MTH 211 -Calculus II - Assignments

Fall 2012

Note on Reading, Exercises and Sample Problems

Assignments have two main sections: Reading and Exercises with sample problems being part of the reading assignment.
Sample problems, found in the text book at the end of each section, are very important because they show how to solve typical problems, and they show the type of information and the presentation desired. Therefore, they are considered as part of the reading assignment.

The exercises are very important in learning mathematics and you should do them promptly after each section. You may not be able to do every exercise, but it is important to try each exercise, and you should let the instructor know if you have trouble with an exercise.

Assignment 1

Reading

Section 6.1 – Primitives and Integrals

Exercises

Section 6.1

1-13

Assignment 2

Reading

Section 6.2 – Area and Integration

Exercises

Section 6.2

1 – 15

Assignment 3

Reading

Section 6.3 – Definite Integrals and the Fundamental Theorem of Calculus

Exercises

Section 6.3

1 – 10

Assignment 4

Reading

Section 6.4 – Important Properties of Integrals

Exercises

Section 6.4

1 – 8

Optional, Advanced Exercises

Section 6.4

1, 2

Assignment 5

Reading

Section 6.5 – More on Computing Area

Exercises

Section 6.4

9, 11 – 16

Section 6.5

1 – 3

Assignment 6

Reading

Section 6.6 – Riemann Sums and Integrals

Exercises

Section 6.6

1 – 4

Assignment 7

Reading

Section 6.6 – Riemann Sums and Integrals

Exercises

Section 6.6

6 – 12

Assignment 8

Reading

Section 6.7 – The Average Value of a Function

Exercises

Section 6.7

1 – 3

Chapter 6 Exercises

1 – 6, 18

Assignment 9

Reading

Section 6.8 – Substitution and Change of Variables for Integration

Exercises

Section 6.8

1 – 8

Assignment 10

Reading

Section 6.8 – Substitution and Change of Variables for Integration

Exercises

Section 6.8

9 – 14

Chapter 6 Exercises

9 – 13

Assignment 11

Reading

Section 6.9 – Riemann Integration and the Integration of Discontinuous Functions

Exercises

Section 6.8

15 – 18

Section 6.9

1 – 7

Assignment 12

Reading

Section 6.9 – Riemann Integration and the Integration of Discontinuous Functions

Exercises

Section 6.9

8 – 12

Chapter 6 Exercises

7, 8, 14, 15, 17

Assignment 13

Reading

Sections 7.1, 7.2

Exercises

Section 7.1

1 – 5

Section 7.2

1, 4, 5, 6

Assignment 14

Reading

Section 7.3, pages 642 – 647

Section 7.3, Sample Problem 1, page 658

Exercises

Section 7.2

2, 3, 7, 8

Section 7.3

1, 2, 3, 7

Assignment 15

Reading

Section 7.3, pages 648 – 660

Exercises

Section 7.2

9, 10

Section 7.3

4, 5, 6, 9, 10

Assignment 16

Reading

Section 7.5

Do the following exercises

We need to define the number e by the equation:

$$e = \exp(1).$$

This is an important number and we use it several times in this exercise set.

1. Show that $\ln(e) = 1$.

2. Find $\int_{1}^{17} \frac{1}{t} dt$.

3. Find and simplify $\int_{2^3}^{4^5} \frac{1}{x} dx$.

4. Express the integral $\int_{1}^{\sqrt{2}} \frac{1}{t} dt$ in terms of the $\ln$ function.

5. The following integral is an integer. Find it: $\int_{1}^{\exp(5)} \frac{1}{t} dt$.

6. The $\ln$ function is called the **natural logarithm function**. We will see later why it is a logarithm and why it is natural.

   Write the following expressions as single natural logarithm.

   a. $5\ln(2) + 3\ln(7)$.
   b. $\frac{1}{2}\ln(16) - \frac{1}{3}\ln(27)$.

7. Let $f(x) = \ln\left(x^2\right)$.

   a. Find the domain of f.
   b. Find the derivative of f.

8. Let $f(x) = \ln\left(4 - x^2\right)$.

   a. Find the domain of f.
   b. Find the derivative of f.
   c. Find and simplify $f\left(4 - e^7\right)$.

9. Simplify $\sqrt{\exp(16)}$.

10. Simplify $\sqrt{e^{12}\ln^{-1}(24)}$.

11. If $\ln(y) = 7$, then express y in terms of the constant e.

12. Simplify $\ln\left((\exp(4))^9\right)$.

13. Find and simplify the derivatives.

    a. $\ln\left(x^2+3\right)$.
    b. $\exp(\sin(x))$.
    c. $\ln(\sin(x))$.
    d. $\exp\left(\ln\left(x^2-1\right)-\ln(x+1)\right)$.
    e. $\ln(\sin(\exp(x)))$.

Assignment 17

Reading

Section 7.6 – Logarithmic and Exponential Functions

Exercises

Section 7.6 Exercises

1. Show that $\log_a x = y$ if and only if $a^y = x$.

2. Use exercise 1 to solve for x.

   a. $\log_{10}(1000) = x$
   b. $\log_{10}(x) = 5$
   c. $\log_2(16) = x$
   d. $\log_7(x+5) - \log_7(x) = 2$.
   e. $\log_3(x+3) + \log_3(x-3) = 3$.

3. Use the formula $\log_a x = \frac{\ln(x)}{\ln(a)}$ and a calculator to approximate the following.

   a. $\log_2(1024)$
   b. $\log_{16}\left((254)^3\right)$, hint: simplify first.

4. Solve for x exactly, unless a particular logarithm is specified.

   a. $7^x = 3$, in terms of $\log_7$.
   b. $3^{x^2-1} = 27$
   c. $11^{x^3} = 15$, in terms of $\log_{11}$.
   d. $9^x = \sqrt{2}$, in terms of $\ln$.

5. Find derivatives

   a. $D(2^x)$
   b. $\frac{d}{dx}\left(3^{x^3+1}\right)$
   c. $\left(x^{\sqrt{2}}\right)'$
   d. $D(5^x \log_5(x))$

6. Find integrals

   a. $\int 3^x dx$
   b. $\int 5^{x^4} x^3 dx$
   c. $\int 2^{\sin(x)} \cos(x)\, dx$

7. Draw rough graphs.

   a. $f(x) = 3^x$
   b. $f(x) = \left(\frac{1}{4}\right)^x$
   c. $f(x) = \ln(x)$
   d. $f(x) = \log_2(x)$
   e. $f(x) = \log_{0.3}(x)$

Assignment 18

Reading

Section 7.7 – Applications of Exponentials and Logarithms

Exercises

Section 7.7 Exercises

In questions 1 – 5 use logarithmic differentiation to find the derivatives of each function.

