if you have both you do them from left to right. The same is true of addition and subtraction: if you have both you do them from left to right. This is not clear from the trick.

The order of operations is a convention, like driving on the right side of the road: some other order could have been used instead. But it is necessary to agree on some particular order so that we all get the same answer when we do the same problem (or don't crash into one another when driving).

Examples

$$\begin{aligned} 2 \times 3 + 4 &= 6 + 4 = 10 \\ 3 + 4(5) &= 3 + 20 = 23 \\ 5(8+1) &= 5(9) = 45 \\ (2+3)(7-4) &= (5)(3) = 15 \\ 5 \cdot 2^3 &= 5(8) = 40 \end{aligned}$$

$$\begin{aligned} 3^2 2^3 &= (3 \times 3) \times (2 \times 2 \times 2) = 9 \times 8 = 72 \\ 5 + 2(3-1) &= 5 + 2(2) = 5 + 4 = 9 \\ 2(5-2) + 3 &= 2(3) + 3 = 6 + 3 = 9 \\ (18 - 3^2)4 + 7 \cdot 4 - 2 \cdot 3 &= (18-9)4 + 28 - 6 = 9 \cdot 4 - 28 - 6 = 36 - 28 - 6 = 2 \end{aligned}$$

Exercise 16 *Order of operations*

1. Compute.

   (**a**) $2 \cdot 4 + 8$
   (**b**) $2 + 7 \cdot 5$
   (**c**) $(7-4)^3 + 1$
   (**d**) 3×5^2
   (**e**) $8 - 5(4^2 + 3 \cdot 5)$
   (**f**) $(3^2 - 2^3)5$
   (**g**) $3 + 4 \cdot 5/2$
   (**h**) $8 - 6/2$
   (**i**) $2(15 - 3^2)$
   (**j**) $12 \cdot 3 - 8 \cdot 4 + 6 \div 2$
   (**k**) $2 + 6 \cdot 3$
   (**l**) $1 + 7 \cdot 2$

2. Compute.

   (**a**) $2 + 4(8+1)$
   (**b**) $(3+7)5$
   (**c**) $2(6-4)^2 + (5-2)$
   (**d**) $(1+2)^2 - (8-6)$
   (**e**) $(9-5)(2+3)^2 + 6 \div 3 \cdot 5$
   (**f**) $(24-16) \div (8-6)$
   (**g**) $3 + 12 \div 4 \cdot 3$
   (**h**) $12 - 8 \div 4 \cdot 2$
   (**i**) $(8-5)/(21 - 2 \cdot 3^2)$
   (**j**) $12 \div 3 - 8 \div 4 + 6 \div 2 \times 3$
   (**k**) $4 + 3(10-6) \div 4 \cdot 3$
   (**l**) $(5+7)(21 \div 7 \cdot 2)$

0.4 Basic laws

More fundamental than the conventions for order of operations are the laws that govern the operations with numbers. These laws are like the law of gravity—they're just part of the way things are. We all need to get used to them. But once we do, we can use them to advantage.

The *commutative law for addition* says that if you add two numbers you can do it in either order and get the same answer. In fact, any string of numbers can be added in any order to give the same result.

Examples

$$\begin{aligned} 2 + 3 &= 3 + 2 = 5 \\ 7 + 9 + 11 &= 9 + 7 + 11 = 11 + 9 + 7 = 27 \end{aligned}$$

The *associative law for addition* says that if you have three numbers to add, you can group them differently and still get the same answer.

Example

$$\begin{aligned} (2+3) + 4 &= 2 + (3+4) \\ 5 + 4 &= 2 + 7 \\ 9 &= 9 \end{aligned}$$

It turns out that because of the associative and commutative laws you can add any string of numbers you have in any order you want. (It's not obvious why having the associative and commutative laws for just two or three numbers guarantees this, but it does.) We can add them left to right or in any order we want, and get the same answer.
Example

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = (1 + 9) + (2 + 8) + (3 + 7) + (4 + 6) = 45$$

Notice in this example that it's easier to add the numbers when they are grouped the way they are on the right, since the pairs within parentheses each add up to ten. If you have a string of numbers to add, you can sometimes find pairs that add up to ten, and so make the addition easier.

The *commutative law for multiplication* says that if two numbers are multiplied, the order doesn't matter; the answer is the same either way.
Examples

$$\begin{aligned} 2 \cdot 3 &= 3 \cdot 2 = 6 \\ 8 \cdot 7 &= 7 \cdot 8 = 56 \end{aligned}$$

The *associative law for multiplication* says that if three numbers are multiplied they can be grouped in different ways and the answer is still the same.
.
Examples (In these examples, you see what happens to the expression on the left side of the = sign by looking below it, and what happens to the expression on the right by looking below it.)

$$\begin{aligned} (2 \cdot 3)4 &= 2(3 \cdot 4) \\ (6)4 &= 2(12) \\ 24 &= 24 \end{aligned}$$

$$\begin{aligned} 7(2 \cdot 3) &= (7 \cdot 2)3 \\ 7(6) &= (14)3 \\ 42 &= 42 \end{aligned}$$

The main result of the commutative and associative laws for multiplication is that you can multiply a string of numbers in any order you want. As with addition, this is true no matter how many numbers there are, even though the laws are stated for only two or three numbers.

Examples

1)
To multiply
$2 \times 3 \times 5$
we can multiply 2×3, get 6,
multiply 6×5 to get 30
or
multiply 2×5, get 10,
multiply 10×3, and get 30.
Any other order gives the same result.

$$\begin{aligned}
& 2) \\
& 2 \times 2 \times 2 \times 5 \times 5 \times 5 \\
& \text{can be done from left to right,} \\
& \text{but if we pair each 2 with a 5 to get} \\
& (2 \times 5)(2 \times 5)(2 \times 5) \\
& \text{the multiplication is easier.} \\
& 10 \times 10 \times 10 \\
= \; & 10000
\end{aligned}$$

Try computing $2 \times 2 \times 2 \times 5 \times 5 \times 5$ from left to right to see if you get the same answer as above.

