(Chapter head:)More numbers

The first numbers anyone learns are the *natural numbers.* also called the *counting numbers*: 1, 2, 3, *etc.* These serve to keep track of how many people are at the dinner table, how many apples are in the bin, how many sheep went out to graze, and so on. As societies and science evolved it became important to enlarge the concept of number; for example, to measure lengths and distances when they are not exact multiples of the units used, to divide a loaf into equal pieces, *etc* Such work requires fractions. Also, the invention of debt requires the invention of negative numbers Signed numbers and fractions are essential for science, business, and and many other aspects of modern life.

Besides measuring things with such numbers we have to be able to combine them by addition, subtraction, multiplication and division, as we do with whole numbers. For this to work they need to obey the same laws as the whole numbers. In advance, it's not obvious this is possible. Fortunately it is (fortunately since otherwise we wouldn't have modern science, medicine, technology, *etc.*). But the operations with signed numbers and fractions have to be defined carefully to ensure that the basic laws still work. Some of the definitions are not intuitive, and may even seem unreasonable at first. But they have to be the way they are in order to make arithmetic possible. This chapter reviews the rules for operations with signed numbers and fractions.

1 Signed numbers

First we look at negative numbers.
Definition We define the *opposite* of a number n to be a number, denoted $-n$, which when added to n gives zero; that is, $n+(-n)=0$. The *opposite* of zero is zero.

Of course there is no other counting number you can add to any given counting numer to get zero, so we have introduced a new kind of number.
DefinitionThe opposites of the positive numbers are the *negative* numbers.
Definition All the positive and negative numbers together with zero are called the *signed numbers.* The whole numbers and their opposites are the *integers.* (We will get to positive and negative fractions a little later.)

Often the term *negative* of a number is used instead of "opposite." This makes no trouble when the original number is positive, but you need to remember that the opposite of a negative number is positive, and you may hear this said as "The negative of a negative is positive.' Foe example, the number you add to -3 to get 0 is 3, so the opposite of -3 is 3. This can be written $-(-3)=3$.

Exercise 1 .

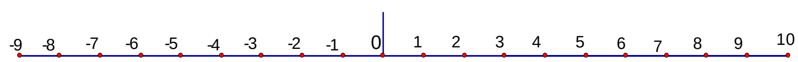
Give the opposite of each number

1.

-8	3. 0	5. 17	7. 18	9. -18
2. 15	4. -43	6. -3	8. -4	10. -7

Inequalities and the number line

One way of getting a feel for the signed numbers is to consider their relative sizes. A number line is useful visual aid for this. Here is a (piece of a) number line:



A number line is imagined to go forever in both directions. Each point on the line is associated with a number, and each number with a point on the line, as indicated by the labeling. (The points between the integers correspond to fractions and decimals. We will return to the number line when we discuss those numbers.) We speak loosely and do not always make the distinction between a number and the point that represents it. Note that the point representing the integers are equally space on the line.

We use the inequalities $<$ and $>$ to describe the relative positions of numbers on this line:

Definition 1 *We say that one number is greater than another if the point representing the first is to the right of that representing the second on the number line, and we use the symbol $>$ between the numbers to indicate this.*

For example, $3 > -5$ because the point corresponding to 3 is to the right of the point corresponding to -5. If a vertical number line is used, "is greater than" means "is higher than."

Definition 2 *Correspondingly, we say that one integer is less than another if the first is to the left of the second on the number line, and we use the symbol $<$ between the numbers to indicate this.*

With a vertical number line, "is less than" means "is lower than."

Examples

$$\begin{aligned} 7 &> 4 \\ 2 &> -4 \\ -1 &> -5 \\ 2 &< 5 \\ -6 &< 1 \\ -7 &< -3 \end{aligned}$$

For the whole numbers (the positive numbers and zero), the concepts of "greater than" and "less than" are familiar, although we usually say "bigger than" or "larger than" instead of "greater than." and sometimes "smaller than" instead of "less than." When it comes to signed numbers, the change in terminology corresponds to a necessary shift in preception: we have defined "greater than' and "less than" in terms of left and right on the number line, and for the negative numbers this point of view may clash with intuitive perceptions.

A negative number a closer to zero is greater than one b farther away from zero, because the one farther away is to the left of it. This may take a lttle getting used to, because in ordinary usage a debt of \$11 is greater than a debt of \$5, and in financial matters debts are regarded as negative. The way to think of it is that if you're \$5 in debt you're better off than if you're \$11 in debt. ..

$$3 > 2 \text{ says “3 is greater than 2”}$$

The symbol for “is less than” is

$$<$$

$$1 < 2 \text{ says “1 is less than 2”}$$

The phrases “is greater than” and “is less than” are used instead of “is larger than” and “is smaller than” because when we compare sizes of negative numbers as well as positive, the way we (have to) do it does not match everyone's everyday conception of *larger* and *smaller*, as you will shortly see.

The symbol $\geq$ (sometimes written $\geqq$) means “is greater than or equal to” and the symbol $\leq$ (sometimes written $\leqq$) means “is less than or equal to” These symbols are not generally used when comparing two specific numbers, since it is obvious from looking at the numbers whether one is greater than the other or the two are equal. But there are times when we need to use a variable to describe a set of numbers; for example, “all numbers greater than or equal to 3.” This set is different from the set of all numbers greater than three, since it includes three, whereas set of all numbers greater than three does not. Amazing as it may seem, this sort of thing sometimes makes a big difference mathematically. It can make a big difference in everyday life, too, when rules go by age; for example, drivers under 25 pay higher insurance rates—if your age is greater than or equal to 25 you pay lower rates.

Example

$$\begin{aligned} -1 &> -2 \\ -2 &\leq 0 \\ -2 &\leq -2 \\ -10 &\geq -100 \\ -6 &\geq -6 \end{aligned}$$

Exercise 2 .