1. $f(x) = x^x$.
2. $f(x) = x^{\sin(x)}$
3. $f(x) = \sqrt[x]{(x-3)^3}$
4. $f(x) = \frac{x^3(x-1)^4(x+1)^5}{(x-2)^3}$
5. $f(x) = \frac{\sqrt{x+1}(x-1)^2}{\sqrt[5]{x^2-x^3}}$.

In the following questions find the derivatives using either logarithmic differentiation, or another method.

6. $f(x) = x^9$
7. $f(x) = x^\pi$
8. $f(x) = \pi^x$
9. $f(x) = 2^{\left(x^2+1\right)}$

Find the integrals

10. $\int \sec(x)\, dx$
11. $\int \tan(x)$
12. $\int \sec^2(x)\, dx$
13. $\int \sin(x)\, dx$
14. $\int \tan\left(x^3\right) x^2 dx$
15. $\int \sec\left(x^2+1\right) x dx$

Assignment 19

Reading

Section 7.7 – Applications of Exponentials and Logarithms

Exercises

Section 7.7 Exercises

Use logarithmic differentiation to find the derivatives of each function.

1. $f(x) = x^{(x^x)}$
2. $f(x) = (x^x)^x$

   Find the integrals

3. $\int \sec(\cos(x)) \tan(\cos(x)) \sin(x)\, dx$
4. $\int \frac{1}{\sec(x)+5} \sec(x) \tan(x)\, dx$
5. $\int e^{\tan(x)} \sec^2(x)\, dx$

   Solve the differential equations. Find all solutions.

6. $y' + 3y = 0$
7. $4y' - 8y = 0$
8. $y' + 4x^3 y = 0$

   Solve the differential equations. Find the solution satisfying the initial condition.

9. $y' - 2y = 0$, initial condition $y(0) = 3$.
10. $y' + 5y = 0$, initial condition $y(1) = 2$.
11. A radioactive substance decays at a rate $y = 5e^{-2t}$ grams per year. What is the half life of the substance?
12. Let y represent the amount of bacteria present in an experiment at time x. If there are 20 bacteria at the start of the experiment, and if the bacteria increase at a rate $y' = 3y$, then how many bacteria will be present 24 hours later?
13. If a radioactive element decays in such a way that $y' = -y$ where y represents the amount present in grams at time t in hours, then what is the half life of the substance?

Assignment 20

Reading

Section 7.8 – Inverses of Trigonometric Functions

Exercises

Section 7.8

1. Verify the following integral formulas. The constant a is positive.

$$\begin{aligned}
\int \frac{1}{\sqrt{a^2-x^2}}dx &= \sin^{-1}\left(\frac{x}{a}\right)+C \\
\int \frac{1}{a^2+x^2}dx &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C \\
\int \frac{1}{|x|\sqrt{x^2-a^2}} &= \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+C
\end{aligned}$$

2. Prove that $\cos^{-1}$ is differentiable on $(-1,1)$ and

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}.$$

3. Find
   a. $\sin^{-1}\left(\frac{1}{2}\right)$
   b. $\arccos\left(\frac{\sqrt{3}}{2}\right)$
   c. $\arctan(1)$
   d. $\tan^{-1}\left(\sqrt{3}\right)$
   e. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right)$

4. Find
   a. $\cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$
   b. $\cos^{-1}\left(\cos\left(\frac{\sqrt{3}}{2}\right)\right)$
   c. $\sin^{-1}\left(\sin\left(\frac{\sqrt{2}}{2}\right)\right)$
   d. $\arctan\left(\tan\left(\frac{\pi}{4}\right)\right)$
   e. $\cos^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$
   f. $\sin^{-1}\left(\sin\left(\frac{38\pi}{6}\right)\right)$
   g. $\sin\left(\arccos\left(\frac{3}{5}\right)\right)$
   h. $\tan\left(\operatorname{arcsec}\left(\frac{13}{5}\right)\right)$

5. Differentiate:
   a. $\sin^{-1}(x)+\cos^{-1}(x)$
   b. $x^2\arctan\left(x^3\right)$
   c. $e^{\arcsin\left(x^2\right)}$
   d. $\ln\left(\tan^{-1}(x)\right)$.

6. Find the integrals:

a. $\int \frac{1}{\sqrt{1-16x^2}}dx$
b. $\int \frac{x}{\sqrt{1-16x^2}}dx$
c. $\int \frac{1}{\sqrt{81-x^2}}dx$
d. $\int \frac{1}{\sqrt{16-25x^2}}dx$
e. $\int \frac{1}{9+x^2}dx$
f. $\int \frac{1}{x\sqrt{x^2-1}}dx$, if $x > 0$
g. $\int \frac{1}{x\sqrt{x^2-1}}dx$, if $x < 0$
h. $\int \frac{1}{x^2+2x+10}dx$

Quiz 5, Tuesday, November 20

Topics: assignments 13 – 17

Section 7.2 Inverses of Functions

Section 7.3 Inverses of Functions

Section 7.5 The ln Function and Its Inverse

Section 7.6 Logarithmic and Exponential Functions

Assignment 21

Reading

Section 7.9 – Hyperbolic Functions and their Inverses

Section 7.10 – Elementary Functions

Section 7.11– Algebraic and Transcendental Functions

Exercises

Section 7.9

1 – 9, 13, optional: 10, 11, 12

Chapter 7 Exercises

1 – 9, and Advanced Exercises 2, 3

Assignment 22

Reading

Section 8.1 – Table of Basic Integrals

Learn the complete table of integrals

Exercises

Section 8.1 – Table of Basic Integrals

1 – 28

Assignment 23

Reading

Section 8.1 – Table of Basic Integrals

Review the complete table of integrals, make sure you know it by heart

Section 8.2 – Trigonometric Integrals (reduction formulae and powers of sines and cosines)

Exercises

Section 8.2 – Trigonometric Integrals

1 – 5

Assignment 24

Reading

Section 8.1 – Table of Basic Integrals

Review the complete table of integrals, make sure you know it by heart

Section 8.2 – Trigonometric Integrals (powers of tangents and secants)

Section 8.3 – Trigonometric Substitutions (sine and tangent substitutions)

Exercises

Section 8.3 – Trigonometric Substitutions

1, 6

Assignment 25

Reading

Section 8.1 – Table of Basic Integrals

Review the complete table of integrals, make sure you know it by heart

Section 8.2 – Trigonometric Integrals (half angle method and miscellaneous methods)

Section 8.3 – Trigonometric Substitutions (secant substitutions)

Exercises

Section 8.2 – Trigonometric Integrals

6 – 9

Section 8.3 – Trigonometric Substitutions

2, 4, 5, 7, 8, 9, 11

Assignment 26

Reading

Section 8.4 – Integration of Rational Functions

pages 787 – 795

Exercises

Section 8.4 – Integration of Rational Functions

1 – 4

Assignment 27

Reading

Section 8.4 – Integration of Rational Functions

pages 796 – 804

Section 8.5 – Integration by Parts

Section 8.6 – Exercises Preliminary to Section 8.7

Exercises

Section 8.5 – Integration by Parts

Try to find the following integrals using the techniques we have learned through integration by parts. Do not devote more than ten minutes to any of these exercises.

