

Problem 1 Find eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

Solution. The characteristic equation $\det(\lambda I - A) = 0$ in our case becomes

$$\begin{vmatrix} \lambda - 1 & 0 & -1 \\ -2 & \lambda + 1 & -1 \\ 1 & -2 & \lambda - 1 \end{vmatrix} = 0$$

Expanding the determinant we obtain the equation $\lambda^3 - \lambda^2 - 2\lambda = 0$. The solutions of this equation, i.e. the eigenvalues of A are $-1, 0, 2$. Notice that because all the eigenvalues of A are distinct the corresponding eigenvectors are linearly independent and therefore A is diagonalizable (see Theorems 5.2.1 and 5.2.2 on pages 306 and 309, respectively). Let us find the eigenvectors. If $\lambda = -1$ then we have to find a nontrivial solution of the homogeneous system $(I + A)\mathbf{x} = \mathbf{0}$ which in scalar form can be written as

$$\begin{aligned} 2x + z &= 0 \\ x - 2y - 2z &= 0 \end{aligned}$$

A particular solution is $x = 2, y = 5, z = -4$ and we can take as an eigenvector corresponding to eigenvalue -1 the vector $2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$.

Similarly, if $\lambda = 0$ we have to find a nontrivial solution of the system

$$\begin{aligned} x + z &= 0 \\ 2x - y + z &= 0 \\ -x + 2y + z &= 0 \end{aligned}$$

It is easy to see from these equations that $y = x$ and $z = -x$; therefore an eigenvector corresponding to eigenvalue 0 is the vector $\mathbf{i} + \mathbf{j} - \mathbf{k}$.

Finally, if $\lambda = 2$ the corresponding system of equations is

$$\begin{aligned}x - y &= 0 \\-2x + 3y - z &= 0 \\x - 2y + z &= 0\end{aligned}$$

We see immediately that $x = y = z$ and thus we can take as the third eigenvector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Problem 2. Diagonalize (if possible) matrix A from Problem 1, i.e. find a matrix P such that $P^{-1}AP$ is a diagonal matrix.

Solution. As the columns of matrix P we can take the eigenvectors of A found in the previous problem. Then

$$P = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 1 & 1 \\ -4 & -1 & 1 \end{bmatrix}$$

We compute $P^{-1} = \frac{1}{6} \begin{bmatrix} -2 & -2 & 0 \\ 9 & -6 & -3 \\ 1 & 2 & 3 \end{bmatrix}$ and $D = P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Problem 3 Evaluate the matrix $B = A^{10}$ where A is the matrix from Problem 1.

Solution. $B = PD^{10}P^{-1} = \begin{bmatrix} 170 & 342 & 512 \\ 169 & 343 & 512 \\ 172 & 340 & 512 \end{bmatrix}$.

Problem 4. Find eigenvalues, eigenvectors, and diagonalize the matrix

$$C = \begin{bmatrix} -1 & 1-i \\ -1-i & 1 \end{bmatrix}$$

Solution. The characteristic equation is $\begin{vmatrix} \lambda + 1 & -1 + i \\ 1 + i & \lambda - 1 \end{vmatrix} = 0$ which expands to $\lambda^2 + 1 = 0$

whence the eigenvalues are i and $-i$. If $\lambda = i$ the corresponding homogeneous system is reduced to one equation $(1+i)x + (-1+i)y = 0$ whence $y = \frac{1+i}{-1+i}x = -ix$

and we can take as an eigenvector $\begin{pmatrix} 1 \\ -i \end{pmatrix}$. Similarly, if $\lambda = -i$, we get an eigenvector

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus we can take $P = \begin{pmatrix} 1 & 1 \\ -i & 1 \end{pmatrix}$. Then $P^{-1} = \frac{1}{1-i} \begin{pmatrix} 1 & -1 \\ -i & 1 \end{pmatrix}$ and a simple computation

shows that $P^{-1}CP = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.