The *distributive law* gives a relation between addition and multiplication. Without using variables (which come in a later section) it is easier to give examples than to state it in English.

Example

$$\begin{aligned}
3(4+6) &= 3 \times 4 + 3 \times 6 \\
3(10) &= 12 + 18 \\
30 &= 30
\end{aligned}$$

An interpretation of this example: if you gave your brother four apples and six oranges, and decide to give your sister three times as much of each as you gave him, then you give her three times as many apples ($3 \times 4 = 12$) and three times as many oranges ($3 \times 6 = 18$), and the total number of pieces of fruit is three times as much.—thirty pieces of fruit instead of ten.

Example

$$\begin{aligned} (2+3)4 &= 2 \times 4 + 3 \times 4 \\ (5)4 &= 8 + 12 \\ 20 &= 20 \end{aligned}$$

Example

$$5(2+4) = 5 \cdot 2 + 5 \cdot 4 = 30$$

It's a little less work to do the computation as it appears on the left, but not so much easier that if you started with the one on the right it would be worth changing it before doing the arithmetic. Sometimes it does save a little work to do so, as in the example below.

Example

By the distributive law $173(-82)+173(82) = 173(-82+82) = 173(0) = 0$. Here you save yourself work if you notice the original expression can be re-written as done here: you don't even need a pencil to find the answer after it's re-written.

Example

Compute $4 \cdot 2 + 4 \cdot 8$ in the easiest way

Answer: $4 \cdot 2 + 4 \cdot 8 = 4(2+8) = 4 \cdot 10 = 40$

This isn't the easiest way unless you see it. When doing arithmetic it's sometimes helpful to look for numbers that add up to ten or a multiple of ten. Note that the number outside the parentheses can be on either side.

Exercise 17 *Basic laws*

1. Work out each side of the equation in the order indicated by the parentheses to check that the two sides of the equation are equal.

   (**a**) $(3+4)+5 = 3+(4+5)$

   (**b**) $2+(8+3) = (2+8)+3$

(**c**) $3 + (7 + 2) = (3 + 7) + 2$
(**d**) $10 + (5 + 6) = (10 + 5) + 6$
(**e**) $(7 + 8) + 5 = 7 + (8 + 5)$
(**f**) $7 + (3 + 9) = (7 + 3) + 9$
(**g**) $10 + (20 + 0) = (10 + 20) + 0$
(**h**) $(13 + 0) + 6 = 13 + (0 + 6)$

2. Add.

(**a**) $3 + 6 + 7 + 4$
(**b**) $6 + 9 + 4 + 1$
(**c**) $1 + 8 + 3 + 2 + 9 + 7$
(**d**) $3 + 8 + 2 + 5 + 7$
(**e**) $6 + 3 + 6 + 7 + 2$
(**f**) * $22 + 56 + 78 + 44$

3. Multiply the numbers on each side of the equation as indicated by the parentheses. See whether you get the same answer on each side.

(**a**) $(2 \times 3)4 = 2(3 \times 4)$
(**b**) $2 \times (3 \times 5) = (2 \times 3) \times 5$
(**c**) $4 \times (1 \times 7) = (4 \times 1) \times 7$

4. Multiply the numbers in the order shown on each side of the equation, and check that you get the same answer either way.

(**a**) $2 \times 3 \times 5 \times 2 = 2 \times 5 \times 2 \times 3$
(**b**) $3 \times 10 \times 10 = 10 \times 10 \times 3$
(**c**) $2 \times 5 \times 5 \times 2 \times 2 \times 5 = (2 \times 5) \times (2 \times 5) \times (2 \times 5)$

5. Choose a string of numbers of your own and multiply them in two different ways; make sure you get the same answer each way.

6. Do the calculation as shown on each side of the = sign, and check that you get the same answer each way.

(**a**) $2(5+7)=2\times 5+2\times 7$

(**b**) $(6+1)3=6\times 3+1\times 3$

(**c**) $3(7+4)=3\cdot 7+3\cdot 4$

(**d**) $(9+1)5=9\times 5+1\times 5$

(**e**) $5(10+2)=5\times 10+5\times 2$

7. Re-write each expression using the distributive law.

(**a**) $(5+2)3$

(**b**) $4(7+2)$

(**c**) $2(8+1)$

(**d**) $(6+2)3$

(**e**) $(3+2)7$

(**f**) $4\times 2+4\times 3$

(**g**) $5\times 2+5\times 1$

(**h**) $3\times 7+2\times 7$

(**i**) $2\times 1+3\times 1$

(**j**) $5\cdot 3+4\cdot 3$

(**k**) $(1+4)3$

(**l**) $*(4+3)(5+2)$

8. Work out each side of the equation in the order indicated by the parentheses to check that the two sides of the equation are equal.

(**a**) $(3+4)+5=3+(4+5)$

(**b**) $2+(8+3)=(2+8)+3$

(**c**) $3+(7+2)=(3+7)+2$

(**d**) $10+(5+6)=(10+5)+6$

(**e**) $(7+8)+5=7+(8+5)$

(**f**) $7+(3+9)=(7+3)+9$

(**g**) $10+(20+0)=(10+20)+0$

0.5 Properties of 0 and 1

Zero

The basic property of zero is that if you add it to any number the answer is that number. Zero has another property that no other number has: if you multiply any number by zero, you get zero.

Examples

$$\begin{aligned} 3+0 &= 3 \\ 0+87 &= 87 \\ 5\times 0 &= 0 \\ 0(67965) &= 0 \\ (5-1)(5-2)(5-3)(5-4)(5-5) &= 0 \end{aligned}$$

In the last equation above you don't have to do any of the subtractions except the last one: since $5-5=0$ it doesn't matter what the other factors are— the product will be zero.

Example

$$(1654-345)(768-451)(678-678)=0$$

Again, you don't have to do all the computations; just notice that since any number minus itself is zero, the last factor is zero.

