Math 172

Review for the final exam.

1. Find the integrals. (8 points each)
   
   (a) \( \int x \arctan x \, dx \);
   
   Solution. We perform integration by parts taking \( u = \arctan x \) and \( dv = x \, dx \). Then
   
   \[ du = \frac{1}{1 + x^2} \, dx \quad \text{and} \quad v = \frac{x^2}{2}. \]
   
   Therefore
   
   \[ \int x \arctan x \, dx = \frac{x^2}{2} \arctan x - \int \frac{x^2}{2(1 + x^2)} \, dx = \]
   
   \[ = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \left( 1 - \frac{1}{1 + x^2} \right) = \frac{x^2}{2} \arctan x - \frac{x}{2} + \frac{1}{2} \arctan x + C = \]
   
   \[ = \frac{x^2}{2} \arctan x - \frac{x}{2} + C. \]
   
   (b) \( \int \frac{1}{t^2 + 2t + 2} \, dt \);
   
   Solution. By completing the square we get
   
   \( \int \frac{1}{t^2 + 2t + 2} \, dt = \int \frac{1}{(t + 1)^2 + 1} \, dt. \)
   
   Let \( u = t + 1 \) then \( du = dt \) and
   
   \[ \int \frac{1}{(t + 1)^2 + 1} \, dt = \int \frac{1}{u^2 + 1} \, du = \arctan u + C = \arctan(t + 1) + C. \]
(c) \[ \int \cos^3 x \sin^3 x \, dx; \]

**Solution.** First we write this integral in the form

\[ \int \cos^2 x \sin^3 x \cos x \, dx = \int (1 - \sin^2 x) \sin^3 x \cos x \, dx \]

and then make the substitution \( u = \sin x \), whence \( du = \cos x \, dx \).

Thus our integral becomes

\[ \int (1 - \sin^2 x) \sin^3 x \cos x \, dx = \int (1 - u^2) u^\frac{3}{2} \, du = \]

\[ = \int u^\frac{3}{2} - u^\frac{5}{2} \, du = \frac{2}{3} u^\frac{5}{2} - \frac{2}{7} u^\frac{7}{2} + C = \]

\[ = \frac{2}{3} \sin^\frac{5}{2} x - \frac{2}{7} \sin^\frac{7}{2} x + C. \]

(d) \[ \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \, dx. \]

**Solution.** Probably the simplest way to solve the problem is to notice that

\[ \frac{d}{dx} (e^{2x} - e^{-2x}) = 2e^{2x} - (-2)e^{-2x} = 2(e^{2x} + e^{-2x}). \]

Thus, the numerator of the fraction we integrate equals to one half of the derivative of the denominator whence

\[ \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \, dx = \frac{1}{2} \ln(e^{2x} + e^{-2x}) + C. \]

**Remark** The answer also can be written in the form

\[ \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \, dx = \frac{1}{2} \cosh 2x + C. \]
2. Find the value of the improper integral (10 points)

$$\int_{2}^{\infty} \frac{1}{x \sqrt{x^2 - 1}} \, dx.$$ 

**Solution.** We perform the substitution $x = \sec t$. Then $dx = \sec t \tan t \, dt$ and $\sqrt{x^2 - 1} = \sqrt{\sec^2 t - 1} = \tan t$. We have to find out how the limits of integration will change under the substitution. If $x = 2$ then $\sec t = 2$ whence $\cos t = \frac{1}{2}$ and $t = \arccos \frac{1}{2} = \frac{\pi}{3}$. If $x \to \infty$ then $\cos t = \frac{1}{\sec t} = \frac{1}{x}$ whence $t = \arccos \frac{1}{x} \to \arccos 0 = \frac{\pi}{2}$. Therefore after the substitution we get a proper integral

$$\int_{\frac{\pi}{2}}^{\infty} \frac{1}{\sec t \tan t} \, dt = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

3. Find the volume of the solid generated when the region bounded by $y = 2 - x, y = \sqrt{x}$ and $x = 0$ is revolved about $x$-axis. (10 points)

**Solution.** The graph of the region is shown below. (On the test you do not have to provide the graph) To find the point of intersection of the curves $y = 2 - x$ and $y = \sqrt{x}$ we have to solve the equation $2 - x = \sqrt{x}$. After we square both parts we get $4 - 4x + x^2 = x$ or $x^2 - 5x + 4 = (x - 1)(x - 4) = 0$. Only $x = 1$ is the solution of our original equation.

