

Problem 1. Find the matrix corresponding to the reflection \mathbf{R} in R^3 about the plane $x + 2y - 3z = 0$.

Solution. We have to find out what happens to the basic vectors \mathbf{i} , \mathbf{j} , and \mathbf{k} under the specified reflection. Notice that a vector orthogonal to the plane is

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

Applying Formula (10) on page 147 we see that the orthogonal projection of \mathbf{i} along \mathbf{a} is $\mathbf{i}_1 = \frac{\mathbf{i} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a} = \frac{1}{14}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ whence the component of \mathbf{i} orthogonal to \mathbf{a}

(and therefore located in the plane) is $\mathbf{i}_2 = \mathbf{i} - \mathbf{i}_1 = \frac{13}{14}\mathbf{i} - \frac{1}{7}\mathbf{j} + \frac{3}{14}\mathbf{k}$. Next notice that

$\mathbf{R}\mathbf{i} = \mathbf{R}(\mathbf{i}_1 + \mathbf{i}_2) = \mathbf{R}\mathbf{i}_1 + \mathbf{R}\mathbf{i}_2 = -\mathbf{i}_1 + \mathbf{i}_2 = \frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$ (because the reflection does not change the component in the plane but changes the sign of the component orthogonal to the plane).

In a similar way we find that $\mathbf{R}\mathbf{j} = -\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$ and $\mathbf{R}\mathbf{k} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k}$.

Now we see that the matrix corresponding to the rotation \mathbf{R} is

$$[\mathbf{R}] = \begin{bmatrix} \frac{6}{7} & -\frac{2}{7} & \frac{3}{7} \\ -\frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & \frac{6}{7} & -\frac{2}{7} \end{bmatrix}$$

Problem 2. Consider composition of two rotations. Rotation in the xz -plane by 30° is followed by the rotation in the yz -plane by 45° . Find the axis and the angle of the composite rotation.

Solution. The matrix of rotation by angle θ in the xz -plane, i.e. about the y -axis is (see page 255)

$$\begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

In our case ($\theta = 30^\circ$) we have

$$A = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{3} & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & \sqrt{3} \end{bmatrix}$$

The matrix of rotation by angle θ in yz -plane, i.e. about the x -axis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

In the case when $\theta = 45^\circ$ we have

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} \end{bmatrix}$$

The matrix of the composite rotation is

$$C = BA = \frac{1}{4} \begin{bmatrix} 2\sqrt{3} & 0 & 2 \\ \sqrt{2} & 2\sqrt{2} & -\sqrt{6} \\ -\sqrt{2} & 2\sqrt{2} & \sqrt{6} \end{bmatrix}$$

To find the axis of rotation we will use the formula Exercise 28 on page 262, namely if \mathbf{x} is any non-zero vector from R^3 then a vector in the direction of the axis of rotation can be found by the formula $\mathbf{u} = C\mathbf{x} + C^T\mathbf{x} + (1 - \text{tr}(C))\mathbf{x}$. Taking e.g. $\mathbf{x} = \mathbf{i}$ we get

$$\mathbf{u} = \left(\frac{\sqrt{3}}{2} + 1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{4} \right) \mathbf{i} + \frac{\sqrt{2}}{4} \mathbf{j} + \left(-\frac{\sqrt{2}}{4} + \frac{1}{2} \right) \mathbf{k}$$

The angle of rotation can be computed by the formula from Problem 27 on page 262

$$\cos \theta = \frac{\text{tr}(C) - 1}{2} \approx 0.5928$$

Hence $\theta \approx \arccos(0.5928) \approx 53.6^\circ$.

Problem 3. Find the rank of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution. The determinant of the matrix is obviously 0 whence its rank is less than 3. On the other hand the minor made of the corner elements, $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ is not 0 whence the rank of the matrix is 2.

Problem 4. Find a basis for the row space and a basis for the column space of the matrix in Problem 3.

Solution. The dimension of both the row space and the column space equals to the rank of the matrix that is 2. The row vectors $(1,0,0)$ and $(0,0,1)$ are obviously linearly independent and make a basis of the row space.

The column vectors $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ are also linearly independent and make the basis of the column space.

Problem 5. For the linear transformation A corresponding to the matrix in Problem 3 find a basis of the image $\text{Im } A$ and of the null-space $\ker A$.

Solution. The dimension of $\text{Im } A$ equals to the rank of the matrix i.e. 2. Therefore any basis of this space will consist of two linearly independent vectors. To find such vectors consider for example $A\mathbf{i} = \mathbf{i} - 2\mathbf{j}$ and $A\mathbf{j} = -2\mathbf{j} + \mathbf{k}$.

The dimension of the null space $\ker A$ is 1 whence any non-zero vector from this space will make its basis. In other words we have to find a nontrivial solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$. Clearly $A\mathbf{j} = \mathbf{0}$ and vector \mathbf{j} is a basis of $\ker A$.