1. Apply the Gram-Schmidt process to the given basis in R^3 to obtain an orthonormal basis.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

Solution. We will first construct an orthogonal basic applying the procedure described on page 359.

$$\mathbf{v}_1 = \mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\| \mathbf{v}_1 \|^2} \mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix},$$

$$\mathbf{v}_2 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\| \mathbf{v}_1 \|^2} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\| \mathbf{v}_2 \|^2} \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

It remains to normalize these vectors.

$$\mathbf{w}_1 = \begin{bmatrix} \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} \frac{3}{\sqrt{22}} \\ \frac{-2}{\sqrt{22}} \\ \frac{3}{\sqrt{22}} \end{bmatrix} \mathbf{w}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

2. Apply the Gram-Schmidt process to the given basis in C^3 to obtain an orthonormal basis.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

Solution. We apply the same procedure as in Problem 1 (keeping in mind the definition of the inner product in C^3) to obtain

$$\mathbf{w}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} i \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} \frac{\sqrt{6}}{6} i \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} \frac{\sqrt{6}}{6}(1+i+ \\ \frac{\sqrt{6}}{6}(-1+i) \\ \frac{\sqrt{6}}{6}(1-i) \end{bmatrix}$$

3. Find eigenvalues and a set of orthonormal eigenvectors for the following symmetric matrix.

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -2 \\ 2 & -2 & 5 \end{bmatrix}$$

Solution. The characteristic equation $\begin{vmatrix} \lambda-2 & 1 & -2 \\ 1 & \lambda-2 & 2 \\ -2 & 2 & \lambda-5 \end{vmatrix} = 0$ can be expanded as $\lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$. The last equation has solutions $\lambda = 1, 1, 7$. If we put $\lambda = 1$ we have to solve the following homogeneous system

$$-x + y - 2z = 0$$
$$x - y + 2z = 0$$
$$-2x + 2y - 4z = 0$$

Clearly, the rank of the matrix of this system is 1 (all three equations are proportional) and the dimension of its null space is 2. We can easily notice that the vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ make a basis of the null space. Applying the Gram – Schmidt process to these two vectors we get two orthonormal eigenvectors corresponding to the eigenvalue 1

$$\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{pmatrix}$$

If $\lambda = 7$ we have the homogeneous system

$$\begin{aligned} 5x + y - 2z &= 0 \\ x + 5y + 2z &= 0 \\ -2x + 2y + 2z &= 0 \end{aligned}$$

In this case the dimension of the null space is 1 and as a basis of the null space we can take the vector $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. The corresponding normalized vector is $\begin{pmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{2\sqrt{6}}{6} \end{pmatrix}$.

4. Find orthogonal diagonalization of matrix A from Problem 3.

Solution. To construct an orthogonal matrix P that will diagonalize matrix A we can use the orthonormal eigenvectors we found in the previous problem.

$$P = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{3}}{3} & \frac{2\sqrt{6}}{6} \end{bmatrix}$$

5. Find a unitary diagonalization of the matrix

$$A = \begin{bmatrix} 1 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & 1 \end{bmatrix}$$

Solution. The matrix is Hermitian and therefore it can be unitary diagonalized. We find that the eigenvalues of matrix A are $0, -1, 2$ and the corresponding normalized eigenvectors are $\begin{pmatrix} -\frac{i}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} \frac{i}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. We construct the unitary matrix U as

$$U=\begin{bmatrix} -\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Then

$$U^{-1}=U^{*}=\begin{bmatrix} \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and

$$U^{*}AU=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

6. Prove that if A and B are Hermitian matrices then AB is a Hermitian matrix if and only if $AB = BA$.

Solution. Because A and B are Hermitian we have $(AB)^{*} = B^{*}A^{*} = BA$. Therefore $(AB)^{*} = AB \Leftrightarrow AB = BA$.