We also have the **Principle of Zero Products**:

If a product is zero, then one of the factors must be zero. This is very useful sometimes, as we shall see a little later when we work with variables.

Another way in which zero is unique is that division by zero is impossible (as mentioned in the section on division). To understand this, consider the following:

A division can always be checked by a multiplication; for example, $6\div 2=3$ is checked by $2\times 3=6$. Now suppose we think we can divide 6 by 0, and the answer is "something." Then $6\div 0=\textit{something}$, and the check is $0\times \textit{something}=6$. This can't be true, because 0 times anything is 0, so that last equation would turn into $0=6$, which is nonsense. This could be applied to any number, not just 6, so either we say all numbers equal zero, which defeats the purpose of having numbers, or we say that division by zero is impossible, which is what we do.

One

The number one has the special property that if you multiply any number by one you get that number for an answer.

Examples

$$\begin{aligned}
1 \cdot 5 &= 5 \\
1 \cdot 5643 &= 5643 \\
789 \cdot 1 &= 789 \\
1 \cdot 1 &= 1 \\
1^3 &= 1 \times 1 \times 1 = 1 \\
1^8 &= 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 = 1 \\
1^{36} &= 1 \\
(3-2)(4-3) &= 1 \cdot 1 = 1 \\
(9-8)(7-6)(6-5) &= 1 \cdot 1 \cdot 1 = 1 \\
(5-4)^4 &= 1^4 = 1 \\
83(4-3)^2 &= 83 \times 1 = 83 \\
2 \times 3 \times (9-8) &= 2 \times 3 \times 1 = 6
\end{aligned}$$

Exercise 18 *Properties of zero and one*

1. Which products are equal to zero? You don't have to do much arithmetic here; just look at each exercise carefully before you start it.

(**a**) $(7-5)(6-3)(9-9)$

(**b**) $(12-4)(11-5)(10-10)$

(**c**) $(101-101)23$

(**d**) $(12343-7865)(345-345)$

(**e**) $5233(5-5)$

(**f**) $(33-23)(42-42)$

(**g**) $123(675-675)$

2. Compute. You don't have to do a lot of arithmetic here; look at each problem carefully before you start it.

(**a**) $(10-9)6$

(**b**) $(79-78)(54)$

(**c**) $(5-4)^2$

(**d**) $(3-1)^2(7-6)^3$

(**e**) $(6-5)^3$

(**f**) $3(4-3)^2$

(**g**) $2(7-6)^{24}$

(**h**) $967-(67-66)^2$

(**i**) $(1001-1000)^4(54-53)^2(58-56)$

3. Solve for x.

(**a**) $1x = 9$

(**b**) $x - 0 = 1x$

(**c**) $x(x-1) = 0$

0.6 Variables and formulas

As was said at the beginning of the chapter, a *variable* is a symbol (in this text always a letter of the alphabet and usually x) that is meant to stand for a number when we do not want to specify the number. There are two reasons we might want to do this: either because we don't know what it is but do know something about it and want to find out what it is, or because we want to make a general statement about all numbers, not just a particular number. The first situation is comparable to knowing something about a person, such as that he sits next to you in math class, but not knowing his name. In the case of a number, you might know that a number is the largest factor of 12345678 less than 1000—there is exactly one number answering that description—but it would take some work to find out what it is. The second reason is that you might want to make a general statement about all numbers. We have already done that in words, and it is simpler (once you get used to it) to do so with variables. For example, instead of saying that two numbers can be added in either order to give the same answer, we can say $a + b = b + a$ for any numbers replacing a and b.

Definition 19 *A formula is an expression involving one or more variables.*

For example, if a price is represented by p and the usual 7% Philadelphia sales tax applies, the amount you pay is given by the

formula $p + .07p$ (since 7% is .07) If you're shopping in Philadelphia it's good to keep this formula in mind.

Recall that when replacing a variable in an expression or an equation by a number you use the same number every place the variable appears.

Also recall that when a variable is multiplied by a number or another variable, we usually omit the multiplication sign. If you see a number and a variable next to each other without any operation between them, it means they are to be multiplied, and the same is true if you see two variables together. For example, $2x$ means $2 \cdot x$, and ab means $a \cdot b$ (but of course 23 still means twenty three).

Example

Find the cost of an item with a $4.00 price tag after the 7% sales tax is added.

Solution: Use the formula $p + .07p$ substituting 4.50 for p. This give $4.00 + .07 \cdot 4.00 = 4.28$. So you pay $4.28.

Exercise 20 *Use the formula above for Philadelphia sales tax.*

1. Find the price after sales tax of an item with the given price tag:

(**a**) $15

(**b**) $50

(**c**) $100

(**d**) $17.95

2. Instead of the formula $p + .07p$ for sales tax use $1.07p$. Compare the results to what you got before. What do you notice. Why does this happen?

3. Evaluate:

(**a**) $a + 2b$ for $a = 3, b = 9$

(**b**) $a + 2b$ for $a = 5, b = 6$

(**c**) $2x + 3(y + 1)$ for $x = 4, y = 5$

0.7 Some terminology

Definition 21 *Terms are numbers or variables or products that are being added.*

Examples

In the expression $7a + b + 1$, the terms are $7a$, b and 1

In the expression $8x + 9y$ the terms are $8x$ and $9y$.

Definition 22 *Factors are numbers, variables or expressions that are being multiplied.*

Examples

In the expression $7ab$, the factors are 7, a and b.

In the expression $2xy + 9uv$, the terms are $2xy$ and $9uv$, the factors of $2xy$ are 2, x and y, and the factors of $9uv$ are 9, u and v. The entire expression is not a product; it is a sum (of two products).

Definition 23 *Like terms are terms that can be combined into one term because the distributive law can be applied. The terms have to have the same variable or variables, to the same power.*

This is called *combining like terms*. It gives a shorter expression that has the same answer for any value of the variable(s).