1. Between each pair of numbers put $<$ or $>$ to make a true statement:

(a) 8____2
(b) 9____1
(c) 3____4
(d) 3____ − 1
(e) 14____ − 24
(f) −4____ − 8
(g) −114____89
(h) 12____ − 431
(i) 13____24
(j) 7____16
(k) −7____ − 4
(l) 18____ − 999
(m) 13____ − 24
(n) 14____ − 500
(o) −15____28
(p) −7____16
(q) 17____ − 4
(r) 1800____999
(s) −19____30
(t) 20____ − 8

2. True or false?

(a) $-9 \le 2$
(b) $9 \ge -11$
(c) $13 \le -40$
(d) $2 \ge -5$
(e) $1 \ge -8$
(f) $-3 \ge 1$
(g) $-4 \le 2$
(h) $-4 \ge 8$
(i) $1 \ge -6$
(j) $1 \ge -2$
(k) $14 \le 9$
(l) $-122 \ge 43$
(m) $13 \ge -2$
(n) $-14 \ge -5$
(o) $-5 \le 2$
(p) $7 \le 16$
(q) $-7 \ge -14$
(r) $-8 \ge -99$
(s) $19 \ge -30$
(t) $20 \le 8$

3. Draw a number line from -10 to 10 then plot and label the points $1, 7, -3, -9, 9\frac{1}{2}$. Use a straightedge (ruler or other straight object) and space the integers evenly on the line.

Absolute value

Definition The *absolute value* of a number x , denoted $|x|$, is its distance from zero.

The absolute value of a number is always positive whether the number itself is positive or negative, with the exception that the absolute value of zero is zero.

Example

$$\begin{aligned} |2| &= 2 \\ |-4| &= 4 \\ |0| &= 0 \end{aligned}$$

A little later in your mathematical education you will consider the absolute value of a variable, and you have to be careful, since a variable can represent either a positive or a negative number. If x is positive or zero; *i.e.,* $x \ge 0$, then $|x| = x$. What's easy to forget is that x may be negative—we don't know. If $x < 0$, then $|x|$ is the opposite of x, which is $-x$. For example, if $x = -3$, then $|x| = 3$, which is not x but its opposite, $-x$ $(= -(-3) = \dot{3})$.

Exercise 3 *Give the absolute value of each number:*

1.

-8	4. -43	7. -4	10. -987
2. 15	5. 17	8. 30	11. $\frac{1}{2}$
3. 0	6. -3	9. 5000	12. -0.7

13. Give the negative of each number in Exercise 3.

Operations on signed numbers

When negative numbers were first introduced in mathematics, even some mathematicians balked, because their intuitions about numbers didn't work for negative numbers, and they didn't like to call them numbers. If some of the rules for operations on signed numbers seem strange to you, you have illustrious company. But signed numbers and these rules for their operations have been accepted for a long time—because they work.

A point of notation before we begin: if an operation is followed by a negative number, put parentheses around the negative number.

Example

Write $9 - (-5)$, not $9 - -5$.

Also, it is still the case that if there is no operation symbol between two numbers or expressions, the operation is multiplication.

Example

$9(-5)$ means $9 \times (-5)$

Addition

The number line is useful for understanding addition of numbers. Think of the integers marked on the line as markers on a road, and the numbers you're adding as telling you how far to move and in which direction. Begin at zero, and for a positive number move to the right; for a negative, move to the left. In this way it is possible to add not just two, but many numbers.

If you make two moves to the right, then you end up to the right. So the sum of two positive numbers is positive, as ever.

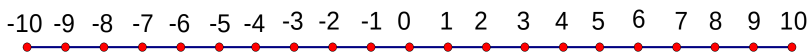
If you make two moves to the left, then you end up to the left. So the sum of two negative numbers is negative.

If you make one move to the left and one to the right, then whether you end up to the left or the right depends on the relative sizes of the moves; that is, on which has the greater absolute value. Explicitly: if adding a positive and a negative number (in either order) take the difference of the two numbers and use the sign of the one with greater absolute value.

Examples

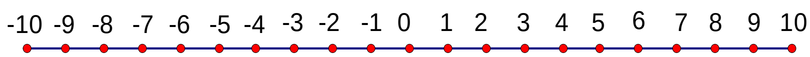
$$\begin{aligned} 3 + 8 &= 11 \\ -6 + (-2) &= -8 \\ -4 + 3 &= -1 \\ 5 + (-1) &= 4 \end{aligned}$$

Example $5 + (-6)$:
Start at 0, go five units to the right, then six to the left. This brings you to -1.



The answer: is $5 + (-6) = -1$

Example$5 + (-6) + 3 + (-4)$:
Start at 0, go five units to the right, then six to the left, then three to the right, then four to the left. This brings you to -2.



The answer: is $5 + (-6) + 3 + (-4) = -2$

Exercise 4 .

1. Do the following additions by hand.

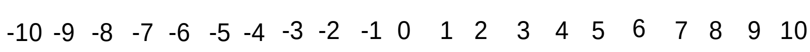
a) $8 + (-2)$	f) $-3 + 1$	k) $-14 + 9$	p) $-7 + 16$
b) $9 + (-1)$	g) $14 + (-24)$	l) $12 + (-1)$	q) $17 + (-4)$
c) $5 + (-4)$	h) $4 + (-8)$	m) $1 + (-24)$	r) $18 + (-28)$
d) $20 + (-5)$	i) $1 + (-6)$	n) $14 + (-5)$	s) $-19 + 30$
e) $11 + (-8)$	j) $12 + (-18)$	o) $15 + (-28)$	t) $20 + (-8)$

2. Do the following additions by hand.

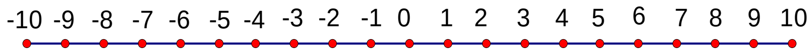
a) $-8 + (-2)$	f) $-3 + (-1)$	k) $-14 + (-9)$	p) $-70 + 10$
b) $-4 + (-1)$	g) $-4 + (-20)$	l) $-3 + (-1)$	q) $17 + (-7)$
c) $-7 + (-4)$	h) $-3 + (-6)$	m) $-3 + (-13)$	r) $38 + (-28)$
d) $8 + (-5)$	i) $1 + (-4)$	n) $-4 + (-5)$	s) $-10 + 30$
e) $11 + 8$	j) $12 + (-1)$	o) $-25 + (-28)$	t) $20 + (-10)$

Exercise 5 *Use motions on the number line to do each of the following additions.*