1. Find $\int xe^{x^2}dx$.
2. Find $\int x^3e^{x^2}dx$.
3. Find $\int x^2e^{x^2}dx$.

Use integration by parts to find the integrals.

1. $\int x\ \sin(x)\,dx$
2. $\int xe^{-2x}dx$
3. $\int \sin(x)\cos(x)\,dx$
4. $\int x^2e^xdx$

Assignment 28

Reading

Section 8.7 – Elementary Functions, Revisited

Section 8.8 – What Do We Do if an Integral is not Elementary?

Exercises

Section 8.7 – Elementary Functions, Revisited

1 – 7

Section 8.5 – Integration by Parts

1. $\int x^3 e^x dx$
2. $\int x \cos(x)\, dx$
3. $\int \arctan(x)\, dx$
4. $\int \ln(x)\, dx$
5. $\int \sin(\ln(x))\, dx$

Lecture Notes, Monday, November 26

Partial Fraction Decomposition

Given: a rational function $f(x) = \frac{g(x)}{h(x)}$.

Step 1. If necessary divide:

If $\deg$ (numerator) $<$ $\deg$ (denominator), then proceed to step 2. If not, then use long division to divide by the denominator obtain $f(x) = p(x) + \frac{q(x)}{h(x)}$.

The term $p(x)$ is called the polynomial part of the decomposition.

Step 2. Factor the denominator into irreducible linear and quadratic factors.

$$h(x) = (x-a)^k \ldots \left(x^2 + bx + c\right)^n \ldots$$

Step 3. For every linear term of the form $(x-a)^k$ form the partial fraction decomposition

$$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$$

and for every quadratic term of the form $\left(x^2+bx+c\right)^n$ form the partial fraction decomposition

$$\frac{B_1 x + C_1}{(x^2+bx+c)} + \frac{B_2 x + C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_n x + C_n}{(x^2+bx+c)^n}$$

Step 4. Add up the polynomial part plus the linear and quadratic partial fractions.

Step 5. Determine the unknown coefficients and write the original rational function decomposed into partial fractions.

Example. $f(x) = \frac{x^4}{x^4 - x^3 - x + 1}$

Step 1. If necessary divide: In this example it is necessary to divide. We did this in class obtaining

$$f(x) = 1 + \frac{x^3 + x + 1}{x^4 - x^3 - x + 1}$$

The polynomial part is $p(x) = 1$. We proceed to step 2 where we decompose $\frac{x^3+x+1}{x^4-x^3-x+1}$ into partial fractions.

Step 2. Factor the denominator into irreducible linear and quadratic factors.

$$x^4 - x^3 - x + 1 = (x-1)^2 \left(x^2 + x + 1\right)$$

Note: we must be careful that none of the quadratic terms can be factored into linear terms. This is equivalent to having an irreducible quadratic. This can be determined as follows: let the quadratic be $ax^2 + bx + c$. This is irreducible if and only if the discriminant $b^2 - 4ac$ is negative. In our case $b^2 - 4ac = -3$ is negative.

Step 3. Form the partial fraction decomposition

$$\frac{x^3 + x + 1}{x^4 - x^3 - x + 1} = \frac{A}{(x-1)^2} + \frac{D}{(x-1)} + \frac{Bx + C}{x^2 + x + 1}$$

Step 4. Recombine the decomposition into normal form, obtaining

$$\frac{x^3 + x + 1}{x^4 - x^3 - x + 1} = \frac{x^3(B+D) + x^2(A - 2B + C) + x(A + B - 2C) + (A + C - D)}{x^4 - x^3 - x + 1}$$

Step 5. Determine the unknown coefficients and write the original rational function decomposed into partial fractions.

We equate the coefficients of the numerators in step 4 to obtain the system of equations

$$\begin{aligned} B+D &= 1 \\ A-2B+C &= 0 \\ A+B-2C &= 1 \\ A+C-D &= 1 \end{aligned}$$

The solution (no details) is: $A=1, B=\frac{2}{3}, C=\frac{1}{3}, D=\frac{1}{3}$. Therefore the answer is

$$\begin{aligned} f(x) &= \frac{x^4}{x^4-x^3-x+1} = 1 + \frac{x^3+x+1}{x^4-x^3-x+1} \\ &= \underset{\text{this is the polynomial part from step 1}}{1} + \underbrace{\frac{A}{(x-1)^2} + \frac{D}{(x-1)} + \frac{Bx+C}{x^2+x+1}}_{\text{this is the partial fraction decomposition from step 3}} \\ &= 1 + \frac{1}{(x-1)^2} + \frac{\frac{1}{3}}{(x-1)} + \frac{\frac{2}{3}x+\frac{1}{3}}{x^2+x+1} \end{aligned}$$

Module on Arc Length

- Write up lecture notes on arc length through derivation of integral formula
- Write some routine exercises
- Challenge problem:

a. Let f be continuously differentiable on the interval $[a,b]$. Let $L(x) = \int_a^x \sqrt{1+f'(t)^2}\,dt$ be the arc length function for f.

i Find $L'(x)$ and show that L is increasing.

ii Find $\lim_{x\to c} \frac{L(x)}{x-c}$

Solution (b). L'Hospital's rule applies

$$\lim_{x\to c} \frac{L(x)}{x-c} = \lim_{x\to c} \frac{L'(x)}{1} = \sqrt{1+f'(c)^2}$$

Assignment Template

Assignment ?

Reading

Section

Exercises

Section

Optional, Advanced Exercises

Section

Modules

Arc Length

Definition 0.1 (arclength) *The arclength of the graph of a function f on an interval $[a,b]$ is defined to be*

$$\Lambda_f(a,b) = \lim_{\|P\|\to 0}\sum_{j=1}^{n}\sqrt{(x_j - x_{j-1})^2 + (f(x_j) - f(x_{j-1}))^2}$$

where the limit runs over all partitions $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$, if this limit exists.