Example

Combine the like terms in $2x + 5x$.

Solution. The two terms are like terms. By the distributive law, $2x+5x = (2 + 5)x$, and by arithmetic, this is $7x$.

Example

In the expression $2x + 3y + 4x^2 + 9x + 4y$, $2x$ and $9x$ are like terms, and so are $3y$ and $4y$. There is no like term for $4x^2$. The expression can be simplified to $11x + 7y + 4x^2$.

Example

Given the expression $2x + 3y$, combining like terms is not an option, because the two terms do not have a *common factor*; that is, a factor that appears in both of them (in contrast to $2x + 5x$—in this expression, the two terms have the common factor x, so it is possible to use the distributive law.

Definition 24 *Using the distributive law to change a sum to a product, is called factoring, and using it to rewrite a product as a sum, is called expanding.*

Example
Factor $2x + 2y$. Solution: $2x + 2y = 2(x + y)$.
Example
Expand $2(x + y)$. Solution: $2(x + y = 2x + 2y$

Exercise 25 *Terminology HERE*

0.8 Equality

It is customary in arithmetic to write the problem on the left and the answer on the right side of an equation; for example, $2 + 2 = 4$. But you need to remember that an equation can be written either way: it is also true that $4 = 2 + 2$. This comes up in algebra from time to time. For example, it may happen that we want to figure out what x is and do some algebra that comes out $2 = x$. We usually re-write this as $x = 2$, not because it says anything different but only because $x = 2$ is in the "Question on the left, answer on the right" format that we're used to. In any equation the two sides can be switched. Sometimes the question and the answer switch roles; for example, if the problem is to find the product of 3 and 7 we write $3 \cdot 7 = 21$, but if the problem is to find a factorization of 21 we write $21 = 3 \cdot 7$.

0.9 Equivalence

Definition 26 *Two expressions involving variables are equivalent if they always give the same answer no matter what number is substituted for the variable (or what numbers for each variable, if there are two or more).*

Definition 27 *If the two sides of an equation are equivalent, the equation is called an equivalence.*

The same sign, =, is used for an equivalence and for what are called conditional equations, such as $x + 5 = 9$, where not just any number makes the equation true. It might be better to have different symbols for the two situations, but it is not the custom. We will be working for a while with equivalences, and later, when we consider conditional equations, it will generally be clear from the context that they are conditional, so no confusion should result.

You will be asked to decide in various cases whether two expressions are equivalent.

Example Is $3x + 6$ equivalent to $6 + 3x$? Answer: Yes, by the commutative law for addition $3x + 6 = 6 + 3x$ is an equivalence

Example

Is $3x + 6$ equivalent to $6x + 3$? Answer: No, it is not true for all numbers; for example, it isn't true for $x = 2$. If $x = 2$ we have $3 \cdot 2 + 6 = 12$ and $6 \cdot 2 + 3 = 15$, and $3x + 6 = 6x + 3$ is not and equivalence since $12 \neq 15$.

The first example gives a *proof*; that is, an explanation of why the expressions are equivalent. The second gives a *counterexample*;that is, a number for which the two expressions are not equal. An equivalence must be true for all replacements of the variable, so if even one number doesn't work an equation is not an equivalence. But be careful: an equation can be true for quite a few numbers and still not be an equivalence. For example, $(x-1)(x-2)(x-3)(x-4) = 0$ is true for 1, 2, 3, and 4, but not for any other numbers. To show that an equation is an equivalence it's necessary to give a justification in terms of the laws and properties of numbers. This may take a number of steps.

Example

Show that $x(y + z) = (z + y)x + 0$ is an equivalence.

Solution:

We start with the expression on the left side of the equation and make changes in it, justifying each by some law or property and trying at each step to get closer to the expression on the right.

$x(y + z) = (y + z)x$ by the commutative law for addition ($y + z = z + y$; ignoring the other symbols for the moment)

$(y + z)x = (z + y)x$ by the commutative law for multiplication (again ignoring the other symbols for the moment)

$(z + y)x = (z + y)x + 0$ by the property of 0.

Each step has been justified, so we can be sure the equation is an equivalence.

There are a number of exercises on equivalences at the end of the chapter, because it is important that you understand why we are allowed to do the things we do in algebra. You may find the work of deciding what's an equivalence interesting in itself, and, even if you don't, the practice and also the realization that everything has a reason will make it easier for you to remember various procedures that you need to be able to do, not only in this course but in other courses and possibly on the job (depending on the job). There are exercises on establishing equivalence in the last exercise set in the chapter.

0.10 Simplifying expressions

To simplify an expression is to write an equivalent expression that is shorter. This requires use of the laws and properties of numbers.

In the examples that follow, the original expression is on the first line, on the left, and the first change is on the first line, on the right. After that, each new expression on the right is a change from the expression directly above it on the right.

Example

$$\begin{aligned} 2x + 5x &= (2+5)x \text{ by the distributive law} \\ (2+5)x &= 7x \text{ by arithmetic} \end{aligned}$$

Example

$$1x + 0y = x$$

This simplification uses the property of 1 and one of the properties of 0.

Example

$$\begin{aligned} x + x + x + x &= 1 \cdot x + 1 \cdot x + 1 \cdot x + 1 \cdot x \\ &= (1+1+1+1)x \\ &= 4x \end{aligned}$$

Example

$$\begin{aligned} 3x + 5y + 2x + 4y &= 3x + 2x + 5y + 4y \\ &= (3+2)x + (5+4)y \\ &= 5x + 9y \end{aligned}$$

In this example the associative and commutative laws of addition are used to rearrange the terms of the expression and then the distributive law is used to collect like terms.

Example

$$\begin{aligned} 2a + 7b + 5a + 3b &= 2a + 5a + 7b + 3b \\ &= (2+5)a + (7+3)b \\ &= 7a + 10b \end{aligned}$$

Example

$$\begin{aligned} 5a + 8b - 5a &= 5a - 5a + 8b \\ &= 0 - 8b \\ &= 8b \end{aligned}$$

Simplifying expressions is something we often do, and we do not write out all the steps every time. Often we don't write any of the intermediate steps.