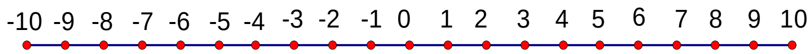
1. $-3 + 7 + (-4)$



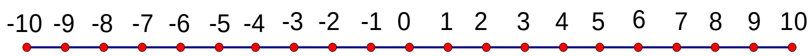
2. $5 + (-2) + 4 + (-1)$



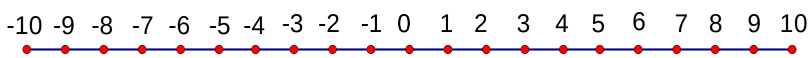
3. $4 + (-3) + 2 + (-1)$



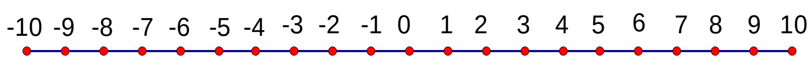
4. $-8 + 6 + (-7) + 5 + (-6) + 4$



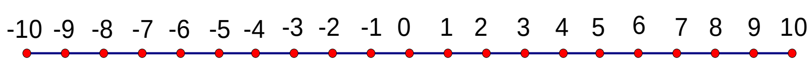
5. $5 + 4 + (-3) + (-1)$



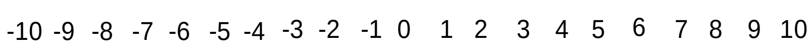
6. $-9 + 9 + (-4) + 4) + (-1)$



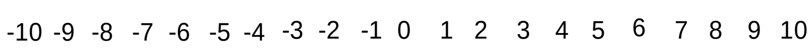
7. $-5 + 8 + (-2) + 4$



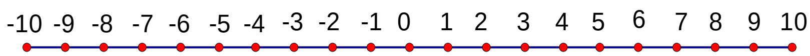
8. $-6 + 3 + 7 + (-9)$



9. $1 + (-1) + 1 + (-1) + 1 + (-1) + 5$



10. $2 + (-3) + 4 + (-7) + 5$



Adding signed numbers quickly without a calculator

Since the associative and commutative laws hold for addition, you can rearrange any string of numbers you're adding so that all the positive ones come first and the negative ones afterwards (If you do the exercises above that way you do all the moves to the right first then all the moves to the left—though in some of the exercises doing so would take you beyond the segment of the number line shown—you would have had to extend it). The quickest way to add signed numbers (even quicker than with a calculator, if the numbers aren't too big) is to add all the positive ones together, add all the negative ones together, and then add the two results.

To add two signed numbers of opposite sign take the difference of their absolute values and use the sign of the one with larger absolute value. This can be interpreted in terms of the number line by observing that if the sum of the moves to the right is greater than the sum of the moves to the left, the result is positive, whereas if the reverse is true, the result is negative.

$$10 + (-7) + 6 + (-1) + (-4)$$

Add the positive numbers together:

$$10 + 6 = 16$$

Add the negative numbers together:

$$-7 + (-1) + (-4) = -12$$

Add the two results:

$$16 + (-12) = 16 - 12 = 4$$

Since the absolute value of 16 is greater than the absolute value of -12, the result is a positive number.

$$-10 + 7 + (-6) + 1 + 4$$

Add the positive numbers together:

$$7 + 1 + 4 = 12$$

Add the negative numbers together:

$$-10 + (-6) = -16$$

Add the two results:

$$\begin{aligned} -16 + 12 &= \\ 12 - 16 &= -4 \end{aligned}$$

Since the absolute value of -16 is greater than the absolute value of 12, ($|-16| = 16 > |12| = 12$) the result is a negative number. To do the computation subtract 12 from 16; *i.e.* take $16 - 12$, then use a negative sign, since the negative outweighs the positive. (In terms of the number line, the interpretation is that the distance traveled to the left is more than the distance traveled to the right.)

Exercise 6 *Add the following, without a calculator. Look over each problem first for the shortest way to do it.*

1. $-3 + 7 + (-4)$
2. $-8 + 7 + (-1)$
3. $-1 + 4 + (-2)$
4. $5 + (-9) + 8 + (-7)$
5. $15 + (-3) + 8 + (-6)$
6. $-4 + 90 + (-4)$
7. $-8 + 90 + (-9)$
8. $-9 + 4 + (-2)$
9. $5 + (-9) + 8 + (-9)$
10. $95 + (-4) + 8 + (-6)$
11. $-9 + 7 + (-17)$
12. $-1 + 7 + (-1)$
13. $-1 + 17 + (-2)$
14. $6 + (-9) + 1 + (-7)$
15. $16 + (-9) + 1 + (-6)$
16. $-17 + 9 + (-17)$
17. $-1 + 9 + (-9)$
18. $-9 + 17 + (-2)$
19. $6 + (-9) + 1 + (-9)$
20. $-6 + (-17) + 1 + (-6)$
21. $-4 + 8 + (-4)$
22. $-8 + 8 + (-2)$
23. $-2 + 4 + (-2) + (-4) + (-4)$
24. $6 + (-9) + 8 + (-8)$
25. $26 + (-4) + 8 + (-6)$
26. $-4 + 19 + (-4) + 9$
27. $-8 + 90 + (-9) + 8 + (-5)$
28. $-9 + 4 + (-2) + 6 + (-4) + 5$
29. $(-9) + 8 + (-9) + (-8) + 3$
30. $6 + (-4) + 8 + (-6)$
31. $-9 + 8 + (-28)$
32. $-2 + 8 + (-2)$
33. $-2 + 28 + (-2)$
34. $6 + (-9) + 2 + (-8)$
35. $-8 + 5 + (-3) + 4$
36. $-1 + (-2) + (-3) + (-4) + (-5)$
37. $8 + (-8) + 7 + (-7) + 6 + (-6) + 5 + (-5)$
38. $8 + 7 + 6 + (-6) + 5 + (-5) + (-8) + (-7)$

39. $16 + (-9) + 2 + (-6) + 6 + (-9) + 2 + (-8)$

40. $-28+9+(-28)+6+(-9)+2+(-8)$

41. $-8+0+8+(-6)+0+5+0+(-4)+(-3)$

42. $-15 + (-32) + 31 + 16$

43. $-100+100+(-300)+200+(-100)+400$

44. $-87+86+(-59)+58+(-101)+100$

45. $-1+2+(-3)+4+(-5)+6+(-1)+8$

46. $-2+9+(-9)+9+8+7+6+5+4+(-4)+(-5)+(-6)+(-7)+(-8)+(-9)$

47. $-9 + 28 + (-2)$

48. $6 + (-9) + 2 + (-9)$

49. $-6 + (-28) + 2 + (-6)$

50. $-599+(-231)+599+(-67)+67+231$

Subtraction

What happens if you subtract a negative number from a given number, say $5 - (-2)$? Would you expect it to be the same as $5 - 2$? In one case we take -2 from 5, and in the other we take 2 from 5. Since we're taking different numbers from 5, it should seem reasonable that we get different answers. To many students the correct way of getting the answer, established long ago, does not seem reasonable. However, it works. It is this:

Rule for subtraction of signed numbers: Subtracting a number b from a number a gives the same result as adding its opposite. In symbols:

$$a - b = a + (-b)$$

In this particular example, $5 - 2 = 5 + (-2)$ and $5 - (-2) = 5 + 2 = 7$. Taking 2 from 5; *i.e.*, computing $5 - 2$, is an old-fashioned problem you did in second grade, and it can be thought of in terms of apples, pennies, whatever. But taking -2 from 5; *i.e.*, computing $5 - (-2)$, involves considerations that came later in human history and are not addressed in second grade; namely, considerations of finance. You can't take -2 apples away; in fact, it's hard to say what you might mean by -2 apples (anti-apples?). But when it comes to money, there is an easy definition for a negative number: it stands for a debt. And forgiving (*i.e.*, taking away) a debt comes to the same thing as giving a person money. Suppose your brother is figuring out his financial situation, adding the money he has in his pocket, the money he has on the bureau, the money he has in his savings account, *etc.*, then subtracting off the money he owes on his car, the money he owes the friend who lent him bus fare and the money he owes you. If you tell him to forget about the money he owes you, then he's better off. If he owed you two dollars, he's better off by two dollars. Taking away a debt of two dollars amounts to subtracting -2, and it has the same effect as adding 2.

$$\begin{aligned} -4 - (-2) &= -4 + 2 \\ &= -2 \end{aligned}$$

You can think of this as describing a situation in which someone owes four dollars but two dollars of that is forgiven. so he's only two dollars in debt.

If you have worked with signed numbers in the past you may have seen positive numbers written with a + sign in front to indicate they are positive; for example, $+4 - (+5)$. This is done as a reminder that we are now in a number system that contains negative numbers as well. In this text the + sign is omitted; + always stands for addition. But be sure you are clear that when you see only the − sign between two numbers, it indicates subtraction, not that the second number is negative.

Example

$$4 - 3$$

is the subtraction of positive 3 from positive 4.

$$4(-3)$$

is multiplication of 4 by -3 (which we discuss in the next section).

Exercise 7 *Perform the following subtractions by hand.*

1. $9 - (-2)$
2. $8 - (-1)$
3. $7 - (-4)$
4. $6 - (-3)$
5. $8 - (-7)$
6. $-9 - (-2)$
7. $-8 - (-1)$
8. $-7 - (-4)$
9. $-6 - (-3)$
10. $-8 - (-7)$
11. $11 - (-4)$
12. $-12 - (-1)$
13. $7 - (-9)$
14. $6 - (-11)$
15. $8 - (-15)$
16. $-29 - (-20)$
17. $-28 - (-10)$
18. $-17 - (-4)$
19. $-16 - (-3)$
20. $-18 - (-6)$
21. $7 - 4$
22. $-6 - 3$
23. $-8 - 7$
24. $11 - 5$
25. $-12 - 1$
26. $7 - 9$
27. $6 - 11$
28. $8 - 15$
29. $-29 - 20$
30. $-28 - 10$

A string of additions and subtractions can usually be done most easily by changing it to an addition problem: replace every subtraction by the addition of its opposite.

Example

$$\begin{aligned} -5+8-(-3) &= -5+8+3 \\ &= -5+11 \\ &= 6 \end{aligned}$$

$$\begin{aligned} -3+(-5)-(-6)-1-(-4)+8 &= -3+(-5)+6+(-1)+4+8 \\ &= -3+(-5)+(-1)+6+4+8 \\ &= -13+18 \\ &= 5 \end{aligned}$$

Exercise 8 *Perform the following subtractions by hand.*

1. $5-(-4)-(-3)$
2. $-8-(-2)-(-1)$
3. $9-(-4)+(-3)$
4. $5-(-3)-(-8)$
5. $10+(-4)-(-4)$
6. $-7-(-1)+(-3)$
7. $50-(-43)-7$
8. $16-(-4)-3$
9. $-8-(-8)-(-1)$
10. $5-4-(-9)$
11. $11-(-6)-(-3)$
12. $-8-(-4)-(-1)$
13. $9-(-6)+(-3)$
14. $11-(-3)-(-8)$
15. $10+(-6)-(-6)$
16. $-7-(-1)+(-3)$
17. $10-(-6)-7$
18. $16-(-6)-3$
19. $-8-(-8)-(-1)$
20. $11-6-(-9)$
21. $1-(-1)-(-2)-2-(-3)-3$
22. $9-6-(-3)-9-(-2)$
23. $8-(-2)+4-5+7-(-6)-(-2)$
24. $-5-8+9-2+(-2)-3-(-3)+(-3)$
25. $5-(-5)-(-2)-2-(-3)-3$
26. $9-6-(-3)-9-(-2)$

27. $11 - (-2) + 9 - 5 + 1 - (-6) - (-2) + 5$

28. $-4 - 6 + 9 - 2 + (-2) - 3 - (-3) + (-1)$

29. $12 - (-3) + (-1) - 7 + 1 + (-8) - (-2)$

30. $100 - (-30) + (-40) - (-20) - 10 + 90$

Multiplication

To understand why products of signed numbers have the signs that they do, it is necessary to understand that the distributive law reigns. If it didn't work for negative as well as positive numbers, we wouldn't get far with negative numbers. Remember, the distributive law says that for all replacements of x, y and z,

$$x(y + z) = xy + xz.$$

For example, for $x = 2$, $y = 3$, $z = 4$,

$$2(3 + 4) = 2 \cdot 3 + 2 \cdot 4.$$

Now consider the following multiplication:

$$2(2 + (-2)) = 2 \cdot 2 + 2(-2).$$

The left side of the equation is zero, since $2 + (-2) = 0$ and $2 \cdot 0 = 0$.
On the right we have $2 \cdot 2 + 2(-2) = 4 + 2(-2)$. If this is to equal zero, we must have that

$$2(-2) = -4.$$

This is an example of a general rule: the product of a positive number and a negative number is always a negative number.