Theorem 0.1 (arclength theorem.) *If f is continuously differentiable on $[a,b]$, then the graph of f has a finite arclength given by the integral formula*

$$\Lambda_f(a,b) = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

We covered the definition and the proof of the theorem in lecture.
Example. Find the length of the graph of $y = e^t$ on the interval $[0,x]$.
Solution. The function $f(t) = e^t$ satisfies the hypothesis of the arclength theorem. We first find the indefinite integral which is $\int \sqrt{1 + (e^x)^2}dx = \int \sqrt{1+e^{2x}}dx$

$$\begin{aligned} u^2 &= 1 + e^{2x} \\ e^{2x} &= u^2 - 1 \\ \sqrt{1+e^{2x}} &= u \\ udu &= \left(u^2 - 1\right)dx \\ dx &= \frac{u}{u^2-1}du \end{aligned}$$

So

$$\begin{aligned} \int \sqrt{1+e^{2x}}dx &= \int \frac{u^2}{u^2-1}du = 1 + \int \frac{1}{u^2-1}du \\ &= u + \frac{1}{2}\ln(u-1) - \frac{1}{2}\ln(u+1) \\ &= \sqrt{1+e^{2x}} + \frac{1}{2}\ln\left(\sqrt{1+e^{2x}} - 1\right) - \frac{1}{2}\ln\left(\sqrt{1+e^{2x}} + 1\right) \\ &= \sqrt{1+e^{2x}} + \frac{1}{2}\ln\left(\frac{\sqrt{1+e^{2x}} - 1}{\sqrt{1+e^{2x}} + 1}\right) + C. \end{aligned}$$

Thus

$$G(x) = \sqrt{1+e^{2x}} + \frac{1}{2}\ln\left(\frac{\sqrt{1+e^{2x}} - 1}{\sqrt{1+e^{2x}} + 1}\right)$$

is a primitive for the arclength integral. This shows that

$$\begin{aligned} \Lambda_f(0,x) &= G(x) - G(0) \\ &= \sqrt{1+e^{2x}} + \frac{1}{2}\ln\left(\frac{\sqrt{1+e^{2x}} - 1}{\sqrt{1+e^{2x}} + 1}\right) - \left(\sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) \end{aligned}$$

Solutions to Additional Exercises in Assignments

Assignment 16 Solutions

1. Show that $\ln(e) = 1$.

   Solution. By the cancellation of inverse functions,
$$\ln(e) = \ln(\exp(1)) = 1.$$

2. Find $\int\limits_1^{17} \frac{1}{x} dx$.

   Solution. We recognize this immediately as $\ln(17)$.

3. Find and simplify $\int\limits_{2^3}^{4^5} \frac{1}{x} dx$.

   Solution. Because $\ln'(x) = \frac{1}{x}$ on the given interval, then $\ln(x)$ is a primitive and by the fundamental theorem
$$\int\limits_{2^3}^{4^5} \frac{1}{x} dx = \ln\left(4^5\right) - \ln\left(2^3\right).$$
   We simplify this using the quotient rule:
$$\ln\left(4^5\right) - \ln\left(2^3\right) = \ln\left(\frac{4^5}{2^3}\right) = \ln\left(2^7\right) = \ln(128)$$
   An alternate answer using the power rule is $7\ln(2)$.

4. Express the integral $\int\limits_1^{\sqrt{2}} \frac{1}{t} dt$ in terms of the $\ln$ function.

   Solution. By definition of the $\ln$ function, $\int\limits_1^{\sqrt{2}} \frac{1}{t} dt = \ln\left(\sqrt{2}\right)$.

5. The following integral is an integer. Find it: $\int\limits_1^{\exp(5)} \frac{1}{t} dt$.

   Solution. We recognize this as a logarithm, then we use the fact that $\exp(x)$ is the inverse of $\ln(x)$:
$$\int\limits_1^{\exp(5)} \frac{1}{t} dt = \ln(\exp(5)) = 5$$
   by cancellation of inverse functions.

6. The $\ln$ function is called the **natural logarithm function**. We will see later why it is a logarithm and why it is natural.

Write the following expressions as single natural logarithm.

a. $5\ln(2)+3\ln(7)$.
b. $\frac{1}{2}\ln(16)-\frac{1}{3}\ln(27)$.

Solution.

$$5\ln(2)+3\ln(7)=\ln\left(2^5 7^3\right)=\ln(10\,976)$$

7. Let $f(x)=\ln\left(x^2\right)$.

   a. Find the domain of f.
   b. Find the derivative of f.

   Solution. By the definition of $\ln(x)$, we need $x^2>0$ and this happens if and only if $|x|>0$. Thus the domain is $(-\infty,0)\cup(0,\infty)$.

   By the chain rule,
$$f'(x)=\ln'\left(x^2\right)\left(x^2\right)'=\frac{1}{x^2}2x=\frac{2}{x}.$$

8. Let $f(x)=\ln\left(4-x^2\right)$.

   a. Find the domain of f.
   b. Find the derivative of f.
   c. Find and simplify $f\left(\sqrt{4-e^7}\right)$.

   Solution. $x\in domain(f)$ if and only if $4-x^2>0$. The solution is $(-2,2)$ and thus $domain(f)=(-2,2)$.
$$f'(x)=\frac{1}{4-x^2}(-2x)=\frac{2}{x^2-4}.$$
   Finally
$$\begin{aligned} f\left(\sqrt{4-e^7}\right) &= \ln\left(4-\left(\sqrt{4-e^7}\right)^2\right) \\ &= \ln\left(e^7\right)=7\ln(e)=7. \end{aligned}$$

9. Simplify $\sqrt{\exp(16)}$.

   Solution.
$$\sqrt{\exp(16)}=\underbrace{(\exp(16))^{\frac{1}{2}}=\exp\left(16\cdot\frac{1}{2}\right)}_{\text{power rule for }\exp(x)\text{ used here}}=\exp(8).$$

10. Simplify $\sqrt{e^{12}\ln^{-1}(24)}$.

    Solution. Using the facts that $e^{12}=\exp(1)^{12}=\exp(12)$, by the power rule, and $\ln^{-1}(24)=\exp(24)$, we get
$$\sqrt{e^4\ln^{-1}(24)}=(\exp(12)\exp(24))^{\frac{1}{2}}=\exp(36)^{\frac{1}{2}}=\exp\left(36\cdot\frac{1}{2}\right)=\exp(18)$$

11. If $\ln(y)=7$, then express y in terms of the constant e.

    Solution. Apply the inverse function $\exp$ to both sides to get $y=\exp(7)=\exp(1\cdot 7)$. By the power rule for $\exp(x)$ we get $\exp(1)^7=e^7$.

12. Simplify $\ln\left((\exp(4))^9\right)$.

Solution.

$$\ln\left((\exp(4))^9\right) = 9\ln(\exp(4)) = 36.$$

13. Find the derivatives.

a. $\ln\left(x^2+3\right)$.
b. $\exp(\sin(x))$.
c. $\ln(\sin(\exp(x)))$.