Next we apply the disks’ method to find the volume.

$$V_x = \pi \int_{0}^{1} [(2 - x)^2 - (\sqrt{x})^2] \, dx = \pi \int_{0}^{1} (x^2 - 5x + 4) \, dx =$$

$$= \pi \left( \frac{x^3}{3} - \frac{5x^2}{2} + 4x \right) \bigg|_{0}^{1} = \pi \left( \frac{1}{3} - \frac{5}{2} + 4 \right) = \frac{11}{6} \pi.$$ 

4. Find the volume of the solid generated when the region bounded by $y = 2 - x, y = \sqrt{x}$ and $x = 0$ is revolved about $y$-axis. (10 points)
Solution. We apply the cylindrical shells’ method.

\[ V_y = 2\pi \int_0^1 x[(2 - x) - \sqrt{x}] \, dx = 2\pi \int_0^1 (2x - x^2 - x^3) \, dx = \]

\[ = 2\pi \left( x^2 - \frac{x^3}{3} - \frac{2}{5}x^5 \right) \bigg|_0^1 = \frac{8\pi}{15}. \]
Graphmatica 2.0e © 2005 kSoft, Inc. - region.gr

Equations on screen:
1. $y = \sqrt{x}$
2. $y = 2 - x$
3. $x = 0$

Data Plots:
- Data plot 1

Centroid
5. Find the coordinates of the centroid of the region described in the previous problem. (10 points) **Solution.** First we have to find the area of the region.

\[
A = \int_0^1 (2 - x - \sqrt{x}) \, dx = \left. \left(2x - \frac{x^2}{2} - \frac{2}{3}x^{3/2}\right)\right|_0^1 = 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}.
\]

Now we can compute the coordinates of centroid.

\[
x_c = \frac{V_y}{2\pi A} = \frac{8\pi}{15} \div \frac{5\pi}{3} = \frac{8}{25} = 0.32,
\]

\[
y_c = \frac{V_x}{2\pi A} = \frac{11\pi}{6} \div \frac{5\pi}{3} = \frac{11}{10} = 1.1.
\]

6. Determine whether the series absolutely converges, conditionally converges, or diverges. (5 points each)

(a) \[
\sum_{k=2}^{\infty} \frac{1}{\sqrt{k^6 - 3k}};
\]

**Solution.** We apply the limit comparison test. By keeping only the leading terms in the numerator and the denominator we get the following series

\[
\sum_{k=2}^{\infty} \frac{1}{\sqrt{k^6}} = \sum_{k=2}^{\infty} \frac{1}{k^2}.
\]

The last series converges because the exponent of \(k\) in the denominator is greater than 1. By limit comparison test our original series also converges.

(b) \[
\sum_{k=1}^{\infty} (-1)^k \frac{k}{k^2 + 2};
\]

**Solution.** This is an alternating series. Indeed,

- the sign of the terms alternates;
• \[ \lim_{k \to \infty} \frac{k}{k^2 + 2} = 0; \]

• The absolute values of the terms of the series are decreasing. To prove it it is enough to prove that the function \( g(x) = \frac{x}{x^2 + 2} \) is decreasing. Indeed, by the quotient rule
\[
\frac{dg}{dx} = \frac{x^2 + 2 - x(2x)}{(x^2 + 2)^2} = \frac{2 - x^2}{(x^2 + 2)^2}.
\]
The last expression is negative if \( x > \sqrt{2} \) and therefore the absolute values of the terms of the series are decreasing starting from \( k = 2 \).

So the series converges as an alternating series. To see whether it converges absolutely or conditionally let us look at the series of absolute values,

\[ \sum_{k=1}^{\infty} \frac{k}{k^2 + 2}. \]

The limit comparison test tells us that this series converges or diverges at the same time as the series

\[ \sum_{k=1}^{\infty} \frac{k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k}. \]

But the last series diverges and therefore the series of absolute values diverges and our original series converges conditionally.

