171 Calculus 1 Review for Test 1

Find the following limits (if the limit is positive or negative infinity, or does not exist state it explicitly).

1 $\lim\limits_{x\to 4}\frac{x^2-16}{x^3-64}$. This is an indeterminate form 0/0. To solve the problem we factor the numerator as difference of two squares
$x^2-16=x^2-4^2=(x+4)(x-4)$ and the denominator as difference of two cubes (according to the formula $a^3-b^3=(a-b)(a^2+ab+b^2)$),
$x^3-64=x^3-4^3=(x-4)(x^2+4x+16)$.
Therefore,

$$\lim_{x\to 4}\frac{x^2-16}{x^3-64}=\lim_{x\to 4}\frac{(x-4)(x+4)}{(x-4)(x^2+4x+16)}=\lim_{x\to 4}\frac{x+4}{x^2+4x+16}=\frac{8}{48}=\frac{1}{6}.$$

2. $\lim\limits_{x\to\infty}(\sqrt{x^2-3x+1}-\sqrt{x^2+3x+1})$**.** Here we have an indeterminate form $\infty-\infty$.

To solve the problem we will first convert it to an indeterminate form $\frac{\infty}{\infty}$ in the following way.

$$\lim_{x\to\infty}(\sqrt{x^2-3x+1}-\sqrt{x^2+3x+1})=\lim_{x\to\infty}\frac{\left(\sqrt{x^2-3x+1}-\sqrt{x^2+3x+1}\right)\left(\sqrt{x^2-3x+1}+\sqrt{x^2+3x+1}\right)}{\sqrt{x^2-3x+1}+\sqrt{x^2+3x+1}}=$$

$$=\lim_{x\to\infty}\frac{(x^2-3x+1)-(x^2+3x+1)}{\sqrt{x^2-3x+1}+\sqrt{x^2+3x+1}}=\lim_{x\to\infty}\frac{-6x}{\sqrt{x^2-3x+1}+\sqrt{x^2+3x+1}}.$$

Because it is now an indeterminate form $\frac{\infty}{\infty}$ the value of the limit will not change if we leave only the leading terms in the denominator, i.e.

$$\lim_{x\to\infty}\frac{-6x}{\sqrt{x^2-3x+1}+\sqrt{x^2+3x+1}}=\lim_{x\to\infty}\frac{-6x}{\sqrt{x^2}+\sqrt{x^2}}=\lim_{x\to\infty}\frac{-6x}{2x}=-3.$$

3. $\lim\limits_{x\to 0}\frac{2^{3x}-1}{3^{2x}-1}$**.** Again we have an indeterminate form 0/0. We will solve the problem with the help of one of our basic exponential limits $\lim\limits_{u\to 0}\frac{e^u-1}{u}=1$. To do it notice that $2=e^{\ln 2}$ and $3=e^{\ln 3}$ whence $2^{3x}=e^{(3\ln 2)x}$ and $3^{2x}=e^{(2\ln 3)x}$. Next,

$\lim_{x\to 0}\frac{2^{3x}-1}{3^{2x}-1}=\lim_{x\to 0}\frac{e^{(3\ln 2)x}-1}{e^{(2\ln 3)x}-1}=\lim_{x\to 0}\frac{e^{(3\ln 2)x}-1}{3\ln 2x}\times\frac{2\ln 3x}{e^{(2\ln 3)x}-1}\times\frac{3\ln 2}{2\ln 3}$. The limits of the first and the second factor are equal to one (put in the first case $u=3\ln 2x$, in the second $u=2\ln 3x$ and apply the basic $\lim_{u\to 0}\frac{e^u-1}{u}=1$). Therefore the answer in this problem is $\frac{3\ln 2}{2\ln 3}$.

4. $\lim_{s\to 0}\frac{\cos(3s)-1}{\sin^2(5s)}$**.** We deal with this indeterminate form 0/0 in the following way

$$\lim_{s\to 0}\frac{\cos(3s)-1}{\sin^2(5s)}=\lim_{s\to 0}\frac{[\cos(3s)-1][\cos(3s)+1]}{\sin^2(5s)[\cos(3s)+1]}=\lim_{s\to 0}\frac{\cos^2(3s)-1}{\sin^2(5s)[\cos(3s)+1]}=$$
$$=\lim_{s\to 0}-\frac{\sin^2(3s)}{\sin^2(5s)[\cos(3s)+1]}.$$

Because

$\lim_{s\to 0}[\cos(3s)+1]=\cos 0+1=2$ we have to compute $\lim_{s\to 0}-\frac{1}{2}\times\frac{\sin^2(3s)}{\sin^2(5s)}$. We will do it with the help of our basic trigonometric limit $\lim_{u\to 0}\frac{\sin u}{u}=1$ as follows.

$\lim_{s\to 0}-\frac{1}{2}\times\frac{\sin^2(3s)}{\sin^2(5s)}=\lim_{s\to 0}-\frac{1}{2}\times\frac{\sin^2(3s)}{(3s)^2}\times\frac{(5s)^2}{\sin^2(5s)}\times\frac{9}{25}=-\frac{9}{50}$. (The second and the third factors in the computation above have limits equal to 1 because $\lim_{u\to 0}\frac{\sin^2 u}{u^2}=1^2=1$.)

Review of problems similar to extra credit problems.

5. Consider the following function

$$f(x)=\frac{\tan x-x}{x^3},\ x\neq 0$$

Compute the values $f(0.1), f(0.01), f(0.001), \text{ and } f(0.0001)$.

Based on your calculations what seems to be the value of

$$\lim_{x\to 0}\frac{\tan x-x}{x^3}\ ?$$

Remember that your calculator must be in the radian mode. We compute

$$f(0.1) \approx 0.33467209$$
$$f(0.01) \approx .33334667$$
$$f(0.001) \approx 0.33333347$$
$$f(0.0001) \approx 0.33333333$$

Now we can guess that $\lim_{x \to 0} \frac{\tan x - x}{x^3} = \frac{1}{3}$

6. Consider the function

$$g(x) = \frac{\sqrt{x}}{\ln x}, x > 1 \quad .$$

Compute $g(10), g(10^3), g(10^6), g(10^9)$.

Based on your calculations what seems to be the value of

$$\lim_{x \to \infty} \frac{\sqrt{x}}{\ln x} \ ?$$

We compute

$$g(10) \approx 1.4$$
$$g(10^3) \approx 4.6$$
$$g(10^6) \approx 72.4$$
$$g(10^9) \approx 1525$$

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Now we can guess that $\lim_{x \to 0} \frac{\sqrt{x}}{\ln x} = \infty$