1. The curve $y=\frac{\ln x}{x}, x\geq 1$ is revolved about the $x$-axis. Is the volume of the resulting solid finite or infinite?

**Solution.** We apply the disks method and see that the question is about the convergence or divergence of the following improper integral. $\pi\int_1^\infty \frac{\ln^2 x}{x^2}dx$ **.** We will prove that the integral converges and find its value using integration by parts. Let $u=\ln^2 x, dv=\frac{1}{x^2}dx$. Then $du=2\ln x\frac{1}{x}dx,\ v=-\frac{1}{x}$, and $\pi\int_1^\infty \frac{\ln^2 x}{x^2}dx=-\pi\frac{\ln^2 x}{x}\Big|_1^\infty+2\pi\int_1^\infty\frac{\ln x}{x^2}dx$. The first term in this expression is equal to 0. Indeed, at the lower limit we have $\ln 1=0$ and at the upper limit we have to look at $\lim_{x\to\infty}\frac{\ln^2 x}{x}$. The last limit is an indeterminate form $\frac{\infty}{\infty}$ and we can apply the L'Hospital's rule. $\lim_{x\to\infty}\frac{\ln^2 x}{x}=\lim_{x\to\infty}\frac{2\ln x\cdot\frac{1}{x}}{1}=\lim_{x\to\infty}\frac{2\ln x}{x}=\lim_{x\to\infty}\frac{2/x}{1}=0$. Therefore $\pi\int_1^\infty\frac{\ln^2 x}{x^2}dx=2\pi\int_1^\infty\frac{\ln x}{x^2}dx$. We apply integration by parts once again; this time we take $u=\ln x, dv=\frac{1}{x^2}dx$. Then $du=\frac{1}{x}dx, v=-\frac{1}{x}$, and $2\pi\int_1^\infty\frac{\ln x}{x^2}dx=-2\pi\frac{\ln x}{x}\Big|_1^\infty+2\pi\int_1^\infty\frac{1}{x^2}dx$. Like in the previous expression we see that the first term is 0 and it remains to evaluate $2\pi\int_1^\infty\frac{1}{x^2}dx=-2\pi\frac{1}{x}\Big|_1^\infty=2\pi$. So, the volume is finite and its value is $2\pi$.

2. The same question about the area of the surface of revolution.

**Solution.** The area of the surface of revolution can be expressed as the following integral. $2\pi\int_1^\infty y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$. We will prove, using the comparison test that the integral diverges to $\infty$. To this end notice the simple inequality $1+\left(\frac{dy}{dx}\right)^2\geq 1$ whence $\sqrt{1+\left(\frac{dy}{dx}\right)^2}\geq\sqrt{1}=1$ and $2\pi\int_1^\infty y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx\geq 2\pi\int_1^\infty y(x)dx=2\pi\int_1^\infty\frac{\ln x}{x}dx$. Let $u=\ln x$ then

$du = \frac{1}{x}dx$ and $2\pi\int_1^\infty \frac{\ln x}{x}dx = 2\pi\int_0^\infty u\,du = \pi u^2\Big|_0^\infty = \infty$. Therefore $2\pi\int_1^\infty y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \infty$ and the area of the surface of revolution is infinite.

3. Prove that the integral $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}dx$ converges and find its value.

**Solution.**

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}dx = \int_{-\infty}^{\infty} \frac{1}{(x+1)^2+1}dx = \int_{-\infty}^{\infty} \frac{1}{u^2+1}du \text{ (where } u = x+1) =$$

$$= \arctan u\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

(Recall that $\lim_{u\to\pm\infty} \arctan u = \pm\frac{\pi}{2}$)

4. Use the Ratio test to find out whether the series $\sum_{n=0}^{\infty} \frac{n!e^n}{(2n)!}$ converges or diverges.

**Solution.** Let $a_n = \frac{n!e^n}{(2n)!}$ be the nth term of our series. Then the ratio $\frac{a_{n+1}}{a_n}$ can be simplified as

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!e^{n+1}}{[2(n+1)]!} \div \frac{n!e^n}{(2n)!} = \frac{(n+1)!}{n!}\cdot\frac{(2n)!}{[2(n+1)]!}\cdot\frac{e^{n+1}}{e^n} =$$

$$= (n+1)\frac{1}{(2n+1)(2n+2)}e = \frac{e}{2(2n+1)} \underset{n\to\infty}{\to} 0.$$

By the ratio test the series converges.

