240 Linear Algebra **Review for the Final Test**

Attention! The solutions of problems below are just outlined. You should work out the details in order to prepare for the final.

For the system of vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

find

1. The cross product $\mathbf{u}_1 \times \mathbf{u}_2$.

Solution. $\mathbf{u}_1 \times \mathbf{u}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = \mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$

2. An equation of the plane parallel to $\mathbf{u}_1$ and $\mathbf{u}_3$ and through the point (1,2,0).

Solution. As a vector orthogonal to the plane we can take $\mathbf{u}_1 \times \mathbf{u}_3 = -7\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ whence an equation of the plane is $7(x-1) - 5(y-2) + z = 0$ or $7x - 5y + z + 3 = 0$.

3. Distance between the line $\mathbf{u}_1 + t\mathbf{u}_2$ and the point $Q(1,0,-3)$.

Solution. We take two points on the line $\mathbf{u}_1 + t\mathbf{u}_2$, e.g. $S(1,2,3)$ and $T(4,3,5)$. Then

$$d = \frac{\| \overrightarrow{ST} \times \overrightarrow{SQ} \|}{\| \overrightarrow{ST} \|} = \frac{2\sqrt{91}}{\sqrt{14}} \approx 5.1$$

4. Apply the Gram-Schmidt process to the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, **and** $\mathbf{u}_3$ to obtain an orthonormal base in R^3.

Solution. $\mathbf{w}_1 = \begin{bmatrix} \frac{\sqrt{14}}{14} \\ \frac{\sqrt{14}}{7} \\ \frac{3\sqrt{14}}{14} \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} \frac{31\sqrt{42}}{210} \\ -\frac{4\sqrt{42}}{105} \\ -\frac{\sqrt{42}}{42} \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} \frac{\sqrt{3}}{15} \\ \frac{7\sqrt{3}}{15} \\ \frac{\sqrt{3}}{3} \end{bmatrix}.$

5. The matrix of a linear transformation ***A***: $\mathrm{R}^3 \to \mathrm{R}^3$ in the standard basis is

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 3 \end{bmatrix}$$

Find (a) *Rank* and *Nullity* of ***A***,

(b) a basis in ker(***A***)**,**

(c) a basis in the range of ***A*.**

Solution. The rank is 2 whence the nullity is 1. To find a basis in $\ker A$ we solve the homogeneous system

$$3x + 2y + z = 0$$
$$x + y - z = 0$$
$$x + 3z = 0$$

It is easy to see that $x = -3z$ and $y = 4z$ whence a basis in $\ker A$ is the vector $-3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$. The dimension of the range is 2 so any basis will consist of two vectors. We can take for example the vectors $\mathbf{u}_1 = A\mathbf{i} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{u}_2 = A\mathbf{j} = 2\mathbf{i} + \mathbf{j}$.

6. Find eigenvectors, eigenvalues, and orthogonal diagonalization of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Solution. Eigenvalues: 3, -1, 3. Corresponding orthonormal eigenvectors:

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix},$$

Then $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ and $P^{-1}AP = P^T AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

7. What kind of matrix (Hermitian, unitary, normal, neither) is

$$\mathbf{B} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}?$$

Explain why or why not.

Solution. $\mathbf{B}^* = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$

Therefore the matrix $\mathbf{B}$ is neither Hermitian ($\mathbf{B} \neq \mathbf{B}^*$) nor unitary (because $\mathbf{B}^{-1} \neq \mathbf{B}^*$).

It is normal because $\mathbf{B}^*\mathbf{B} = \mathbf{B}\mathbf{B}^* = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

8. Diagonalize matrix ***B***, if possible find a unitary diagonalization, if not explain why.

Solution. The matrix can be unitary diagonalized because it is normal. Characteristic equation: $(\lambda - 1)^2 + 1 = 0$; eigenvalues: $1 - i, 1 + i$; corresponding orthonormal eigenvectors: $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$; $U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$; $U^{-1}BU = U^*BU = \begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix}$.