Now consider

$$-2(2 + (-2))$$

Expanding (*i.e.*, applying the distributive law) gives

$$-2(2 + (-2)) = -2(2) + (-2)(-2).$$

On the left we get zero, as in the previous case. On the right, since $-2(2) = -4$, we get

$$-2(2) + (-2)(-2) = -4 + (-2)(-2)$$

In order for the right side to equal zero, we must have that

$$(-2)(-2) = 4.$$

The product of any two negative numbers is positive. This is necessary for the distributive law to hold, and it does not violate any of our other cherished laws.

Rules for multiplication of signed numbers:

$$\begin{aligned}
\text{positive} \times \text{positive} &= \text{positive} \\
\text{positive} \times \text{negative} &= \text{negative} \times \text{positive} = \text{negative} \\
\text{negative} \times \text{negative} &= \text{positive}
\end{aligned}$$

Since the product of two negative numbers is positive, if we multiply a string of negative numbers, we can pair negative factors to give positive products. Therefore the product of an even number (2, 4, 6, etc.) of negative factors is positive, and the product of an odd number (1, 3, 5, etc.) of negative factors is negative, because there's one negative factor left over.

1) $(-1)(-1)(-1)(-1) = [(-1)(-1)][(-1)(-1)] = 1 \cdot 1 = 1$

2) $(-1)(-1)(-1)(-1)(-1) = [(-1)(-1)][(-1)(-1)](-1) = 1 \cdot 1(-1) = -1$

Exercise 9 *Perform the following multiplications by hand.*

1. $(-2)8$
2. $(-3)(-4)$
3. $(-4)(-7)$
4. $(-6)(-1)$
5. $(-1)1$
6. $(-3)(-9)$
7. $(-7)6$
8. $(-9)(-2)$
9. $(-10)(-5)$
10. $(-10)(-2)$
11. $(-12)8$
12. $(-2)(-12)$
13. $5 \cdot 6$
14. $(-3)(-11)$
15. $(-9)8$
16. $(-2)(-2)(-3)$
17. $(-4)(-1)(-2)$
18. $(-3)(-1)(-2)2 \cdot 3$
19. $(-1)(-1)(-1)(-1)(-1)$
20. $(-1)(-1)(-3)$

Division

Division by any number but zero is the equivalent of multiplication by its reciprocal; *i.e.*,by one over the number. For example, the reciprocal of 3 is $\frac{1}{3}$, and dividing by 3 is equivalent to multiplying by $\frac{1}{3}$. So every division is equivalent to a multiplication (by the reciprocal), and because of this, the rules for the signs of quotients (answers to division) are the same as for products:

Rules for division of signed numbers:

$$\begin{aligned}
\text{positive} \div \text{positive} &= \text{positive} \\
\text{positive} \div \text{negative} &= \text{negative} \div \text{positive} = \text{negative} \\
\text{negative} \div \text{negative} &= \text{positive}
\end{aligned}$$

So far we have confined out attention to integers (the counting numbers, zero, and the negatives of the counting numbers). When we do division, we often get fractions. Some of the exercises below have fractional answers. Write these in fractional form (not decimal) and reduce to lowest terms. (Also, for practice, think about which parentheses are necessary and which could be omitted). A later section of this chapter gives a brief review of operations with fractions.

Exercise 10 *Perform the following divisions by hand.*

1. $8 \div (-2)$
2. $(-3)(-2)(-4)$
3. $(-4) \div (-2)$
4. $-6 \div (-1)$
5. $(-1) \div 1$
6. $-9 \div (-3)$
7. $(-72) \div 6$
8. $(-9) \div (-1)$
9. $(-10) \div (-5)$
10. $(-10) \div (-2)$
11. $(-12) \div 8$
12. $(-2) \div (-12)$
13. $5 \div 6$
14. $(-3) \div (-12)$
15. $(-9) \div 8$
16. $(-2) \div (-2)$
17. $(-4) \times (-1) \div (-2)$
18. $(-3) \div (-1) \times (-2) \div 2 \cdot 3$
19. $(-1) \div (-1) \div (-1) \div (-1) \div (-1)$
20. $(-1) \div (-3)$

Combined operations with signed numbers

Since the associative, commutative and distributive laws apply to the set of integers, as do the properties of zero and one, and the order of operations used for natural numbers, we have the rules for doing combined operations.

Example

$$\begin{aligned} -4 + (-5)[-2 - (-3)^2] &= \\ -4 + (-5)[-2 - 9] &= \\ -4 + (-5)(-11) &= \\ -4 + 55 &= 51 \end{aligned}$$

Definition 3 *Negation is the term for taking the opposite of a number or expression. It comes after exponentiation unless parentheses indicate otherwise.*

Example

$$\begin{aligned} -1^2 &= -(1^2) = -1 \\ (-1)^2 &= 1 \end{aligned}$$

Exercise 11 *Compute by hand.*

1. $(-1) + 3(-2 - 1)$
2. $[(-4) + 3]^2$
3. $(-2)^4$
4. $-1 - 12 - (-5)$
5. $(-3)(-2) - 4$
6. $-8 - (-2)(-3)$
7. $1 + 5(-2)(-3)$
8. $(-2)3 - (-5)2$
9. $(-1)(-2)(-3) - 4$
10. $-1 - 2 - 3(-4)$
11. $-9 - 6(-5) - 2 \cdot 3$
12. $(-5) + 3(-2 - 5)$
13. $[(-4) + 3]^2$
14. $(-2)^4$
15. $-5 - 5 \cdot 2 - (-7)$
16. $(-3)(-2)[1 - 2^2] - 4$
17. $-9 - (-2)(-3) + 1$
18. $5 + 7(-2)(-3)$
19. $(-2)3 - (-7)2$
20. $(-5)(-2)(-3) - 4$
21. $-5 - 2 - 3(-4)$
22. $-1(-2) - 3(-4) - 5(-6)$
23. $-1 - 5^2 - (-3)^2$
24. $(-2 - 3)(-4 - 5)$
25. $(-2)(-3)(-4) - 5$
26. $(-1 - 0)(-2 - 0)(3 - 0)$
27. $12 \div 2 + 8 \div (-4)$
28. $12 \cdot (-2) + 8 \div (-4)$
29. $(9 - 1)^2 - (11 - 2)(3^2 - 4/2)$
30. $17 \cdot 15 - 16^2$

2 Factors and primes

In some ways multiplication is the most interesting of the basic operations on numbers, at least on the counting numbers. This is because of special features of primes and factorization, which we now consider.