5. Find the radius of convergence of the power series $\sum_{n=2}^{\infty} \frac{x^n}{n\ln n}$.

**Solution.** We will apply the ratio test to find the radius of convergence. Let $a_n = \frac{x^n}{n\ln n}$. Then $\frac{|a_{n+1}|}{|a_n|} = \frac{n\ln n}{(n+1)\ln(n+1)}|x|$. Next notice that $\lim_{n\to\infty}\frac{n\ln n}{(n+1)\ln(n+1)} = 1$.

Indeed, $\lim\limits_{n\to\infty}\frac{n}{n+1}=1$. To find the limit $\lim\limits_{n\to\infty}\frac{\ln n}{\ln(n+1)}$ we will apply the L'Hospital's rule, $\lim\limits_{x\to\infty}\frac{\ln(u+1)}{\ln u}=\lim\limits_{x\to\infty}\frac{1/(u+1)}{1/u}=\lim\limits_{x\to\infty}\frac{u}{u+1}=1$. Therefore $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|x|$. By the ratio test the series converges absolutely if $|x|<1$ and diverges if $|x|>1$.

6. Find the value of $\arcsin(0.499)$ with accuracy $10^{-7}$ using an appropriate Taylor series.

**Solution.** A "brute force" solution would be to use the Maclaurin series for $\arcsin$ at the point 0.499. The solution suggested below is much more effective.

As often in such problems we will try to state the problem in such a way that we can use Maclaurin series of known pattern. To this end notice that $\arcsin 0.5=\frac{\pi}{6}$. Next we will use the formula $\arcsin a-\arcsin b=\arcsin\left(a\sqrt{1-b^2}-b\sqrt{1-a^2}\right)$. We can easily prove this formula if we notice that

$\sin(\arcsin a-\arcsin b)=\sin(\arcsin a)\cos(\arcsin b)-\cos(\arcsin a)\sin(\arcsin b)$

and recall that $\sin(\arcsin x)=x$ and $\cos(\arcsin x)=\sqrt{1-x^2}$. In our case we get

$\frac{\pi}{6}-\arcsin(0.499)=\arcsin\left(\frac{1}{2}\sqrt{1-0.499^2}-0.499\sqrt{1-\left(\frac{1}{2}\right)^2}\right)\approx\arcsin 0.0011543159$. Now we will use the Maclaurin series $\arcsin x=x+\sum\limits_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdot K\cdot(2n-1)}{2^n n!(2n+1)}x^{2n+1}$.

If we compute the sum of the first two terms of the series above for $x=0.0011543159$ we will get

$\arcsin(.499)\approx\frac{\pi}{6}-0.0011543159-\frac{1}{2\cdot3}(.0011543159)^3\approx .5224444594$. All the digits are correct.

**Remark.** You will not be required to do it on the test but it is not difficult to estimate the error of our approximation. Indeed,

$$|Error|\le\sum_{n=2}^{\infty}\frac{1\cdot3\cdot\ldots\cdot(2n-1)}{2^n n!(2n+1)}(0.0011543159)^{2n+1}\le\sum_{n=2}^{\infty}(0.0011543159)^{2n+1}=$$

$$=(0.0011543159)^5\sum_{n=0}^{\infty}[(0.0011543159)^2]^n=\frac{(0.0011543159)^5}{1-(0.0011543159)^2}\approx .2049386957\times10^{-14}$$

