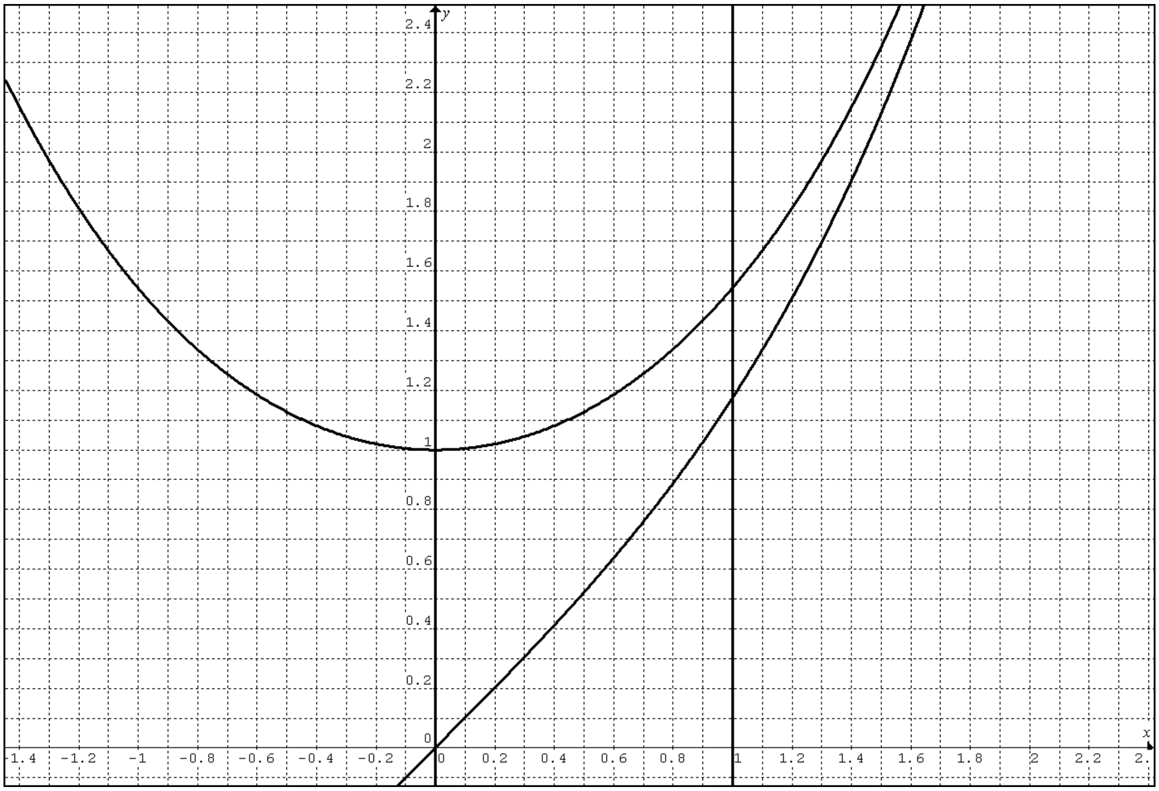
Calculus 2 172 Review 2

Consider the region between curves $y = \cosh x$, $y = \sinh x$, $x = 0$, $x = 1$.

1. Find the area of the region.

Solution. Recall that $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$, whence $\sinh x \le \cosh x$.



Therefore the area is given by the integral

$$A = \int_0^1 (\cosh x - \sinh x)dx = (\sinh x - \cosh x)\Big|_0^1 = \sinh 1 - \cosh 1 - \sinh 0 + \cosh 0 =$$

$$= \frac{e - e^{-1}}{2} - \frac{e + e^{-1}}{2} - 0 + 1 = 1 - \frac{1}{e} = \frac{e-1}{e} \approx 0.63.$$

2. Find the volume of the solid of revolution when the region is revolved about the x-axis.

Solution. We will compute the volume using the washers' method and the identity $\cosh^2 x - \sinh^2 x = 1$.

$$V_x = \pi \int_0^1 (\cosh^2 x - \sinh^2 x)dx = \pi \int_0^1 1 dx = \pi.$$

3. Find the volume of the solid of revolution when the region is revolved about the y-axis.

Solution. We will use the cylindrical shells' method.