If you wanted to write a counting number as the sum of the simplest building blocks possible, you would write it as the sum of however many ones; *e.g.*, $3 = 1+1+1$. This is pretty dull. But if you want to write a counting number as a *product* of the simplest building blocks possible, things get more interesting. You write the number as a product of numbers that can't be broken down any further by factoring; *e.g.*, write 12 as $2 \cdot 2 \cdot 3$. The factors (*i.e.*, the numbers being multiplied) cannot be factored any further, so they are the simplest bluilding blocks for the original number when it's written as a product. We can write 12 in different ways as a product; *e.g.*, $12 = 2 \cdot 6$ or $12 = 3 \cdot 4$, but in each case one of the factors can be factored ($6 = 2 \cdot 3$, $4 = 2 \cdot 2$), so we haven't used the simplest possible building blocks. We need a name for these simplest factors.

Definition 4 *A prime number is a natural number which has exactly two divisors: itself and one.*

This definition is worded carefully so as to exclude one as a prime. If 1 were called prime, then it would always be possible to get another prime factorization but throwing in another factor of 1; *e.g.*, $12 = 2 \cdot 2 \cdot 3 \cdot 1$, and we don't want this. Leaving out 1 as a prime, there is only one way of factoring a number into primes. This is good to know.

If the original number is prime, like 3, then its prime factorization is just itself.

ExampleThe numbers 2, 3, 5, 7,11, 13, 17, 19 and 23 are prime numbers.

Definition 5 *To factor a number is to write it as a product, which is called a factorization of the number. A factor of a natural number is a natural number that goes into it evenly; i.e., without a remainder.*

Example

$$\begin{aligned} & \text{Factorizations of 12 are } 1 \cdot 12,\ 2 \cdot 6,\ 3 \cdot 4 \text{ and } 2 \cdot 2 \cdot 3, \text{ since} \\ 12 &= 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4 = 2 \cdot 2 \cdot 3 \end{aligned}$$

The numbers 1, 2, 3, 4, 6, and 12 are all factors of 12. The numbers 5, 7, 8, 9, 10, 11 and any number bigger than 12 are not.. More generally, a natural number a is a factor of a natural number b if a goes in b evenly, or equivalently, if b can be written as a product $b = ac$, where c is also a natural number.
Example

The factors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, 100.

Definition 6 *The prime factorization of a natural number is its unique factorization into primes.*

There is only one way to factor a number into primes (if order doesn't matter).
Example

The prime factorization of 12 is $2^2 3$.

The prime factors of 100 are 2 and 5.
The prime factorization of 100 is $2^2 5^2$.

There is no end to the primes, and their patterns and properties have been studied for many centuries and are still studied—there are still unanswered questions about them. Besides the interest that the mathematically inclined take in them for their own sake, they have practical uses. For example, if you know the product of two very large primes, you can use it to encode a message in such a way that would take a very long time for anyone to crack who doesn't know those prime factors. This method has been used by governments and industries to send secret messages.

In this course, we use primes in finding the smallest common denominator of fractions to be added or subtracted. In the case of a negative number, factor its opposite into primes and multiply by negative one to get a useful factorization of the negative number. (We usually arrange matters so that denominators are positive, as you will see.)

$$\begin{aligned} & \text{Factor } -28; \\ -28 &= (-1)2^2 7 \end{aligned}$$

This is the most complete factorization of -28: -1 times the prime factorization of 28. The term *prime factorization* is reserved for natural numbers, however.

Another reason to study factorizations of numbers is that we will have reason to factor algebraic expressions, and factoring numbers is a good refresher for the concept.

Exercise 12 .

1. Find all the primes from 1 to 100.

2. Find *all* factors of each of the following:

   (a) 24 (c) 50 (e) 200

   (b) 30 (d) 98 (f) 420

3. Write all possible factorizations for each number below into a product of two numbers.

   (a) 24 (c) 50 (e) 200

   (b) 30 (d) 98 (f) 420

4. Write the prime factorization of each number in Exercise 3.

3 Rational numbers

If you subtract a natural number from another natural number, you don't necessary get an answer unless you allow negative answers (for example, $4 - 9$). Similarly, you can't go far with division before hitting fractions. A *fraction* $\frac{a}{b}$, where a and b are integers, can be thought of in several ways:

1) as another way of expressing division: $\frac{a}{b} = a \div b$.
2) as the number you get by taking a parts of b equal parts
3) as the ratio of a to b.

The number on the top of a fraction is called the *numerator*, and the number on bottom the *denominator*. Since division by zero doesn't work, zero is not a permissible denominator: the expression $a/0$ is undefined for all numbers a.

In everyday conversation, the term "fraction" is taken to mean a number between zero and one. In arithmetic an *improper fraction* is a (positive) fraction whose numerator is bigger than its denominator; for example, $3/2$. Our definition allows any integer for a numerator and any integer except zero for a denominator. The standard mathematical term for such numbers is *rational numbers*. Since 1 can be used as a denominator, and $a/1 = a$ for all numbers a, all integers are rational numbers. (The word *rational* comes from the word *ratio*.)

As an example of taking a of b equal parts, if a pizza is divided into eight equal pieces and you take three, you have taken three eighths of the pizza. In this case, the fraction can be regarded as an amount, three eighths of the pizza. . In general, when speaking of three eighths, you would want to know three eighths of what, just as you would want to know one of what, or two of what, when dealing with whole numbers. But just as with integers, the properties and relations of fractions can be abstracted, and the mathematical operations that give the right answers for pizzas give the right answers for anything else.

An example of the use of a fraction as a ratio: if a batter gets a hit one out of every three times at bat, he gets a hit one third of the time. This could be written $1/3$ or $\frac{1}{3}$, but is more commonly expressed as a decimal (.333).

The fact that there are different ways of regarding a fraction sometimes causes confusion with the concept. But we can all be grateful that no matter how they're thought of, the same rules work. Now we summarize these rules, which you have seen before.

Operations on rational numbers

Reducing fractions

We need to reduce fractions to lowest terms ($\frac{2}{6} = \frac{1}{3}$, $\frac{3}{6} = \frac{1}{2}$), and to perform the standard operations of arithmetic with them.