Exercise 28 *Simplify each of the following expressions.*

1. $7a + 3a$

2. $9x + y + 5x$

3. $2x + 3x^2 + 4x$

4. $1 + 2b + 3b + 5$

5. $8c + 9d + 4 + 2c + 7d^2$

6. $2x + 3y + 9x + 7y$

7. $4 + 5x + 6 + 7x$

8. $18 + 19x + 20$

9. $a + b + c + a$

10. $x + 2x + 3x^2$

0.11 Parentheses

The rules for order of operations says to do operations within parentheses before exponentiation and other arithmetic. Be aware that sometimes nothing can be done. For example, in the expression $2(x+3)$ we can't add x and 3 because we don't have a specific value for x. In such a situation we go ahead and do the next thing. Here the next step would usually be to apply the distributive law, getting $2(x+3) = 2x+6$.

Sometimes parentheses are used to keep track of a little history. For example, if I know that whatever amount x a ticket costs I have to add a processing fee of \$3, and in addition I have to add y for transportation to the show, I might write my costs as $(x+3)+y$, lumping the x and the 3 together because they are tied to each other more closely than either is to the travel cost. But if convenient I could remove those parentheses, because they don't affect the computation once I have specific numbers to work with.

In the statement "For any numbers a, b, c (that is, for any numbers that replace those variables) $a + (b + c) = (a + b) + c$." the parentheses are used in the basic way to say which variables to add first, and the equation says it doesn't matter—the answer is the same either way.

Sometimes an expression has multiple sets of parentheses in it, and it's a chore to figure out which ones go together. To help with this, other symbols besides (and) are used: in particular, square brackets [and], and curly brackets { and }.

Example

Consider the expression $1 + \{2 + [3 + (4 + 5)]\}$. This is a very directive expression telling us to add $4 + 5$, then add 3 to the result, then 2 to that result, then 1 to that result: $1 + \{2 + [3 + (4 + 5)]\} = 1 + \{2 + [3 + 9]\} = 1 + \{2 + 12\} = 1 + 14 = 15$.

In this example the parentheses and their alternate symbols make no difference whatsoever to the answer and they could all be left out. But there are situations in which they're needed.

An expression with many parentheses can be confusing. If you have one to deal with, look at it carefully.

0.12 Summary of properties of numbers

For any numbers a, b, and c:

- $a + b = b + a$
  This is the *commutative law for addition*: if you add two numbers, the order doesn't matter.

- $(a + b) + c = a + (b + c)$
  This is the *associative law for addition*: if you add three numbers it doesn't matter whether you add the third to the sum of the first two or the sum of the second and third to the first.

- It can be shown (though it's a lot of work and we don't do it) that as a result of the associative and commutative laws for addition, any string of numbers to be added can be added in any order.

- $ab = ba$
  This is the *commutative law for multiplication*: if you multiply two numbers, the order doesn't matter.

- $(ab)c = a(bc)$
  This is the *associative law for multiplication*: if you multiply three numbers it doesn't matter whether you multiply the product of the first two by the third or the first by the product of the second and third.

- It can be shown (though it's a lot of work and we don't do it) that as a result of the associative and commutative laws for multiplication, any string of numbers to be multiplied can be multiplied in any order.

- $a(b + c) = ab + ac$.
  This is the *distributive law.*

- Properties of 0: $a + 0 = 0 + a = a$, $a0 = 0a = 0$, division by 0 is impossible

- Property of 1: $1a = a1 = a$

Note that there is a little redundancy in the last two lines: for example, if $1a = a$ then $a1 = a$ by the commutative law for multiplication. In mathematics we generally try to keep this to a minimum, but it seems best to lay it all out explicitly here. Also, the fact that $a0 = 0a = 0$ can be proved from the fact that $a + 0 = a$.

0.13 Abbreviations to use when giving reasons

- C+: the commutative law for addition
- A+: the associative law for addition
- C×: the commutative law for multiplication
- A×: the associative law for multiplication
- D: the distributive law
- P0: a property of 0
- P1: the property of 1
- arith: ordinary arithmetic
- AC+: Use this when repeated applications of the associative and commutative laws for addition are used to rearrange a string of numbers to be added. For example, if you want to re-write $(x+1)+(x+2)$ as $x+x+1+2$ you are just writing the terms in a different order (and removing parentheses that don't affect the calculation).
- AC×: Use this when repeated applications of the associative and commutative laws for addition are used to rearrange a string of numbers to be multiplied, as with addition.
- Not: Notation. If you are merely using a different symbol or way of writing an expression, all you are doing is changing notation. For example, if you change $2(3)$ to $2 \cdot 3$, this is just a change in notation. If you add or remove parentheses in a way that doesn't change the value of the expression and doesn't depend on any of the laws, this is just a change in notation (but be sure it doesn't change the value!). For example, $(3+4) = 3+4$; the parentheses on the left do not give any information about the computation. On the other hand, $2+(3+4) = (2+3)+4$ is true because of the associative law for addition; the change is not just a matter of notation.

- Def: Definition. If you re-write 2×2 as 2^2 or xx as x^2 you are using the definition of exponentiation.