$V = 2\pi\int_0^1 x(\cosh x - \sinh x)dx$. We will integrate by parts taking $u = x$ and $dv = (\cosh x - \sinh x)dx$. Then $du = dx$ and $v = \sinh x - \cosh x$ whence

$$V = 2\pi x(\sinh x - \cosh x)\Big|_0^1 - 2\pi\int_0^1 (\sinh x - \cosh x)dx =$$

$$= 2\pi(\sinh 1 - \cosh 1) + 2\pi\int_0^1 (\cosh x - \sinh x)dx.$$

The last integral we have already computed in Problem 1 and therefore

$$V_y = -2\pi\frac{1}{e} + 2\pi\frac{e-1}{e} = 2\pi\frac{e-2}{e} \approx 1.66$$

4. Find the coordinates of the geometric center of the region.

Solution. By the Pappus's Centroid Theorem we have

$$x_c = \frac{V_y}{2\pi A} = \frac{e-2}{e-1} \approx 0.42, \text{ and}$$

$$y_c = \frac{V_x}{2\pi A} = \frac{e}{2(e-1)} \approx 0.80.$$

5. Find the area of the surface of revolution when the region is revolved about the x-axis.

Solution. The area in question consists of four parts.

(a) The area of the surface generated by the rotation of the segment of the y-axis from 0 to 1 about the x-axis. This surface is a disk with the radius 1 and its area is equal to π.

(b) The area of the surface generated by the rotation of the segment of the vertical line $x = 1$ from $\sinh 1$ to $\cosh 1$ about the x-axis. This surface is an annulus with the radii $\sinh 1$ and $\cosh 1$ and its area is equal to $\pi(\cosh^2 1 - \sinh^2 1) = \pi$.

(c) The area of the surface generated by the rotation of the curve $y = \cosh x$ about the x-axis. This area is given by the formula

$$S_1 = 2\pi\int_0^1 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^1 \cosh x\sqrt{1+\sinh^2 x}\,dx =$$

$$= 2\pi\int_0^1 \cosh x\sqrt{\cosh^2 x}\,dx = 2\pi\int_0^1 \cosh^2 x\,dx = \pi\int_0^1 (\cosh 2x+1)\,dx =$$

$$= \frac{\pi}{2}\sinh 2x\Big|_0^1 + \pi = \pi\left(\frac{e^2-e^{-2}}{4}+1\right) = \pi\frac{e^4+4e^2-1}{4e^2}.$$

(d) The area of the surface generated by the rotation of the curve $y=\sinh x$ about the x-axis. This area is given by the formula

$S_2 = 2\pi\int_0^1 \sinh x\sqrt{1+\cosh^2 x}\,dx$. Let $u=\cosh x$, then $du=\sinh x\,dx$ and therefore

$$S_2 = 2\pi\int_1^{\cosh 1}\sqrt{1+u^2}\,du \overset{formula\,21}{=} 2\pi\left(\frac{u}{2}\sqrt{1+u^2}+\frac{1}{2}\ln(u+\sqrt{1+u^2}\right)\Bigg|_1^{\cosh 1} =$$

$$= \pi\left[\cosh 1\sqrt{1+\cosh^2 1}-\sqrt{2}-\ln(1+\sqrt{2})+\ln(\cosh 1+\sqrt{1+\cosh^2 1})\right].$$

Adding all four areas from (a) – (d) we get that the surface area in question is approximately 20.65

6. Find the area of the surface of revolution when the parametric curve $x=\cosh t,\ y=\sinh t,\ 0\le t\le 1$, is revolved about the x-axis and about the y-axis.

Solution. (a) To find the surface area in case when the curve is revolved about the x-axis we use the formula

$$S_x = 2\pi\int_0^1 y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = 2\pi\int_0^1 \sinh t\sqrt{\sinh^2 t+\cosh^2 t}\,dt = 2\pi\int_0^1 \sinh t\sqrt{2\cosh^2 t-1}\,dt.$$

Taking $u=\cosh t$ we have $du=\sinh t\,dt$ and $S_x = 2\pi\int_1^{\cosh 1}\sqrt{2u^2-1}\,du$. Let $v=\sqrt{2}u$ then

$$S_x = \sqrt{2}\pi\int_{\sqrt{2}}^{\sqrt{2}\cosh 1}\sqrt{v^2-1}\,dv \overset{\text{formula 39}}{=} \frac{\sqrt{2}\pi}{2}\left(v\sqrt{v^2-1}-\ln(v+\sqrt{v^2-1}\right)\Big|_{\sqrt{2}}^{\sqrt{2}\cosh 1} =$$

$$= \pi(\cosh 1\sqrt{2\cosh^2 1-1}-1)+\frac{\sqrt{2}\pi}{2}\ln\frac{\sqrt{2}+1}{\sqrt{2}\cosh 1+\sqrt{2\cosh^2 1-1}} \approx 5.07$$