Solution.

$$\begin{aligned}
\frac{d}{dx}\ln\left(x^2+3\right) &= \frac{2x}{x^2+3} \\
\frac{d}{dx}\exp(\sin(x)) &= \cos(x)\exp(\sin(x)) \\
\frac{d}{dx}\ln(\sin(x)) &= \frac{1}{\sin(x)}\cos(x) = \cot(x) \\
\frac{d}{dx}\exp\left(\ln\left(x^2-1\right)-\ln(x+1)\right) &= \frac{d}{dx}\exp\left(\ln\left(\frac{x^2-1}{x+1}\right)\right) \\
&= x-1 \\
\frac{d}{dx}\ln(\sin(\exp(x))) &= \frac{1}{\sin(\exp(x))}\cdot(\sin(\exp(x)))' \\
&= \frac{1}{\sin(\exp(x))}\cos(\exp(x))\cdot\exp(x) \\
&= \cot(\exp(x))\cdot\exp(x)\,.
\end{aligned}$$

Assignment 17 Solutions

1. Show that $\log_a x = y$ if and only if $a^y = x$.

   Solution. Let $f(x) = a^y$ and let $g(x) = \log_a x$. By the definition of base a logarithms f and g are inverse to each other. Thus $g(x) = y$ if and only if $f(y) = x$ and this is exactly the given statement.

2. Use exercise 1 to solve for x.

   a. $\log_{10}(1000) = x$, answer $x = 3$.
   b. $\log_{10}(x) = 5$, answer $x = 10^5$.
   c. $\log_2(16) = x$, answer $x = 4$.
   d. $\log_7(x+5) - \log_7(x) = 2$.

$$\begin{aligned} 2 &= \log_7\left(\frac{x+5}{x}\right) \\ 49 &= 7^2 = \frac{x+5}{x} \\ 49x &= x+5 \\ x &= \frac{5}{48} \end{aligned}$$

   e. $\log_3(x+3) + \log_3(x-3) = 3$.

   Similarly to the previous problem, laws of logarithms leads to $27 = x^2 - 9$, with solutions: $x = -6, x = 6$. But we cannot compute the logarithms if $x = -6$ (why?) Thus the unique solution is $x = 6$.

3. Use the formula $\log_a x = \frac{\ln(x)}{\ln(a)}$ and a calculator to approximate the following.

   a. $\log_2(1024) = \frac{\ln(1024)}{\ln(2)} = 10.0$, Actually the answer is exact (why?)
   b. $\log_{16}\left((254)^3\right)$, hint: simplify first.

$$\log_{16}\left((254)^3\right) = 3\log_{16}(254) \approx 5.99151351$$

4. Solve for x exactly, unless a particular logarithm is specified.

   a. $7^x = 3$, in terms of $\log_7$. Answer: $x = \log_7(3)$.
   b. $3^{x^2-1} = 27$,
      Taking $\log_3$, $x^2 - 1 = 3$ so $x = \pm 2$.
   c. $11^{x^3} = 15$, in terms of $\log_{11}$.

$$\begin{aligned} x^3 &= \log_{11}(15) \\ x &= \sqrt[3]{\log_{11}(15)} \end{aligned}$$

   d. $9^x = \sqrt{2}$, in terms of $\ln$.

$$\begin{aligned} x\ln(9) &= \frac{1}{2}\ln(2) \\ x &= \frac{1}{2}\frac{\ln(2)}{\ln(9)}. \end{aligned}$$

5. Find derivatives

   a. $D(2^x) = \ln(2)\,2^x$
   b. $\frac{d}{dx}\left(3^{x^3+1}\right) = 3x^2\ln(3)\,3^{x^3+1}$

c. $\left(x^{\sqrt{2}}\right)' = \sqrt{2}x^{\sqrt{2}-1}$

d. $D\left(5^x \log_5(x)\right) = (5^x)' \log_5(x) + 5^x \left(\log_5(x)\right)' = \ln(5)\, 5^x \log_5(x) + 5^x \frac{1}{\ln(5)x}$.

6. Find integrals

a. $\int 3^x dx = \frac{1}{\ln(3)} 3^x + C$

b. $\int 5^{x^4} x^3 dx$

Substitute $u = x^4, \frac{1}{4} du = x^3 dx \Longrightarrow \frac{1}{4} \int 5^u du = \frac{1}{4\ln(5)} 5^{x^4} = \frac{1}{\ln(625)} 5^{x^4} + C$

c. $\int 2^{\sin(x)} \cos(x)\, dx$

Substitute $u = \sin(x)\,, du = \cos(x)\, dx \Longrightarrow \int 2^u du = \frac{1}{\ln(2)} 2^{\sin(x)}$

7. Draw rough graphs.

a. $f(x) = 3^x$

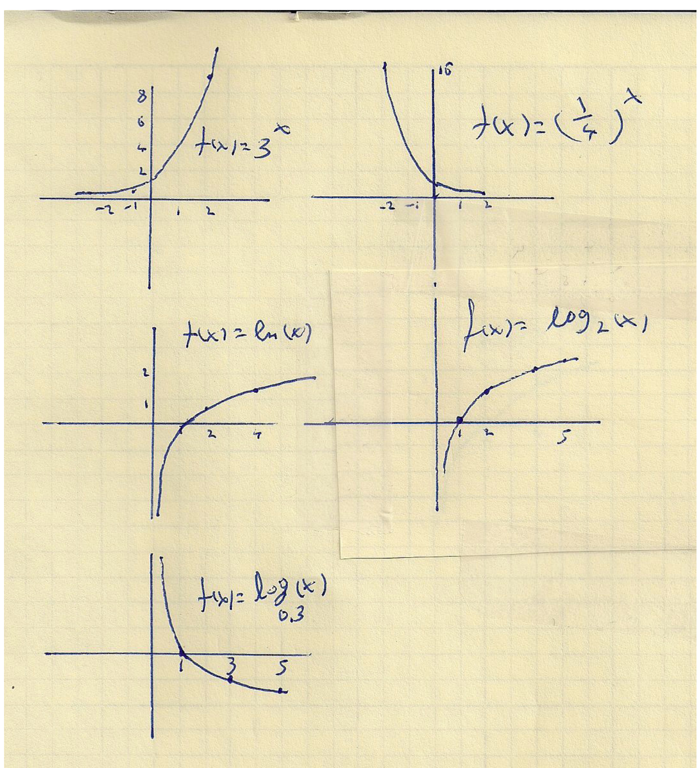
b. $f(x) = \left(\frac{1}{4}\right)^x$

c. $f(x) = \ln(x)$

d. $f(x) = \log_2(x)$

e. $f(x) = \log_{0.3}(x)$

Solution.



1.

Assignment 18 Solutions

In questions 1 – 5 use logarithmic differentiation to find the derivatives of each function.

1. $f(x) = x^x$.

   Solution.

$$\begin{aligned} \ln y &= x \ln(x) \\ \frac{1}{y} y' &= \ln(x) + 1 \end{aligned}$$

   Answer: $y' = x^x (\ln(x) + 1)$.

