1. Find the angles in the triangle with the vertices

$P(1,2,3),\ Q(4,-5,6),\ \text{and } R(7,8,-9).$

Round to one hundredth of a degree.

Solution

$$\cos(\angle P) = \frac{\mathbf{PQ}\bullet\mathbf{PR}}{\|\mathbf{PQ}\|\cdot\|\mathbf{PR}\|} =$$

$$= \frac{\mathbf{(3,-7,3)\bullet(6,6,-12)}}{\|\mathbf{(3,-7,3)}\|\cdot\|\mathbf{(6,6,-12)}\|} =$$

$$= -\frac{60}{6\sqrt{67}\sqrt{6}} \approx -.49875$$

Whence $\angle P \approx \arccos(-0.49875) \approx 119.92^\circ$

Similarly we find $\angle Q \approx 39.39^\circ$ and $\angle R \approx 20.69^\circ$

2. For three given vectors

$\mathbf{U}(1,-2,3), \mathbf{V}(1,2,-3), \text{ and } \mathbf{W}(-1,2,3)$

find $(\mathbf{U}\times\mathbf{V})\bullet\mathbf{W}$ and $(\mathbf{U}\times\mathbf{V})\times\mathbf{W}$.

Solution

(a) The triple scalar product $(\mathbf{U}\times\mathbf{V})\bullet\mathbf{W}$ is equal to the determinant

$$\begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & -3 \\ -1 & 2 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & -3 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ -1 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 24$$

(b) By part (e) of Theorem 3.5.1 on page 162 we have

$$\mathbf{(U \times V) \times W = (W \bullet U)V - (W \bullet V)U} =$$
$$= 4\mathbf{V} + 6\mathbf{U} = (10, -4, 6)$$

3. Four points in R^3 are given as
$P(-1,1,1),\ Q(1,-1,1),\ R(1,1,-1),$ and $S(2,0,0)$. Find

(a) An equation of the plane containing points $P, Q,$ and $R.$

(b) The distance between the plane from part (a) and point S.

(e) The volume of the tetrahedron with vertices P, Q, R and S.

Solution

(a) Vector

$$\mathbf{PQ} \times \mathbf{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

Is perpendicular to the plane PQR whence an equation of this plane can be written as $x + y + z + D = 0$. Plugging into this equation the coordinates of P we see that $D = -1$ and an equation of plane PQR is

$$x + y + z - 1 = 0$$

(b) By formula (13.6.5) on page 686 we have

$$d = \frac{|1 \cdot 2 + 1 \cdot 0 + 1 \cdot 0 - 1|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}$$

(c) The area of a tetrahedron is one third of the product of its height and the area of the base. Therefore,

$$V = \frac{1}{3} \cdot \frac{1}{2} \| \mathbf{PQ} \times \mathbf{PR} \| \cdot d = \frac{4\sqrt{3}}{6} \cdot \frac{1}{\sqrt{3}} = \frac{2}{3}$$

4. Find parametric equations of the line of intersection of planes

$x + y + z = 3$ and $x + 2y + 3z = 6.$

Solution Subtracting the first equation from the second one we get

$y + 2z = 3$

Let $z = t,$ then $y = 3 - 2t,$ and $x = 3 - y - z = t$

Finally we have

$$x = t$$
$$y = 3 - 2t$$
$$z = t$$

5. Find an equation of the plane through the point $(1,1,1)$ and perpendicular to the line $x = 2 - t,\ y = 3 + 2t,\ z = 1 - 4t.$

Solution The directing vector of the line

$$\mathbf{d} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

must be perpendicular to the plane whence an equation of the plane is

$$x - 2y + 4z + D = 0.$$

By plugging into this equation

$x = y = z = 1$

we see that

$D = -3$

and the answer is

$$x - 2y + 4z - 3 = 0$$