(b) If the curve is revolved about the y-axis we have

$$S_y = 2\pi\int_0^1 x(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = 2\pi\int_0^1 \cosh t\sqrt{\sinh^2 t+\cosh^2 t}\,dt = 2\pi\int_0^1 \cosh t\sqrt{2\sinh^2 t+1}\,dt.$$

Taking $u=\sinh t$ we have $du=\cosh t\,dt$ and $S_y = 2\pi\int_0^{\sinh 1}\sqrt{2u^2+1}\,du$. Let $v=\sqrt{2}u$ then

$$S_y=\sqrt{2}\pi\int_0^{\sqrt{2}\sinh 1}\sqrt{v^2+1}\,dv \overset{\text{formula 21}}{=} \frac{\sqrt{2}\pi}{2}\left[v\sqrt{v^2+1}+\ln(v+\sqrt{v^2+1})\right]\Big|_0^{\sqrt{2}\sinh 1} =$$

$$=\pi\sinh 1\sqrt{2\sinh^2 1+1}+\frac{\sqrt{2}\pi}{2}\ln(\sqrt{2}\sinh 1+\sqrt{2\sinh^2 1+1})\approx 10.00$$

7. Find the arc length of the curve $y=3^x,\quad 0\le x\le 1.$

Solution. According to the formula for arc length we have

$$L=\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^1\sqrt{1+(3^x\ln 3)^2}\,dx=\int_0^1\sqrt{1+9^x\ln^2 3}\,dx.$$

Let $u=\sqrt{1+9^x\ln^2 3}$. Then $u^2=1+9^x\ln^2 3$ whence

$2u\,du=9^x(\ln 9)(\ln^2 3)dx=2(\ln^3 3)9^x\,dx$ and, because $9^x=\dfrac{u^2-1}{\ln^2 3}$, we

have $dx=\dfrac{du}{(\ln^3 3)9^x}=\dfrac{du}{(\ln 3)(u^2-1)}$. Therefore

$$L=\frac{1}{\ln 3}\int_{\sqrt{1+\ln^2 3}}^{\sqrt{1+9\ln^2 3}}\frac{u^2\,du}{u^2-1}=\frac{1}{\ln 3}\int_{\sqrt{1+\ln^2 3}}^{\sqrt{1+9\ln^2 3}}\left(1+\frac{1}{2}\frac{1}{u-1}-\frac{1}{2}\frac{1}{u+1}\right)du=$$

$$=\frac{1}{\ln 3}\left[\sqrt{1+9\ln^2 3}-\sqrt{1+\ln^2 3}+\frac{1}{2}\ln\frac{\sqrt{1+9\ln^2 3}-1}{\sqrt{1+9\ln^2 3}+1}-\frac{1}{2}\ln\frac{\sqrt{1+\ln^2 3}-1}{\sqrt{1+\ln^2 3}+1}\right]\approx 2.25.$$

8. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

Solution. According to the Hooke's law the force required to hold the spring stretched x meters beyond its natural length is $f(x)=kx$. When the spring is stretched from 12 cm to 15 cm, the amount stretched is 3 cm = 0.03 m. This means that $f(0.03)=30$, so $0.03k=30, k=\dfrac{30}{0.03}=1000$. Thus $f(x)=1000x$ and the work done in stretching the spring from 12 cm to 20 cm is

$$W = \int_0^{0.08} 1000x\,dx = 500x^2 \big|_0^{0.08} = 3.2J$$

9. A tank full of water has the shape of a paraboloid of revolution; that is, its shape is obtained by rotating a parabola about a vertical axis. If its height is 4 ft and the radius at the top is 4 ft, find the work required to pump the water out of the tank.
Solution. The equation of the revolved parabola is $y = ax^2$. Because $f(4) = 4$ we have $y = \frac{1}{4}x^2$ and $x = 2\sqrt{y}$. Therefore the area of cross-section at height y is

$A(y) = 4\pi y$. Considering a thin slice of water from the height of y to $y + \Delta y$ we see that the work required to raise the water in this slice to the top of the tank is approximately
$\Delta W = 62.5 \times 4\pi y(4 - y)\Delta y$. Finally,

$$W = 250\pi \int_0^4 y(4 - y)\,dy = \frac{8000\pi}{3} \text{ lb-ft.}$$