2. $f(x) = x^{\sin(x)}$

$$\begin{aligned} \ln y &= \sin(x) \ln(x) \\ \frac{1}{y} y' &= \cos(x) \ln(x) + \frac{\sin(x)}{x} \end{aligned}$$

   Answer: $y' = x^{\sin(x)} \left(\cos(x) \ln(x) + \frac{\sin(x)}{x} \right)$

3. $f(x) = \sqrt[x]{(x-3)^3}$

$$\begin{aligned} \ln y &= \frac{3}{x} \ln(x-3) \\ \frac{1}{y} y' &= -\frac{3}{x^2} \ln(x-3) + \frac{3}{x(x-3)} \end{aligned}$$

   Answer: $y' = \sqrt[x]{(x-3)^3} \left(-\frac{3}{x^2} \ln(x-3) + \frac{3}{x(x-3)} \right)$

4. $f(x) = \frac{x^3 (x-1)^4 (x+1)^5}{(x-2)^3}$

$$\begin{aligned} \ln y &= 3\ln(x) + 4\ln(x-1) + 5\ln(x+1) - 3\ln(x-2) \\ \frac{1}{y} y' &= \frac{3}{x} + \frac{4}{x-1} + \frac{5}{x+1} - \frac{3}{x-2} \end{aligned}$$

   Answer: $y' = \left(\frac{3}{x} + \frac{4}{x-1} + \frac{5}{x+1} - \frac{3}{x-2} \right) \frac{x^3 (x-1)^4 (x+1)^5}{(x-2)^3}$.

5. $f(x) = \frac{\sqrt{x+1}(x-1)^2}{\sqrt[5]{x^2 - x^3}}$.

$$\begin{aligned} \ln y &= \frac{1}{2} \ln(x+1) + 2\ln(x-1) - \frac{1}{5} \ln\left(x^2 - x^3\right) \\ \frac{1}{y} y' &= \frac{1}{2(x+1)} + \frac{2}{x-1} - \frac{1}{5(x^2 - x^3)} \left(2x - 3x^2\right) \end{aligned}$$

   Answer: $y' = \frac{\sqrt{x+1}(x-1)^2}{\sqrt[5]{x^2 - x^3}} \left(\frac{1}{2(x+1)} + \frac{2}{x-1} - \frac{2x - 3x^2}{5(x^2 - x^3)} \right)$.

   In the following questions find the derivatives using either logarithmic differentiation, or another method.

6. $f'(x) = (x)' = 9x^8$

7. $f'(x) = (x^\pi)' = \pi x^{\pi - 1}$

8. $f'(x) = (\pi^x)' = \ln(\pi) \pi^x$

9. $f'(x) = \left(2^{(x^2+1)}\right)' = \ln(2) 2^{(x^2+1)} \cdot 2x$

Find the integrals

10. $\int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + C$

11. $\int \tan(x) = \ln|\sec(x)| + C$

12. $\int \sec^2(x)\, dx = \tan(x) + C$

13. $\int \sin(x)\, dx = -\cos(x) + C$

14. $\int \tan\left(x^3\right) x^2 dx = \frac{1}{3}\ln\left|\sec\left(x^3\right)\right| + C$, by a substitution.

15. $\int \sec\left(x^2+1\right) x dx = \frac{1}{2}\ln\left|\sec\left(x^2+1\right) + \tan\left(x^2+1\right)\right| + C$, by a substitution.

Assignment 19 Solutions

Use logarithmic differentiation to find the derivatives of each function.

1. $f(x) = x^{(x^x)}$

   Solution. By the solution to exercise **??**, $(x^x) = x^x(\ln(x)+1)$. Then

$$\begin{aligned} y &= x^{(x^x)} \\ \ln(y) &= (x^x)\ln(x) \\ \frac{1}{y}y' &= (x^x(\ln(x)+1))\ln(x) + (x^x)\frac{1}{x} \\ y' &= x^{(x^x)}\left((x^x(\ln(x)+1))\ln(x) + (x^x)\frac{1}{x}\right). \end{aligned}$$

2. $f(x) = (x^x)^x$

   Solution. By the solution to exercise **??**, $(x^x) = x^x(\ln(x)+1)$. Then

$$\begin{aligned} y &= (x^x)^x \\ \ln(y) &= x\ln(x^x) \\ \frac{1}{y}y' &= \ln(x^x) + x\frac{1}{x^x}(x^x(\ln(x)+1)) \\ y' &= (x^x)^x\left(\ln(x^x) + x\frac{1}{x^x}(x^x(\ln(x)+1))\right). \end{aligned}$$

Find the integrals

3. $\int \sec(\cos(x))\tan(\cos(x))\sin(x)\, dx$

   Solution. Substitute $u = \cos(x), -du = \sin(x)\, dx$

$$-\int \sec(u)\tan(u)\, du = -\sec(\cos(x)) + C.$$

4. $\int \frac{1}{\sec(x)+5}\sec(x)\tan(x)\, dx$

   Solution. Substitute $u = \sec(x) + 5, du = \sec(x)\tan(x)\, dx$

$$\int \frac{1}{u}du = \ln|\sec(x)+5| + C.$$

5. $\int e^{\tan(x)} \sec^2(x)\, dx$

   Solution. Substitute $u = \tan(x) .du = \sec^2(x)$

$$\int e^u du = e^{\tan(x)} + C.$$

   Solve the differential equations. Find all solutions.

6. $y' + 3y = 0$

   Solution. $y = ce^{-3x}$.

7. $4y' - 8y = 0$

   Solution. $y = ce^{2x}$.

8. $y' + 4x^3 y = 0$

   Solution. Let $A(x) = \int 4x^3 dx = x^4$. The solution is $ce^{-A(x)} = ce^{-x^4}$.

   Solve the differential equations. Find the solution satisfying the initial condition.

9. $y' - 2y = 0$, initial condition $y(0) = 3$.

   Solution. $y = ce^{2x}$. Then $3 = y(0) = c$ so the solution is $y = 3e^{2x}$.

10. $y' + 5y = 0$, initial condition $y(1) = 2$.

    Solution. $y = ce^{-5x}$. Then $2 = y(1) = ce^{-5}, c = 2e^5$, so the solution is $y = 2e^5 e^{-5x} = 2e^{5-5x}$.

11. A radioactive substance decays at a rate $y = 5e^{-2t}$ grams per year. What is the half life of the substance?

    Solution. By the half life formula $T_{1/2} = \frac{\ln(2)}{a}$. Here $a = 2$ so the half life is $\frac{\ln(2)}{2} \approx 0.347$ years.

