Find the following limits (if the limit is positive or negative infinity, or does not exist state it explicitly).

1. \( \lim_{x \to -4} \frac{x^2 - 16}{x^3 - 64} \). This is an indeterminate form 0/0. To solve the problem we factor the numerator as difference of two squares 

\( x^2 - 16 = x^2 - 4^2 = (x + 4)(x - 4) \) and the denominator as difference of two cubes (according to the formula \( a^3 - b^3 = (a - b)(a^2 + ab + b^2) \)),

\( x^3 - 64 = x^3 - 4^3 = (x - 4)(x^2 + 4x + 16) \).

Therefore,

\[
\lim_{x \to -4} \frac{x^2 - 16}{x^3 - 64} = \lim_{x \to -4} \frac{(x - 4)(x + 4)}{(x - 4)(x^2 + 4x + 16)} = \lim_{x \to -4} \frac{x + 4}{x^2 + 4x + 16} = \frac{8}{48} = \frac{1}{6}.
\]

2. \( \lim_{x \to \infty} (\sqrt{x^2 - 3x + 1} - \sqrt{x^2 + 3x + 1}) \). Here we have an indeterminate form \( \infty - \infty \).

To solve the problem we will first convert it to an indeterminate form \( \frac{\infty}{\infty} \) in the following way.

\[
\lim_{x \to \infty} (\sqrt{x^2 - 3x + 1} - \sqrt{x^2 + 3x + 1}) = \lim_{x \to \infty} \frac{(\sqrt{x^2 - 3x + 1} - \sqrt{x^2 + 3x + 1})(\sqrt{x^2 - 3x + 1} + \sqrt{x^2 + 3x + 1})}{\sqrt{x^2 - 3x + 1} + \sqrt{x^2 + 3x + 1}} = \lim_{x \to \infty} \frac{x^2 - 3x + 1 - (x^2 + 3x + 1)}{\sqrt{x^2 - 3x + 1} + \sqrt{x^2 + 3x + 1}} = \lim_{x \to \infty} \frac{-6x}{\sqrt{x^2 - 3x + 1} + \sqrt{x^2 + 3x + 1}}.
\]

Because it is now an indeterminate form \( \frac{\infty}{\infty} \) the value of the limit will not change if we leave only the leading terms in the denominator, i.e.

\[
\lim_{x \to \infty} \frac{-6x}{\sqrt{x^2 - 3x + 1} + \sqrt{x^2 + 3x + 1}} = \lim_{x \to \infty} \frac{-6x}{\sqrt{x^2 + 3x + 1}} = \lim_{x \to \infty} \frac{-6x}{2x} = -3.
\]

3. \( \lim_{x \to 0} \frac{2^{3x} - 1}{3^{2x} - 1} \). Again we have an indeterminate form 0/0. We will solve the problem with the help of one of our basic exponential limits \( \lim_{u \to 0} \frac{e^u - 1}{u} = 1 \). To do it notice that \( 2 = e^{ln 2} \) and \( 3 = e^{ln 3} \) whence \( 2^{3x} = e^{(3ln 2)x} \) and \( 3^{2x} = e^{(2ln 3)x} \). Next,
\[ \lim_{x \to 0} \frac{2^{3x} - 1}{3^{2x} - 1} = \lim_{x \to 0} \frac{e^{(3\ln 2)x} - 1}{e^{(2\ln 3)x} - 1} = \lim_{x \to 0} \frac{e^{(3\ln 2)x} - 1}{3 \ln 2 x} \times \frac{2 \ln 3 x}{e^{(2\ln 3)x} - 1} \times \frac{3 \ln 2}{2 \ln 3}. \]

The limits of the first and the second factor are equal to one (put in the first case \( u = 3 \ln 2 \), in the second \( u = 2 \ln 3 \) and apply the basic \( \lim_{u \to 0} \frac{e^u - 1}{u} = 1 \)). Therefore the answer in this problem is \( \frac{3 \ln 2}{2 \ln 3} \).

4. \( \lim_{s \to 0} \frac{\cos(3s) - 1}{\sin^2(5s)} \). We deal with this indeterminate form 0/0 in the following way:

\[
\lim_{s \to 0} \frac{\cos(3s) - 1}{\sin^2(5s)} = \lim_{s \to 0} \frac{\cos(3s) - 1}{\sin^2(5s)} \cdot \frac{\cos(3s) + 1}{\cos(3s) + 1} = \lim_{s \to 0} \frac{\cos^2(3s) - 1}{\sin^2(5s)(\cos(3s) + 1)}.
\]

Because \( \lim_{s \to 0} (\cos(3s) + 1) = \cos 0 + 1 = 2 \) we have to compute \( \lim_{s \to 0} \frac{1}{2} \times \frac{\sin^2(3s)}{\sin^2(5s)} \). We will do it with the help of our basic trigonometric limit \( \lim_{u \to 0} \frac{\sin u}{u} = 1 \) as follows.

\[
\lim_{s \to 0} \frac{1}{2} \times \frac{\sin^2(3s)}{\sin^2(5s)} = \lim_{s \to 0} \frac{1}{2} \times \frac{\sin^2(3s)}{(3s)^2} \times \frac{(5s)^2}{\sin^2(5s)} \times \frac{9}{25} = \frac{9}{50}. \]

(The second and the third factors in the computation above have limits equal to 1 because \( \lim_{u \to 0} \frac{\sin u}{u^2} = 1^2 = 1 \).)

Review of problems similar to extra credit problems.

5. Consider the following function

\[ f(x) = \frac{\tan x - x}{x^3}, \quad x \neq 0 \]

Compute the values \( f(0.1), f(0.01), f(0.001), \) and \( f(0.0001) \).

Based on your calculations what seems to be the value of

\[ \lim_{x \to 0} \frac{\tan x - x}{x^3} ? \]

Remember that your calculator must be in the radian mode. We compute
\[ f(0.1) = 0.33467209 \]
\[ f(0.01) \approx 0.3334667 \]
\[ f(0.001) = 0.3333347 \]
\[ f(0.0001) \approx 0.3333333 \]

Now we can guess that \( \lim_{x \to 0} \frac{\tan x - x}{x^3} = \frac{1}{3} \)

6. Consider the function
\[ g(x) = \frac{\sqrt{x}}{\ln x}, \quad x > 1 \]

Compute \( g(10), g(10^3), g(10^6), g(10^9) \).

Based on your calculations what seems to be the value of \( \lim_{x \to \infty} \frac{\sqrt{x}}{\ln x} \)?

We compute
\[ g(10) \approx 1.4 \]
\[ g(10^3) \approx 4.6 \]
\[ g(10^6) \approx 72.4 \]
\[ g(10^9) = 1525 \]

Now we can guess that \( \lim_{x \to 0} \frac{\sqrt{x}}{\ln x} = \infty \)