1) Show $2(x+1)+3x+4=5x+6$ is an equivalence.
Proof:
$2(x+1)+3x+4=2x+2+3x+4$ by D
$2x+2+3x+4=2x+3x+2+4$ by AC+
$2x+3x+2+4=2x+3x+6$ by arith
$2x+3x+6=(2+3)x+6$ by D
$(2+3)x+6=5x+6$ by arith
QED (QED is an abbreviation of the Latin phrase "Quod erat demonstrandum," which means "Which was to be proved." You can use it if you like at the end of a proof.)
2) Show $1+2+3+4+5+6+7+8+9=45$
Proof: $1+2+3+4+5+6+7+8+9=45$ by AC+ and arith
3) Show $2 \cdot 5 \cdot 2 \cdot 2 \cdot 5 \cdot 5 = 2 \cdot 5 \cdot 2 \cdot 5 \cdot 2 \cdot 5$
Proof: $2 \cdot 5 \cdot 2 \cdot 2 \cdot 5 \cdot 5 = 2 \cdot 5 \cdot 2 \cdot 5 \cdot 2 \cdot 5$ by AC×
Note that the re-arrangement of the numbers in this case is helpful: the next thing to do is to see that on the right we have $10 \cdot 10 \cdot 10$, an easy multiplication. When using AC+ or AC× it's wise to think about what's useful.

Exercise 29 *Summary exercises*

1. Decide which equations are true and which false. Try to do this without actually doing the arithmetic.

(**a**) $(7 \times 5) \times 3 = 7 \times (5 \times 3)$
(**b**) $(7-4)-2=7-(4-2)$
(**c**) $(8+1) \times 5 = 8 \times (1 \times 5)$
(**d**) $2+6=6+2$
(**e**) $113+6798=6798+113$
(**f**) $2 \times (4 \times 9) = 9 \times (4+2)$
(**g**) $9-(4-2)=(9-4)-2$
(**h**) $(1 \times 5) \times 9 = 9 \times (5 \times 1)$
(**i**) $1-5-9=9-5-1$

(**j**) $1 \times [(3+5)+7+9] = \{[(9+7)+5]+3\} \times 1$

2. Go over each true equation in Exercise **1** and decide which parentheses can be left out without changing the answer.

3. Decide which equations below are true and which are false. Try to do this without actually doing the arithmetic.

(**a**) $2(3 \cdot 4) = (2 \times 3)4$

(**b**) $[(2 \times 3) \times 4]5 = 2 \cdot 3(4 \times 5)]$

(**c**) $111(222 \times 333) = (111 \times 222)333$

(**d**) $9 \times 1 \cdot 2 \times 4 \times 7 = 2 \times 7 \times 9 \times 4 \times 1$

(**e**) $[(56)23]92 = 92[56(23)]$

(**f**) $2 \times 3 + 4 = 2(3+4)$

(**g**) $2 \times 3 + 4 = 2 + 3 \cdot 4$

(**h**) $2(3+4) = (2 \times 3) + 4$

(**i**) $(5+9)3 = 5 \cdot 3 \times 9 \cdot 3$

(**j**) $(67+32)43 = 67 \cdot 43 + 32 \cdot 43$

(**k**) $(8+4)3 = 3(8+4)$

(**l**) $(24 \div 3) \div 2 = 24 \div (3 \div 2)$

4. Evaluate the expressions that make sense, and indicate which do not. In this and other exercises, if an expression does not make sense, you may write the abbreviation DNMS to say that. (This is done in the answer key.)

(**a**) $1 + (2 + 3)$

(**b**) $(1 + 2) + \times 3$

(**c**) $(1 + 2) + (3 + 4)$

(**d**) $1 + 2 + 3 + 4)$

(**e**) $5 + 6+$

(**f**) $1 + (2))$

(**g**) $7(2) + 3$

(**h**) $7 \cdot 2 + 3$

(**i**) $(7)(2) + 3$

(**j**) $7(2 + 3)/$

(**k**) $7 - (6 - 3)$

(**l**) $24 \div 6 \div 2$

(**m**) $5 \times 3 + 2$

(**n**) $5 \times (4 - 3$

(**o**) $\dfrac{2 + 3}{3}$

(**p**) $2 \bullet 3 - 4$

(**q**) $\dfrac{1 + 7}{2 + 6}$

(**r**) $\dfrac{1}{2} + \dfrac{7}{6}$

(**s**) $8 \cdot 2 - 1$

(**t**) $0 \div 4$

(**u**) $4 \div 0$

5. Evaluate the expressions that make sense. A few of them don't: indicate which ones those are.

(**a**) $[(7 - 2) - (4 - 3)] - 1$

(**b**) $3\{[7 + 2(3 - 1) - 2] - 2\}$

(**c**) $\dfrac{8 - 6}{9 - 7}$

(**d**) $3 + +2$

(**e**) $\dfrac{2}{3} \div \dfrac{1}{2}$

(**f**) $\frac{\frac{1}{2}}{\frac{1}{4}}$

(**g**) $7 \times \dfrac{1}{3}$

(**h**) $5(8 - 1)$

(**i**) $2(3 + 1)$

(**j**) $2 \cdot 3 + 1$

(**k**) $2 + 3 \cdot 1$

(**l**) $\dfrac{7 - 6}{8 - 4 \cdot 2}$

(**m**) $(\dfrac{2}{3} - \dfrac{1}{4})(\dfrac{1}{3} \div \dfrac{1}{7} - \dfrac{1}{8}$

(**n**) $30 - 2(3 + 4)$

(**o**) $8 - 5(4 - 1)$

(**p**) $\dfrac{7}{8} - \dfrac{(2 - 1}{3}$

(**q**) $9 - 3(4 - 2) + 7$

(**r**) $(\dfrac{2}{9} + \dfrac{2}{3}$

(**s**) $[(20 - 2(3 + 1)(4 - 2) + 7][1 + 2(3 + 4 - 5)]$

(**t**) $\dfrac{2\{(8-1)-[9-(2+3)]\}}{4[8-2(4-3)]}$

(**u**) $\dfrac{6-3)\{(8-1)-[9-(2+3)]\}}{4[8-2(4-3)]}$

6. Re-write by using the distributive law and whatever else is needed to group like terms. Show the steps and give the reasons for them. (This is the same as proving that your answer is equivalent to the original expression.)