12. Let y represent the amount of bacteria present in an experiment at time x. If there are 20 bacteria at the start of the experiment, and if the bacteria increase at a rate $y' = 3y$, then how many bacteria will be present 24 hours later?

    Solution. Exponential growth is governed by the equation $y = ce^{3x}$. Since $20 = y(0) = c$, then at the end of x hours there are $y = 20e^{3x}$ bacteria. Answer: $20e^{72} \approx 3.717 \times 10^{32}$ bacteria.

13. If a radioactive element decays in such a way that $y' = -y$ where y represents the amount present in grams at time t in hours, then what is the half life of the substance?

    Solution. Exponential decay is governed by the equation $y = ce^{-x}$. By the half life equation the half life is $\frac{\ln(2)}{1} \approx$ 0.693 hours.

Assignment 20 Solutions

1. Verify the following integral formulas. The constant a is positive.

$$\begin{aligned} \int \frac{1}{\sqrt{a^2 - x^2}} dx &= \sin^{-1}\left(\frac{x}{a}\right) + C \\ \int \frac{1}{a^2 + x^2} dx &= \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C \\ \int \frac{1}{|x| \sqrt{x^2 - a^2}} &= \frac{1}{a} sec^{-1}\left(\frac{x}{a}\right) + C \end{aligned}$$

Solution.
$$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right)=\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\left(\frac{x}{a}\right)'=\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\frac{1}{a}$$
We can bring the factor $\frac{1}{z}$ into the square root, where the a becomes a^2 giving
$$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right)=\frac{1}{\sqrt{\left(1-\left(\frac{x}{a}\right)^2\right)a^2}}=\frac{1}{\sqrt{a^2-x^2}}$$
and this proves the first formula. The other formulas are proved in a similar way.

2. Prove item **??** of theorem **??**: $\cos^{-1}$ is differentiable on $(-1,1)$ and
$$\frac{d}{dx}\cos^{-1}(x)=\frac{-1}{\sqrt{1-x^2}}.$$
Solution. By definition, $\cos^{-1}(x)=\frac{\pi}{2}-\sin^{-1}(x)$. Therefore,
$$\begin{aligned}\frac{d}{dx}\cos^{-1}(x) &= \frac{d}{dx}\left(\frac{\pi}{2}-\sin^{-1}(x)\right)\\ &= 0-\frac{d}{dx}\left(\sin^{-1}(x)\right)\\ &= \frac{-1}{\sqrt{1-x^2}}.\end{aligned}$$

3. Find
   a. $\sin^{-1}\left(\frac{1}{2}\right)=\frac{1}{6}\pi$
   b. $\arccos\left(\frac{\sqrt{3}}{2}\right)=\frac{1}{6}\pi$
   c. $\arctan(1)=\frac{1}{4}\pi$
   d. $\tan^{-1}\left(\sqrt{3}\right)=\frac{1}{3}\pi$
   e. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right)=\frac{1}{6}\pi$

4. Find
   a. $\cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$
   Solution. $x=\frac{\sqrt{3}}{2}\in[-1,1]$ which is in the domain of the inverse cosine function. Therefore, cancellation applies and $\cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)=\frac{\sqrt{3}}{2}$
   b. $\cos^{-1}\left(\cos\left(\frac{\sqrt{3}}{2}\right)\right)$
   Solution. The cancellation law $\cos^{-1}(\cos(x))=x$ is valid as long as x is in the range $[0,\pi]$ of the inverse cosine function. Because this is true for $x=\frac{\sqrt{3}}{2}$,then $\cos^{-1}\left(\cos\left(\frac{\sqrt{3}}{2}\right)\right)=\frac{\sqrt{3}}{2}$.
   c. $\sin^{-1}\left(\sin\left(\frac{\sqrt{2}}{2}\right)\right)=\frac{\sqrt{2}}{2}$
   d. $\arctan\left(\tan\left(\frac{\pi}{4}\right)\right)=\frac{\pi}{4}$
   e. $\cos^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$
   Solution. Because $x=\frac{13\pi}{6}$ is not in the range $[0,\pi]$ of the inverse cosine function, then cancellation does not work. However, by division, we see that $\frac{13\pi}{6}=2\pi+\frac{\pi}{6}$. Therefore, by periodicity, $\cos\left(\frac{13\pi}{6}\right)=\cos\left(\frac{\pi}{6}\right)$.

Because $x=\frac{\pi}{6}$ is in the range of $\cos^{-1}$, then
$$\cos^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)=\cos^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)=\frac{\pi}{6}.$$

f. $\sin^{-1}\left(\sin\left(\frac{38\pi}{6}\right)\right)=\frac{\pi}{3}$
Because $\frac{38\pi}{6}=6\pi+\frac{\pi}{3}$ and $\frac{\pi}{3}$ is in the range of the inverse sine function.

g. $\sin\left(\arccos\left(\frac{3}{5}\right)\right)=\frac{4}{5}$.
Solution. Let $\arccos\left(\frac{3}{5}\right)=\theta$. Then $\cos(\theta)=\frac{3}{5}$ and from $\sin^2(\theta)+\cos^2(\theta)=1$ we get $\sin\left(\arccos\left(\frac{3}{5}\right)\right)=\pm\frac{4}{5}$. But the range of the inverse cosine is $[0\,,\pi]$ which contains only positive angles. Therefore, $\sin\left(\arccos\left(\frac{3}{5}\right)\right)=\frac{4}{5}$.

h. $\tan\left(\operatorname{arcsec}\left(\frac{13}{5}\right)\right)=\frac{12}{5}$.
Solution. Let $\operatorname{arcsec}\left(\frac{13}{5}\right)=\theta$. Then $\sec(\theta)=\frac{13}{5}$ and from the identity $\tan^2(\theta)=\sec^2(\theta)-1$ we get $\tan^2(\theta)=\frac{144}{25}$ and hence $\tan\left(\operatorname{arcsec}\left(\frac{13}{5}\right)\right)=\pm\frac{12}{5}$. But this cannot be negative, as we see from the graph of the inverse secant function. Thus $\tan\left(\operatorname{arcsec}\left(\frac{13}{5}\right)\right)=\frac{12}{5}$.

5. Differentiate:

a. $\sin^{-1}(x)+\cos^{-1}(x)$
b. $x^2\arctan\left(x^3\right)$
c. $e^{\arcsin\left(x^2\right)}$
d. $\ln\left(\tan^{-1}(x)\right)$.

