240 Linear Algebra **Review 1**

Problem 1

Describe all the solutions of the following system in parametric form

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ -x_1 + 2x_2 - 7x_3 &= -2 \\ 2x_1 + x_2 + 4x_3 &= 9 \end{aligned}$$

Solution. The augmented matrix of the system is $\begin{bmatrix} 1 & 1 & 1 & 5 \\ -1 & 2 & -7 & -2 \\ 2 & 1 & 4 & 9 \end{bmatrix}$**. Performing the row operations** $R_2 + R_1$ and $R_3 - 2R_1$ **we bring it to the form** $\begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 3 & -6 & 3 \\ 0 & -1 & 2 & -1 \end{bmatrix}$**.**

Next, applying the operations $R_2 \div 3$ and $R_3 + R_2$ **we obtain the matrix** $\begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The corresponding system (equivalent to the original one) is

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_2 - 2x_3 &= 1 \end{aligned}$$

Introducing the parameter t **and putting** $x_3 = t$ **we get**

$$\begin{aligned} x_1 &= -3t + 4 \\ x_2 &= 1 + 2t \\ x_3 &= t \end{aligned}$$

Problem 2. Compute $A^3 - 2A^2 + A - I$ **where**

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

Solution. We find $A^2 = \begin{bmatrix} 6 & 4 & 5 \\ 4 & 3 & 4 \\ 5 & 4 & 6 \end{bmatrix}$ **and** $A^3 = \begin{bmatrix} 20 & 15 & 21 \\ 15 & 11 & 15 \\ 21 & 15 & 20 \end{bmatrix}$**, whence**

$$A^3 - 2A^2 + A - I = \begin{bmatrix} 8 & 8 & 13 \\ 8 & 5 & 8 \\ 13 & 8 & 8 \end{bmatrix}.$$

Problem 3. Find the following determinant using the Row Reduction

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 3 & -4 \\ 1 & 4 & 9 & 16 \\ 1 & -8 & 27 & -64 \end{vmatrix}.$$

Solution. The row operations $R_2 - R_1, R_3 - R_1, R_4 - R_1$ **do not change the determinant. Applying them we get the following determinant**

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 2 & -5 \\ 0 & 3 & 8 & 15 \\ 0 & -9 & 26 & -65 \end{vmatrix}$$

Using the method of cofactors and properties of determinants we see that the last determinant is equal to

$$\begin{vmatrix} -3 & 2 & -5 \\ 3 & 8 & 15 \\ -9 & 26 & -65 \end{vmatrix} = (-3)(2)(-5)\begin{vmatrix} 1 & 1 & 1 \\ -1 & 4 & -3 \\ 3 & 13 & 13 \end{vmatrix} = 30\begin{vmatrix} 1 & 1 & 1 \\ 0 & 5 & -2 \\ 0 & 10 & 10 \end{vmatrix} = 30(5 \cdot 10 + 2 \cdot 10) = 2100.$$

Problem 4. . Find the inverse A^{-1} using the adjoint matrix.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

Solution. The matrix of cofactors is

$$A^C = \begin{pmatrix} -5 & 1 & 7 \\ 7 & -5 & 1 \\ 1 & 7 & -5 \end{pmatrix}.$$

The transpose

$$B = \left(A^C\right)^T = \begin{pmatrix} -5 & 7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{pmatrix}.$$

The product

$$AB = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix}.$$

Hence

$$A^{-1} = \frac{1}{18}\begin{pmatrix} -5 & 7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{pmatrix}$$

Problem 5. . Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$**. Compute** A^{-1} **using the elementary row operations (Gauss – Jordan elimination).**

Solution.

$$\left[\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{matrix}\right]\left|\left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]\right. \overrightarrow{R_2 - R_1} \left[\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 9 \end{matrix}\right]\left|\left[\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]\right. \overrightarrow{R_3 - R_1}$$

$$\left[\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{matrix}\right]\left|\left[\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}\right]\right. \overrightarrow{R_3 - 3R_2} \left[\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{matrix}\right]\left|\left[\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -3 & 1 \end{matrix}\right]\right. \overrightarrow{R_3 / 2}$$

$$\left[\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{matrix}\right]\left|\left[\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -3/2 & 1/2 \end{matrix}\right]\right. \overrightarrow{R_2 - 2R_3} \left[\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]\left|\left[\begin{matrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{matrix}\right]\right. \overrightarrow{R_1 - R_3}$$

$$\left[\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]\left|\left[\begin{matrix} 0 & 3/2 & -1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{matrix}\right]\right. \overrightarrow{R_1 - R_2} \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]\left|\left[\begin{matrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{matrix}\right]\right..$$

We got that the inverse matrix is

$$A^{-1} = \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}.$$

It is a good idea to check our answer.

$$A^{-1}A=\frac{1}{2}\begin{bmatrix}6 & -5 & 1\\ -6 & 8 & -2\\ 2 & -3 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{bmatrix}=$$

$$=\frac{1}{2}\begin{bmatrix}6\cdot1+(-5)1+1\cdot1 & 6\cdot1+(-5)2+1\cdot4 & 6\cdot1+(-5)3+1\cdot9\\ (-6)1+8\cdot1+(-2)\cdot1 & (-6)1+8\cdot2+(-2)4 & (-6)1+8\cdot3+(-2)9\\ 2\cdot1+(-3)1+1\cdot1 & 2\cdot1+(-3)2+1\cdot4 & 2\cdot1+(-3)3+1\cdot9\end{bmatrix}=$$

$$=\frac{1}{2}\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$