(c)

\[ \sum_{k=1}^{\infty} \frac{(n!)^2}{(2n)!}; \]

Solution. We will apply the ratio test.

\[
\frac{a_{n+1}}{a_n} = \frac{(n + 1)!^2}{[2(n+1)]!} \div \frac{(n!)^2}{(2n)!} = \left( \frac{(n + 1)!}{n!} \right)^2 \frac{(2n)!}{(2n + 2)!} = \frac{(n + 1)^2}{(2n + 1)(2n + 2)} = \frac{n + 1}{2(2n + 1)} \to \frac{1}{4} < 1.
\]

By ratio test the series converges.
7. Find the function to which the series
\[ \sum_{k=1}^{\infty} (k - 1)x^{k+1} \]
converges. (10 points)

**Solution.** The solution is based on two formulas. The first one is the infinite geometric progression.
\[ \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad -1 < x < 1. \]  \((*)\)

The second we obtain if we differentiate both parts of \((*)\).
\[ \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}, \quad -1 < x < 1. \] \((***)\)

Now we can compute the sum in our problem.
\[ \sum_{k=1}^{\infty} (k - 1)x^{k+1} = \sum_{k=1}^{\infty} kx^{k+1} - \sum_{k=1}^{\infty} x^{k+1} = \]
\[ = x^2 \sum_{k=1}^{\infty} kx^{k-1} - x^2 \sum_{k=1}^{\infty} x^{k-1} = \]
(where \(i = k - 1\))
\[ = x^2 \sum_{k=1}^{\infty} kx^{k-1} - x^2 \sum_{i=0}^{\infty} x^i = \]
(by \((*)\) and \((***)\))
\[ = \frac{x^2}{(1-x)^2} - \frac{x^2}{1-x} = \frac{x^2 - x^2(1-x)}{(1-x)^2} = \frac{x^3}{(1-x)^2}. \]

Of course, the series converges and the formula is correct only on the interval \(-1 < x < 1.\)

8. Find the Taylor series about \(x = a\) for the given function; express your answer in sigma notation \((\Sigma)\); then find its radius of convergence and the interval of convergence. (10 points each)
(a) \[ f(x) = \frac{1}{2 + x}, \]

at 0.

**Solution.** Plugging in \(-x\) instead of \(x\) into formula (\(*\)) we get

\[ \frac{1}{1 + x} = \sum_{k=0}^{\infty} (-1)^k x^k, \quad -1 < x < 1. \quad (\star \star \star) \]

Therefore

\[ \frac{1}{2 + x} = \frac{1}{2(x + \frac{1}{2})} = \frac{1}{2} \frac{1}{1 + \frac{x}{2}} = \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \left( \frac{x}{2} \right)^k = \]

\[ = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{2^{k+1}}, \quad -1 < \frac{x}{2} < 1, \quad \text{or} \quad -2 < x < 2. \]

The radius of convergence of the Maclaurin series above is 2 and the interval of convergence is \((-2, 2)\).

(b) \[ f(x) = \ln x, \]

at 2.

**Solution.** It would be not very difficult to use the formula for Taylor series of function \(f\) at point \(a\)

\[ \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k, \quad \text{where} \quad f^{(0)} = f, \]

but it is easier to reduce the problem to one of our standard Maclaurin series. Recall that integrating both parts in (\(\star \star \star \)) we get

\[ \ln (1 + x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}, \quad -1 < x < 1. \quad (\star \star \star \star) \]

Let \(u = x - 2\) then \(x = u + 2\) and, using (\(\star \star \star \)) we obtain

\[ \ln x = \ln (u + 2) = \ln [2(1 + \frac{u}{2})] = \ln 2 + \ln (1 + \frac{u}{2}) = \ln 2 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(u/2)^k}{k} = \]
\[ \sum_{k=1}^{\infty} (-1)^{k+1} \frac{u^k}{2k} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-2)^k}{2^k}. \]

The series above converges if \(-1 < u/2 < 1\), or \(-2 < u < 2\), or \(-2 < x - 2 < 2\), or \(0 < x < 4\). Therefore its radius of convergence is 2 and its interval of convergence is \((0, 4)\).