Solution.

a.
$$\frac{d}{dx}\left(\sin^{-1}(x)+\cos^{-1}(x)\right)=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}=0.$$
Remark. Therefore $\sin^{-1}(x)+\cos^{-1}(x)$ is constant on $[-1,1]$. Substituting $x=0$ shows that the constant is $\frac{\pi}{2}$ and hence $\sin^{-1}(x)+\cos^{-1}(x)=\frac{\pi}{2}$ which is consistent with our definition: $\cos^{-1}(x)=\frac{\pi}{2}-\sin^{-1}(x)$.

b.
$$\frac{d}{dx}x^2\arctan\left(x^3\right)=2x\arctan\left(x^3\right)+\frac{3x^4}{1+x^6}.$$

c.
$$\begin{aligned}\frac{d}{dx}e^{\arcsin\left(x^2\right)} &= e^{\arcsin\left(x^2\right)}\frac{d}{dx}\arcsin\left(x^2\right)\\ &= e^{\arcsin\left(x^2\right)}\frac{2x}{\sqrt{1-x^4}}.\end{aligned}$$

d.
$$\frac{d}{dx}\ln\left(\tan^{-1}(x)\right)=\frac{1}{\tan^{-1}(x)}\frac{1}{1+x^2}.$$

6. Find the integrals:

a. $\int\frac{1}{\sqrt{1-16x^2}}dx=\frac{1}{4}\arcsin(4x)+C$ (hint: substitute $u=4x$)
b. $\int\frac{x}{\sqrt{1-16x^2}}dx=-\frac{\sqrt{1-16x^2}}{16}+C$
Solution. This is not an inverse sine. Substitute $u=1-16x^2, du=-32xdx$ and we get the answer.
c. $\int\frac{1}{\sqrt{81-x^2}}dx=\arcsin\left(\frac{x}{9}\right)+C$ (by the table formula $\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+C$)
d. $\int\frac{1}{\sqrt{16-25x^2}}dx$
Solution. Substitute $u=5x, du=5dx$ to get $\frac{1}{5}\int\frac{1}{\sqrt{4^2-u^2}}du=\frac{1}{5}\arcsin\left(\frac{1}{4}u\right)=\frac{1}{5}\sin^{-1}\left(\frac{x}{4}\right)+C$

e. $\int \frac{1}{9+x^2} dx = \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$

f. $\int \frac{1}{x\sqrt{x^2-1}} dx$, if $x > 0$

Solution. Because $x > 0$ this is the same as $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1}(x) + C$

g. $\int \frac{1}{x\sqrt{x^2-1}} dx$, if $x < 0$

Solution. Because $x < 0$ this is the same as $-\int \frac{1}{|x|\sqrt{x^2-1}} dx = -\sec^{-1}(x) + C$

h. $\int \frac{1}{x^2+2x+10} dx =$

Solution. Complete the square: $x^2 + 2x + 10 = (x+1)^2 + 3^2 = u^2 + 3^2$ where $u = x + 1, du = dx$, giving $\int \frac{1}{3^2+u^2} du = \frac{1}{3} \arctan\left(\frac{u}{3}\right) = \frac{1}{3} \arctan\left(\frac{x+1}{3}\right) + C.$

Exercises, to be entered in CBM

Problems for Section 7.6

1. Show that $\log_a x = y$ if and only if $a^y = x$.

2. Use exercise 1 to solve for x.

   a. $\log_{10}(1000) = x$
   b. $\log_{10}(x) = 5$
   c. $\log_2(16) = x$
   d. $\log_7(x+5) - \log_7(x) = 2$.
   e. $\log_3(x+3) + \log_3(x-3) = 3$.

3. Use the formula $\log_a x = \frac{\ln(x)}{\ln(a)}$ and a calculator to approximate the following.

   a. $\log_2(1024)$
   b. $\log_{16}\left((254)^3\right)$, hint: simplify first.

4. Solve for x exactly, unless a particular logarithm is specified.

   a. $7^x = 3$, in terms of $\log_7$.
   b. $3^{x^2-1} = 27$
   c. $11^{x^3} = 15$, in terms of $\log_{11}$.
   d. $9^x = \sqrt{2}$, in terms of $\ln$.

5. Find derivatives

   a. $D(2^x)$
   b. $\frac{d}{dx}\left(3^{x^3+1}\right)$
   c. $\left(x^{\sqrt{2}}\right)'$
   d. $D(5^x \log_5(x))$

6. Find integrals

   a. $\int 3^x dx$
   b. $\int 5^{x^4} x^3 dx$
   c. $\int 2^{\sin(x)} \cos(x)\, dx$

7. Draw rough graphs.

   a. $f(x) = 3^x$
   b. $f(x) = \left(\frac{1}{4}\right)^x$
   c. $f(x) = \ln(x)$
   d. $f(x) = \log_2(x)$
   e. $f(x) = \log_{0.3}(x)$

Problems for Section 7.7

In questions 1 – 5 use logarithmic differentiation to find the derivatives of each function.

1. $f(x) = x^x$.
2. $f(x) = x^{\sin(x)}$
3. $f(x) = \sqrt[x]{(x-3)^3}$
4. $f(x) = \frac{x^3(x-1)^4(x+1)^5}{(x-2)^3}$
5. $f(x) = \frac{\sqrt{x+1}(x-1)^2}{\sqrt[5]{x^2-x^3}}$.

   In the following questions find the derivatives using either logarithmic differentiation, or another method.
6. $f(x) = x^9$
7. $f(x) = x^{\pi}$
8. $f(x) = \pi^x$
9. $f(x) = 2^{\left(x^2+1\right)}$

   Find the integrals
10. $\int \sec(x)\,dx$
11. $\int \tan(x)$
12. $\int \sec^2(x)\,dx$
13. $\int \sin(x)\,dx$
14. $\int \tan\left(x^3\right)x^2dx$
15. $\int \sec\left(x^2+1\right)xdx$

Use logarithmic differentiation to find the derivatives of each function.

1. $f(x) = x^{\left(x^x\right)}$
2. $f(x) = (x^x)^x$

   Find the integrals
3. $\int \sec(\cos(x))\tan(\cos(x))\sin(x)\,dx$
4. $\int \frac{1}{\sec(x)+5}\sec(x)\tan(x)\,dx$
5. $\int e^{\tan(x)}\sec^2(x)\,dx$

   Solve the differential equations. Find all solutions.
6. $y' + 3y = 0$
7. $4y' - 8y = 0$
8. $y' + 4x^3y = 0$

Solve the differential equations. Find the solution satisfying the initial condition.

9. $y' - 2y = 0$, initial condition $y(0) = 3$.
10. $y' + 5y = 0$, initial condition $y(1) = 2$.
11. A radioactive substance decays at a rate $y = 5e^{-2t}$ grams per year. What is the half life of the substance?
12. Let y represent the amount of bacteria present in an experiment at time x. If there are 20 bacteria at the start of the experiment, and if the bacteria increase at a rate $y' = 3y$, then how many bacteria will be present 24 hours later?
13. If a radioactive element decays in such a way that $y' = -y$ where y represents the amount present in grams at time t in hours, then what is the half life of the substance?