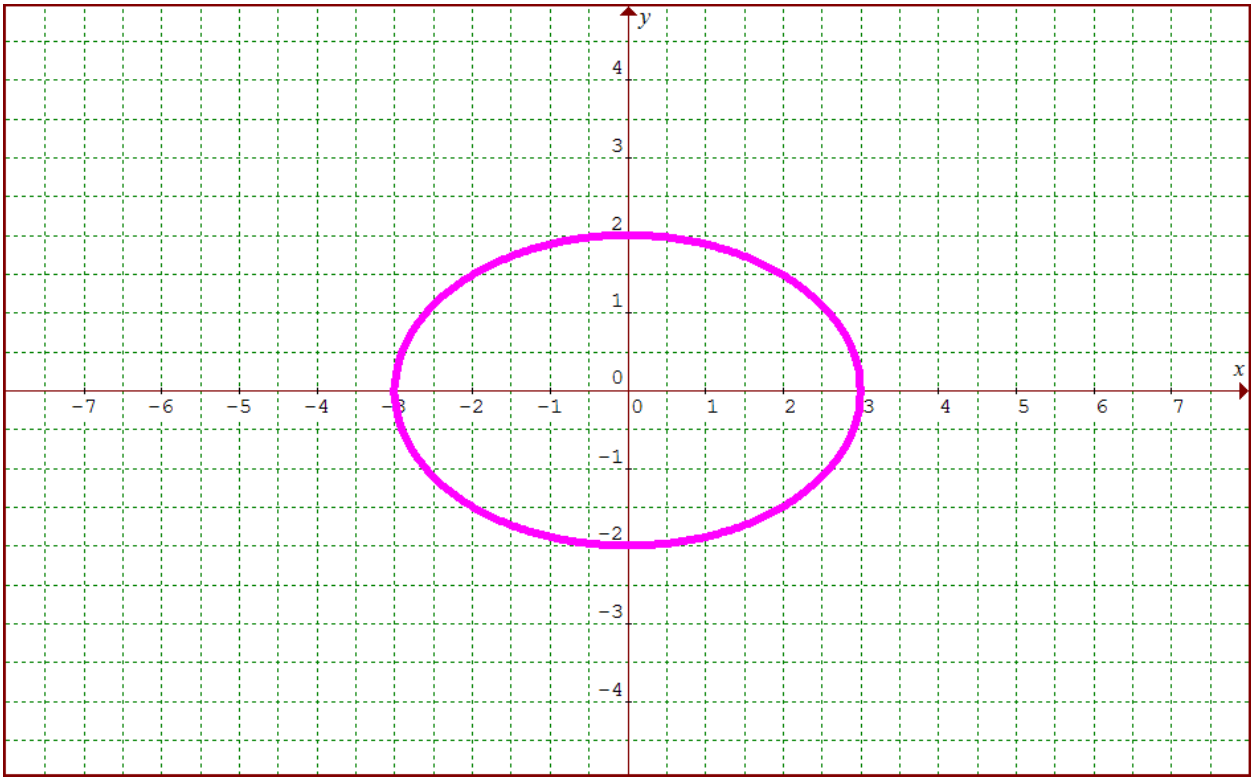
Surface area of ellipsoids of revolution.

A supplement for 172, Calculus 2 course.

We consider an ellipse given by its standard equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

We assume that $a > b$. The graph below shows such an ellipse for $a = 3, b = 2$.



Part 1. We will compute the surface area of the ellipsoid of revolution resulting when the upper half of the ellipse is revolved about the *x*-axis. The parametric equations of that part of the ellipse can be written as

$$x = a\cos t,\ y = b\sin t,\ 0 \le t \le \pi.$$

The surface area is given by the formula

$$S_x = 2\pi \int_{t_1}^{t_2} y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt,$$

which in our case becomes

$$S_x = 2\pi b \int_0^{\pi} \sin t \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\,dt.$$

By symmetry we can write

$$S_x = 4\pi b \int_0^{\pi/2} \sin t \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\,dt =$$

$$= 4\pi b \int_0^{\pi/2} \sin t \sqrt{a^2 - \left(a^2 - b^2\right)\cos^2 t}\,dt =$$

$$= 4\pi ab \int_0^{1} \sin t \sqrt{1 - e^2 \cos^2 t}\,dt,$$

where $e = \sqrt{1 - \frac{b^2}{a^2}}$ is the eccentricity of the ellipse (do not confuse it with the base of natural logarithms).

After performing the substitution $u = e\cos t$ we get

$$S_x = \frac{4\pi ab}{e} \int_0^{e} \sqrt{1-u^2}\,du = \frac{2\pi ab}{e}\left[u\sqrt{1-u^2} + \arcsin u\right]\Big|_0^e =$$

$$= 2\pi ab\left(\sqrt{1-e^2} + \frac{\arcsin e}{e}\right).$$

Notice that if $b \to a$ then $e \to 0$ and because $\lim\limits_{e \to 0} \frac{\arcsin e}{e} = 1$ the expression for S_y approaches $4\pi a^2$ - the well known formula for the surface area of a sphere of radius *a*.

Part 2. Now we will consider the case when the right half of the ellipse is revolved about the *y*-axis. In this case we use the formula

$$S_y = 2\pi \int_{t_1}^{t_2} x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$$

$$= 2\pi a \int_{-\pi/2}^{\pi/2} \cos t \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt =$$

$$= 4\pi a \int_{0}^{\pi/2} \cos t \sqrt{\left(a^2 - b^2\right) \sin^2 t + b^2}\, dt =$$

$$= 4\pi a^2 \int_{0}^{\pi/2} \cos t \sqrt{e^2 \sin^2 t + (b/a)^2}\, dt.$$

After performing the substitution $u = e \sin t$ we get

$$S_y = \frac{4\pi a^2}{e} \int_0^e \sqrt{u^2 + (b/a)^2}\, du =$$

$$= \frac{2\pi a^2}{e} \left[u\sqrt{u^2 + (b/a)^2} + (b/a)^2 \ln\left(u + \sqrt{u^2 + (b/a)^2} \right) \right] \Bigg|_0^e$$

Plugging in the limits and taking into consideration that $e^2 + (b/a)^2 = 1$ we obtain

$$S_y = 2\pi a^2 + \frac{2\pi b^2}{e} \ln(1 + e) - \frac{\pi b^2}{e} \ln(1 - e^2).$$

And finally, using the laws of logarithms:

$$S_y = 2\pi a^2 + \frac{\pi b^2}{e}\ln\left(\frac{1+e}{1-e}\right).$$

It is easy to see (using for example the L'Hospital's rule) that

$$\lim_{e\to 0}\frac{1}{e}\ln\left(\frac{1+e}{1-e}\right) = 2$$

And therefore in the limit case we get again the formula for the surface area of a sphere.