Reducing a fraction to lowest terms:

Example

$$\frac{ac}{bc} = \frac{a}{b}$$

Example

$$\begin{aligned} \frac{2}{6} &= \frac{1\cdot 2}{3\cdot 2} = \frac{1}{3} \\ \frac{3}{6} &= \frac{1\cdot 3}{2\cdot 3} = \frac{1}{2} \end{aligned}$$

It is a common mistake when working with variables in fractions to cancel terms (expressions added , not multiplied) in numerator and denominator.

Example

$$\begin{aligned} \frac{x+2}{x+3} &\neq \frac{2}{3} \text{ unless } x = 0. \\ \text{For instance, if } x &= 1, \\ \frac{x+2}{x+3} &= \frac{1+2}{1+3} = \frac{3}{4} \neq \frac{2}{3} \end{aligned}$$

Summary of rules for operations with fractions

Adding fractions

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

Example

$$\frac{1}{6} + \frac{2}{6} = \frac{3}{6}$$

If two fractions have different denominators, adding them requires getting a common denominator, then adding as above.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{db} = \frac{ad+bc}{bd}$$

Example

$$\frac{3}{8} + \frac{1}{6} = \frac{3\cdot 6}{8\cdot 6} + \frac{1\cdot 8}{6\cdot 8} = \frac{18}{48} + \frac{8}{48} = \frac{26}{48} = \frac{13}{24}$$

or

$$\frac{3}{8}+\frac{1}{6}=\frac{3\cdot 3}{8\cdot 3}+\frac{1\cdot 4}{6\cdot 4}=\frac{9}{24}+\frac{4}{24}=\frac{13}{24}$$

It is best to find the *least* common denominator, 24, in this example, so as to work with smaller numbers and not to have to reduce at the end. In the case of pizzas, this example can be interpreted as taking two pizzas of equal size, dividing one into eight equal pieces and taking three, and dividing the other into six equal pieces and taking one. Getting the common denominator, 48, corresponds to cutting them each into forty-eight equal pieces then taking eighteen of the 48 from the first pizza and 8 of the forty eight from the second. Finding the least common denominator corresponds to cutting the pizzas each into 24 pieces instead of 48, then taking 9 from the first and 4 from the second. In either case, both pizzas have been cut into pieces of the same size. It is less messy to do the mathematics than to cut the pizzas into such tiny slices.

Subtracting fractions

$$\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$$

$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

Multiplying fractions

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

Note that multiplication of fractions is easier than addition or subtraction; you don't need to get a common denominator. However, you should reduce to lowest terms. It's generally easier to do this by cancellation before you do the multiplications in the numerator and denominator.

Example

$$\frac{3}{7}\times\frac{5}{8}=\frac{15}{56}$$

$$\frac{3}{10}\times\frac{5}{9}=$$

Dividing fractions

$$\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$$

The rule for division of fractions usually strikes people as odd. It may seem more reasonable if you recall that the check for division is multiplication, and look at an example with integers:

$$6\div 2=3 \quad \text{Check: } 3\times 2=6$$

With fractions:

$$\frac{3}{7}\div\frac{1}{2}=\frac{3}{7}\times\frac{2}{1}=\frac{6}{7} \quad \text{because } \frac{6}{7}\times\frac{1}{2}=\frac{6}{14}=\frac{3}{7}$$

The fraction bar in the order of operations

When there are computations to be done within a fraction, the fraction bar is a division sign, and the numerator and denominator are each regarded as being within parentheses separated by the fraction bar.

Example

$$\begin{aligned}\frac{10-2^2}{2(2\cdot 3-5)} &= (10-2^2)\div 2(2\cdot 3-5)\\ &= (10-6)\div[2(6-5)]\\ &= 4\div 2\\ &= 2\end{aligned}$$

When using a calculator it is crucial to put numerator and denominator inside parentheses if they have more than one term. Otherwise the calculator will not use the appropriate order of operations, and you will get the wrong answer.

Negative of a rational number

The negative of a rational number is defined as it is for integers:

Definition 7 *The negative of a rational number $\frac{a}{b}$, denoted $-\frac{a}{b}$, is the number with the property that $\frac{a}{b}+(-\frac{a}{b})=0$.*

It can be shown from the descriptions of the operations on rational numbers that

$$-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}$$

It is customary to write the negative sign in front or in the numerator rather than in the denominator. Granted, there is conceptual difficulty in thinking about taking, for example, -2 of 3 equal parts, or 2 of -3 equal parts. Don't try. It's easier to think of negative rational numbers in terms of debts, or as corresponding to points on the number line, which we will discuss again in the next section.

Exercise 13 .

1. Reduce to lowest terms.

(a) $\frac{20}{30}$	(e) $\frac{16}{18}$	(i) $\frac{2+3}{6+3}$
(b) $\frac{12}{18}$	(f) $\frac{27}{144}$	(j) $\frac{2^2+1}{3^2+1}$
(c) $\frac{14}{28}$	(g) $\frac{80}{88}$	(k) $\frac{5-1}{21-1}$
(d) $\frac{26}{52}$	(h) $\frac{20}{30}$	(l) $\frac{2^2+0}{3^3+1}$