7. Estimate the value of $\int_0^{\pi/4} \frac{\sin x}{x} dx$ with accuracy 0.000001 using an appropriate Taylor series.

**Solution.** $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Therefore $\frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$ and

$$\int_0^{\pi/4} \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^{\pi/4} x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n+1}}{2n+1}\Big|_0^{\pi/4} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)(2n+1)!}.$$

Next we have to decide how many terms of the last series we want to use for approximation. The series is an alternating one so when we stop the error we make is not greater than the absolute value of the next term. We can compute that for $n = 5$ the absolute value of the fifth term is $.6543351347 \times 10^{-6}$ which is enough for the accuracy we need. Therefore we use the approximation

$$\int_0^{\pi/4} \frac{\sin x}{x} dx \approx \frac{\pi}{4} - \frac{\pi^3}{1152} + \frac{\pi^5}{614400} - \frac{\pi^7}{578027520} \approx .758976.$$

8. Find the solution of differential equation $\frac{dy}{dx} = \frac{\sinh x}{x}$, $y(0) = 1$, in the form of a Taylor series.

**Solution.** $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$. Thus $\frac{\sinh x}{x} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}$ and

$$y(x) = 1 + \int_0^x \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n+1)!} = 1 + \sum_{n=0}^{\infty} \int_0^x \frac{t^{2n}}{(2n+1)!} = 1 + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)(2n+1)!}.$$

9. Find the Maclaurin polynomial of degree 4 for the function $\sec x$ using the techniques of division of Taylor series. Use this polynomial to approximate $\sec 5^\circ$.

**Solution.** Because $\sec 0 = 1$ and $\sec x$ is an even function its Maclaurin polynomial of degree 4 is of the form $1 + a_2 x^2 + a_4 x^4$. As $\sec x \cos x = 1$ and Maclaurin polynomial of degree 4 for $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ we have

$$(1 + a_2 x^2 + a_4 x^4)(1 - \frac{x^2}{2!} + \frac{x^4}{4!}) \sim 1.$$

From it we obtain, $a_2 - \frac{1}{2} = 0$, $a_2 = \frac{1}{2}$ and $a_4 - \frac{a_2}{2} + \frac{1}{4!} = 0$, $a_4 = \frac{1}{4} - \frac{1}{24} = \frac{5}{24}$ .

Next, $\sec 5^\circ = \sec \frac{\pi}{36} \approx 1 + \frac{1}{2} \frac{\pi^2}{36^2} + \frac{5}{24} \frac{\pi^4}{36^4} \approx 1.003819800$. For comparison, a direct approximation with a calculator provides 1.003819838.

10. Use binomial Maclaurin series to write and exact expression for the arc length of the curve $y = x^3, 0 \le x \le 1/2$ in the form of a Maclaurin series. Use the obtained series to estimate the arc length with the accuracy of $10^{-6}$.

**Solution.**

$$L = \int_0^{0.5} \sqrt{1+9x^4}\, dx = \int_0^{0.5} \left( 1 + \sum_{n=1}^{\infty} \frac{1/2(1/2-1)...(1/2-n+1)}{n!} 9^n x^{4n} \right) =$$

$$= 0.5 + \sum_{n=1}^{\infty} \frac{1/2(1/2-1)...(1/2-n+1)}{n!} 9^n \frac{0.5^{4n+1}}{4n+1}.$$

This is an alternating series and we can check that

$$\left| \frac{1/2(1/2-1)(1/2-2)(1/2-3)(1/2-4)(1/2-5)}{6! \cdot 25} 9^6 (1/2)^{25} \right| < 2.2 \cdot 10^{-7}$$

Therefore for the accuracy we need it is enough to take

$$L \approx 0.5 + \sum_{n=1}^{5} \frac{1/2(1/2-1)...(1/2-n+1)}{n!} 9^n \frac{0.5^{4n+1}}{4n+1} \approx 0.526268$$