(**a**) $4a + 9a$

(**b**) $9b - 4b$

(**c**) $8x + 2x$

(**d**) $5x - 2x$

(**e**) $6x + x$

(**f**) $3x + x$

(**g**) $17x + x$

(**h**) $5x - x$

(**i**) $7x - x$

(**j**) xx

(**k**) $x + x + x$

(**l**) $2x + 3x + 5x$

(**m**) $4t + 9t - 2t$

(**n**) $8r + 5r - r$

(**o**) $5x + x + x$

(**p**) $x + x - x$

(**q**) $(x + x) - x$

(**r**) $x + (x - x)$

7. Simplify as much as possible. You don't need to write the reasons, but you might want to think about them.

(**a**) $3x{+}7x + 9p + 4p$

(**b**) $x{+}7y{+}9x{+}3y - 2y$

(**c**) $5x - x - x$

(**d**) $9x + 5t + x - t$

(**e**) $6x + 9 + 4x + 7t - 3 + t$

(**f**) $x + 2x{+}3x{+}4x{+}5x$

(**g**) $x + 2x + y{+}3y{+}3x + 5y$

(**h**) $m + n + 2m{+}9 + 3m{+}21$

(**i**) $(3m + 2n) + (5n + 7m)$

(**j**) $(x + 2y) + (7x + 3y)$

(**k**) $5(7x{+}6y) + 2y1 + 5x$

(**l**) $[9x{+}(5g{+}3x)1 + y$

(**m**) $(5x + 5y) + 7x - 27)$

(**n**) $5x + 21y{+}(5x + 3y - 8)1$

(**o**) $z + 2y + 3x{+}4w - 3w - 2x - y$

8. Indicate which expressions are equal to $3(4) + 3(5)$. Try to do this without doing the arithmetic, by thinking about the laws and properties that might be used to change each expression to $3(4)+3(5)$, and looking for variations that don't look as if they're likely to produce the same answer. Check by doing the arithmetic.

(**a**) $3 \cdot 4 + 3 \cdot 5$
(**b**) $3(4) + 3 \cdot 5$
(**c**) $5(3 + 4)$
(**d**) $4 + 4(5)$
(**e**) $3(4 + 3)5$
(**f**) $(4 + 5)3$
(**g**) $3 + 4 + 3 \cdot 5$
(**h**) $4 + 4 + 4 + 5 + 5 + 5$
(**i**) $4 + (4 + (4)) + 5 + (5 + (5))$
(**j**) $4 + (4 + (4)) + 5 + (5 + (5))$
(**k**) $3 \cdot 4 + 5$
(**l**) $4(3 + 5)$
(**m**) $3 \cdot 4 \cdot 5$
(**n**) $4 + 5 \cdot 3$
(**o**) $3(4 + 5)$
(**p**) $3 + (4 \cdot 5)$
(**q**) $34 + 35$
(**r**) $3 \times 4 \times 3 \times 5$

9. Indicate which expressions are equal to $2(7 + 3)$. See the instructions for **8**.

(**a**) $2(7) + 3(7)$
(**b**) $2 \times 7 + 3$
(**c**) $7(2) + 7(3)$
(**d**) $2x(7 + 3)$
(**e**) $2 + (7 \times 3)$
(**f**) $(7 + 3)2$
(**g**) $2 \cdot 7 + 2 \cdot 3$
(**h**) $7 + 7 + 3 + 3$
(**i**) $2(7 + 3)$
(**j**) $3 + 7 \times 2$
(**k**) $2 \cdot 7 + 2 \cdot 3$
(**l**) $2 \times 7 \times 3$
(**m**) $2 \times 7 + 3$
(**n**) $2(3 + 7)$
(**o**) $(2 \cdot 7) + 3$
(**p**) $(3 + 7)2$

10. Indicate which expressions are equal to $6 \cdot 5 + 6 \cdot 8$. See the instructions for **8**.

(**a**) $6(5) + 6(8)$
(**b**) $6 \cdot (5 + 6 \cdot 8)$
(**c**) $6 \times 5 \times 6 + 8$
(**d**) $6{\times}(5 + 8)$
(**e**) $6(5 + 8)$
(**f**) $5(6) + 6(8)$
(**g**) $6(8) + 6{\times}5$
(**h**) $5{\times}6{\times}8$
(**i**) $6{\times}5 + 8$
(**j**) $5{\times}(6 + 8)$
(**k**) $6 + 6 + 6 + 6 + 6 + 6{\times}8$
(**l**) $6{\times}(5 + 6+){\times}8$
(**m**) $(6{\times}5 + 8{\times}6)$
(**n**) $6{\times}5 + 6(8)$
(**o**) $8(6) + 5(6)$
(**p**) $(5 + 8)6$

11. Indicate which expressions are equal to $5(8 - 2)$. See the instructions for **8**.

(**a**) $(5)(8 - 2)$
(**b**) $5(8) - 5(2)$
(**c**) $5(2 - 8)$
(**d**) $5(8) - 2$
(**e**) $(5 \cdot 8) - (5 \cdot 2)$
(**f**) $5((8) - (2))$
(**g**) $8(5 - 2)$
(**h**) $8(5) - 2(8)$
(**i**) $2(8) - 5(8)$
(**j**) $2 \cdot 5 - 8 \cdot 5$
(**k**) $5 + 5 + 5 + 5 + 5 + 5$
(**l**) $5(1 + 1 + 1 + 1 + 1 + 1)$
(**m**) $10 \div 2 \times (8 - 2)$
(**n**) $5 \cdot 8 - 2$
(**o**) $(8 - 2)5$
(**p**) $5 \cdot 6$

12. Indicate which expressions are equivalent to $x(y + z)$; i.e., are equal to it for any replacement of x.

(**a**) $x(y) + z(y)$
(**b**) $x \times y + z$
(**c**) $y(x) + y(z)$
(**d**) $(y + z)x$
(**e**) $x + (y \times z)$
(**f**) $(y + z + 0)x$
(**g**) $xy + xz$
(**h**) $xy + z$
(**i**) $x(y + z + 0)$
(**j**) $z + yx$
(**k**) $xy + xz$