2. Perform the indicated operations and reduce to lowest terms.

(a) $\frac{7}{10} + \frac{1}{10}$

(b) $\frac{7}{8} - \frac{3}{8}$

(c) $\frac{2}{3} + \frac{1}{2}$

(d) $\frac{2}{3} + \frac{3}{4}$

(e) $\frac{7}{8} + \frac{1}{2}$

(f) $\frac{5}{8} - \frac{1}{2}$

(g) $\frac{7}{3} \cdot \frac{1}{2}$

(h) $\frac{4}{5} \cdot \frac{1}{2}$

(i) $\frac{2}{9} \cdot \frac{3}{8}$

(j) $\frac{7}{10} \cdot \frac{1}{14} \cdot \frac{25}{28}$

(k) $\frac{9}{24} \cdot \frac{16}{81}$

(l) $\frac{7}{24} - \frac{10}{30}$

(m) $\frac{4}{5} \div \frac{1}{2}$

(n) $\frac{9}{5} \div \frac{10}{3}$

(o) $\frac{9}{5} \div \frac{3}{10}$

3. Perform the indicated operations and reduce to lowest terms.

(a) $8 \div \frac{1}{2}$

(b) $\frac{1}{2} \div \frac{1}{2}$

(c) $\frac{-3}{10} + \frac{1}{10}$

(d) $\frac{7}{8} - \left(-\frac{3}{8}\right)$

(e) $-\frac{2}{3} + \left(-\frac{1}{2}\right)$

(f) $\frac{1}{3} - \frac{3}{4}$

(g) $-\frac{7}{8} + \frac{1}{2}$

(h) $\frac{5}{8}\left(-\frac{1}{2}\right)$

(i) $\frac{7}{3} \cdot \frac{1}{2}$

(j) $\left(-\frac{4}{5}\right) \cdot \left(-\frac{1}{2}\right)$

(k) $\frac{1}{2} + \frac{2}{3} \cdot \left(-\frac{2}{3}\right)$

(l) $-\frac{1}{2} + \frac{1}{2}$

(m) $\frac{1}{2} - \frac{2}{3} + \frac{1}{4}$

(n) $\frac{5}{6} \cdot \frac{1}{2} - \frac{2}{3} \cdot \frac{1}{4}$

(o) $\frac{-5}{6} \cdot \frac{1}{-2} - \frac{2}{3} \cdot \left(-\frac{1}{4}\right)$

4. Re-write each fraction in two ways by putting the negative sign in a different position.

(a) $\frac{-2}{5}$

(b) $\frac{-1}{8}$

(c) $\frac{3}{-7}$

(d) $\frac{9}{-2}$

(e) $-\frac{1}{4}$

(f) $-\frac{10}{11}$

5. Re-write each fraction using as few negative signs as possible.

(a)

$\dfrac{-3}{-7}$ (e) $\dfrac{-11}{-19}$

(b) $-\dfrac{2}{-5}$ (f) $-\dfrac{21}{-35}$

(c) $-\dfrac{-1}{7}$ (g) $-\dfrac{-31}{7}$

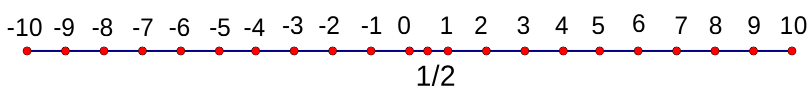
(d) $-\dfrac{-2}{-9}$ (h) $-\dfrac{-20}{-11}$

The number line revisited

The number line has room on it for many more numbers than just the integers. Every number we have worked with can be plotted on the number line, Of course, in the case of fractions it may be difficult to estimate the position exactly if only the integers are marked on the line. We settle for a good approximation. Change an improper fraction to a mixed number before plotting it.

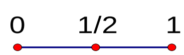
Plot $\dfrac{1}{2}$ on the number line.

Solution:

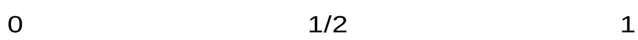


The point corresponding to $\dfrac{1}{2}$ lies halfway between the points for 0 and 1.

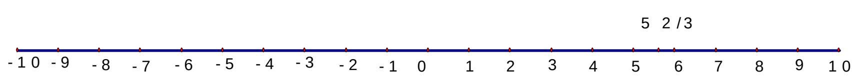
If this is the only point being plotted, a shorter number line will do:



and we may make a larger version:

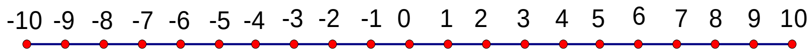


ExampleTo plot the number $\frac{17}{3}$, change it to a mixed number to find which integers it lies between. Since $\frac{17}{3} = 5\frac{2}{3}$, this number lies between 5 and 6; in fact, two thirds of the way from 5 to 6. It is plotted below, with reasonable accuracy:

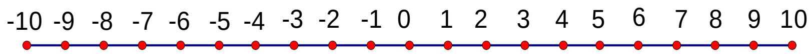


Exercise 14 .

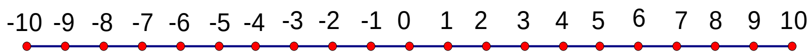
1. Plot and label $-4\frac{1}{2}$, $-3\frac{1}{2}$, $-2\frac{1}{2}$, $-1\frac{1}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$ on the number line.



2. List five numbers between 1 and 2 and plot and label them on the number line.



3. Plot and label the numbers on the number line.



   (a) $3\frac{1}{4}$
   (b) $2\frac{3}{4}$
   (c) $3\frac{3}{4}$
   (d) $4\frac{1}{10}$
   (e) $1\frac{7}{8}$
   (f) $-\frac{1}{4}$
   (g) 0
   (h) $\frac{1}{4}$
   (i) $-3\frac{3}{4}$
   (j) $-2\frac{1}{4}$
   (k) $-1\frac{1}{10}$
   (l) $-\frac{7}{8}$

4. As stated at the beginning of the chapter, the same rules and conventions that apply to whole numbers apply to all rational numbers. The equations below are all equivalences. In each case state what law(s) and/or properties guarantee this. You do not need to do any computation

   (a) $\frac{1}{2}+\frac{1}{3}=\frac{1}{3}+\frac{1}{2}$
   (b) $\left(\frac{1}{2}+\frac{1}{3}\right)+\frac{1}{4}=\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)$
   (c) $\frac{10}{21}\cdot\frac{-5}{3}=\frac{-5}{3}\cdot\frac{10}{21}$
   (d) $\left(\frac{-1}{3}\cdot\frac{2}{5}\right)\cdot\frac{7}{11}=\frac{-1}{3}\cdot\left(\frac{2}{5}\cdot\frac{7}{11}\right)$
   (e) $\frac{5}{5}\cdot\frac{1}{3}=\frac{1}{3}$
   (f) $\left(\frac{2}{3}-\frac{2}{3}\right)\cdot\frac{8}{9}=\frac{8}{9}$

(g) $\frac{6}{7} \cdot \frac{8}{8} = \frac{6}{7}$

(h) $\frac{6}{7}\left(\frac{3}{8} + \frac{-1}{4}\right) = \frac{6}{7} \cdot \frac{3}{8} + \frac{6}{7} \cdot \frac{-1}{4}$

(i) $\frac{3}{8} + \frac{3}{11} + \frac{2}{5} = \frac{2}{5} + \frac{3}{8} + \frac{3}{11}$

(j) $\frac{3}{8} \cdot \frac{-2}{9} + \frac{1}{7} \cdot \frac{3}{11} = \frac{3}{11} \cdot \frac{1}{7} + \frac{-2}{9} \cdot \frac{3}{8}$

5. For each part of 4 compute the answer for the expression on each side of the equation (in cases where there is a computation to do). Make sure you get the same answer for each side.