(**l**) $x \times y \times z$

(**m**) $x \times y + z$

(**n**) $x(z + y)$

13. Indicate which expressions are equivalent to $u(v + w)$; i.e., are equal to it for any replacement of u.

(**a**) $u(v) + w(v)$

(**b**) $u \times v + w$

(**c**) $v(u) + v(w)$

(**d**) $u \cdot 0(v + w)$

(**e**) $u + (v \times w)$

(**f**) $(v + w)u$

(**g**) $u \cdot v + uw$

(**h**) $(w + v)(u + 0)$

(**i**) $(u + 0)(v + w)$

(**j**) $w + v \times u$

(**k**) $uv + uw$

(**l**) $u \times v \times w$

(**m**) $u \times v + w$

(**n**) $u(w + v)$

(**o**) $(uv) + w$

(**p**) $uv + 0 + wv$

14. Indicate which expressions are equivalent to $p(q + w)$; i.e., are equal to it for any replacement of p.

(**a**) $pq + pw$

(**b**) $p(q + 0) + pw$

(**c**) $pqp + w$

(**d**) $p(1q + w)$

(**e**) $p(q + 1w)$

(**f**) $q(p) + p(w + 0)$

(**g**) $p(w) + pq$

(**h**) qpw

(**i**) $pq + w$

(**j**) $q(p + w)$

(**k**) $qp + pw$

(**l**) $(1p + 0)(1q + 1w)$

(**m**) $(pq + wp)$

(**n**) $pq + p(w)$

(**o**) $w(p) + q(p)$

15. Indicate which expressions are equivalent to $ab + ac$; i.e., are equal to it for any replacement of x.

(**a**) $1ab + 1ac$	(**g**) $a + b + ac$	(**m**) abc
(**b**) $a(b + 0) + ac$	(**h**) $ba + bc$	(**n**) $b + ca$
(**c**) $c(a + b)$	(**i**) $aba + c$	(**o**) $a(b + c)$
(**d**) $b + b(c)$	(**j**) $1b + (a + 0)c$	(**p**) $a + (bc)$
(**e**) $a(b + a)c$	(**k**) $ab + c$	(**q**) $ba + ca$
(**f**) $(b + c)a$	(**l**) $b(a + c)$	(**r**) $abac$

16. Indicate which expressions are equivalent to $m(n+p)$; i.e., are equal to it for any replacement of x. .

(**a**) $m(n) + p(n)$	(**f**) $(n + p + 0)m$	(**k**) $mn + mp$
(**b**) $mn + p$	(**g**) $mn + mp$	(**l**) mnp
(**c**) $n(m) + n(p)$	(**h**) $(0 + n)m + 1mp$	(**m**) $mn + p + 0$
(**d**) $m(p + n)$	(**i**) $m[1(n + p)]$	(**n**) $m(p + n)$
(**e**) $m + (np)$	(**j**) $p + nm$	(**o**) $(mn) + p$

17. Indicate which expressions are equivalent to $rs + rt$; i.e., are equal to it for any replacement of x.

(**a**) $r(s + 0) + r(t)$	(**f**) $s(r) + r(t)$	(**k**) $(r + 1)s - rs + rt$
(**b**) $sr + tr$	(**g**) $r(t) + rs$	(**l**) $r(s + r+)t$
(**c**) $rsr + t$	(**h**) srt	(**m**) $(rs + tr)$
(**d**) $r(s + t)$	(**i**) $rs + t$	(**n**) $rs + r(t)$
(**e**) $r(s + t + 0)$	(**j**) $s(r + t)$	(**o**) $t(r) + s(r)$

18. Indicate which expressions are equivalent to $a(c - b)$; i.e., are equal to it for any replacement of x.

(**a**) $(a + 0)(c - b)$	(**c**) $a(b - c)$	(**e**) $(ac) - (ab)$
(**b**) $a(c) - a(b)$	(**d**) $a(c) - b$	(**f**) $a((c) - (b))$

(**g**) $c(a-b)$

(**h**) $c(a)-b(c)$

(**i**) $b(c)-a(c)$

(**j**) $ba-ca$

(**k**) $a+c-b$

(**l**) $a(c1-b1)$

(**m**) $1a+b(c-b)$

(**n**) $ac-b$

(**o**) $(c-b)a$

19. In some places in Exercises **8** through **18** there are pairs of parentheses that are not needed—they make no difference to the calculation. Decide which parentheses these are.

20. Find a number to replace x that makes the equation true.

(**a**) $1+x=9$

(**b**) $7+x=12$

(**c**) $10=x+2$

(**d**) $x+3=7$

(**e**) $7+x=7$

(**f**) $x+23=67$

(**g**) $x+x=2x$

21. In each problem replace x by a number that gives a true equation.

- (**a**) $10 - x = 4$
- (**b**) $x - 5 = 4$
- (**c**) $5 - x = 7$
- (**d**) $x - 1 = 0$
- (**e**) $8 - x = 5$
- (**f**) $8 - x = 8$
- (**g**) $8 - x = 0$
- (**h**) $56,987 - x = 0$
- (**i**) $123,567 - x = 123,567$
- (**j**) $876543 - x = 875543$

22. Find a solution for each equation. (Remember, if there's no operation shown between two symbols, they are to be multiplied (unless they're both numbers, like 23).

- (**a**) $2x = 6$
- (**b**) $3x = 9$
- (**c**) $4x = 10$
- (**d**) $100 = 10x$
- (**e**) $8x = 24$
- (**f**) $7x = 0$
- (**g**) $50 = 5x$
- (**h**) $1000 = 50x$

23. Solve each equation.

- (**a**) $10/x = 5$
- (**b**) $25 \div 5 = x$
- (**c**) $x \div 2 = 6$
- (**d**) $x/4 = 4$
- (**e**) $16/x = 8$
- (**f**) $20/x = 20$
- (**g**) $154/154 = x$
- (**h**) $x \div 1 = 12$
- (**i**) * $1/x = 2$