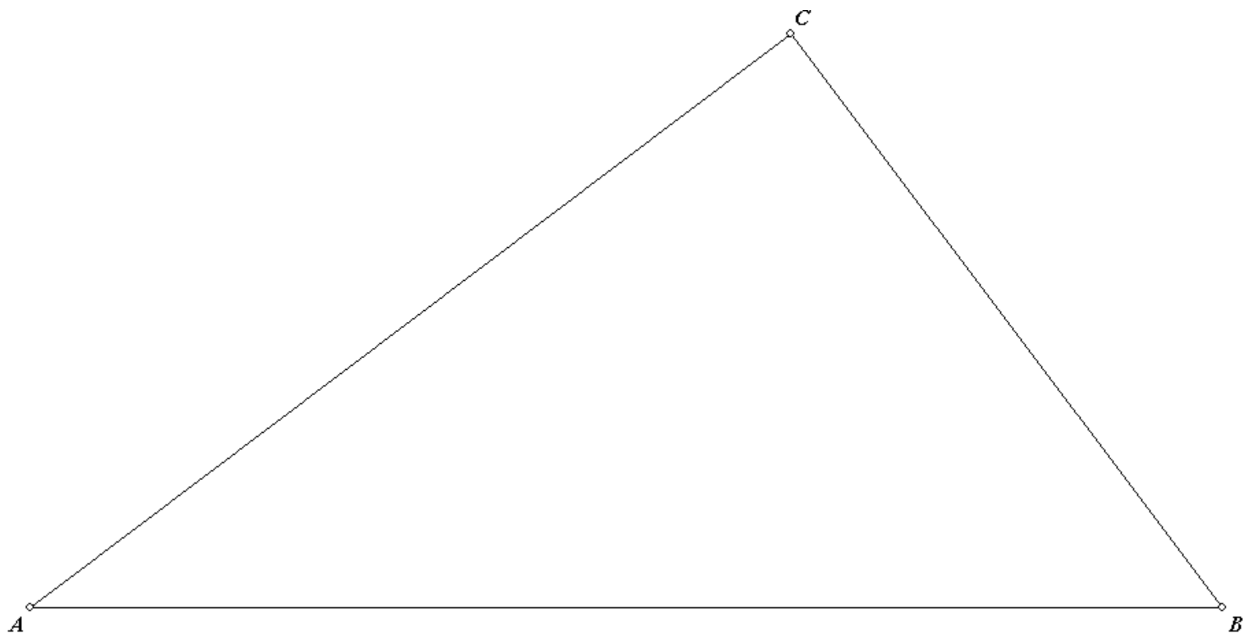
1. Solve the right triangle with an angle 37° and opposite side 14.

**Solution**

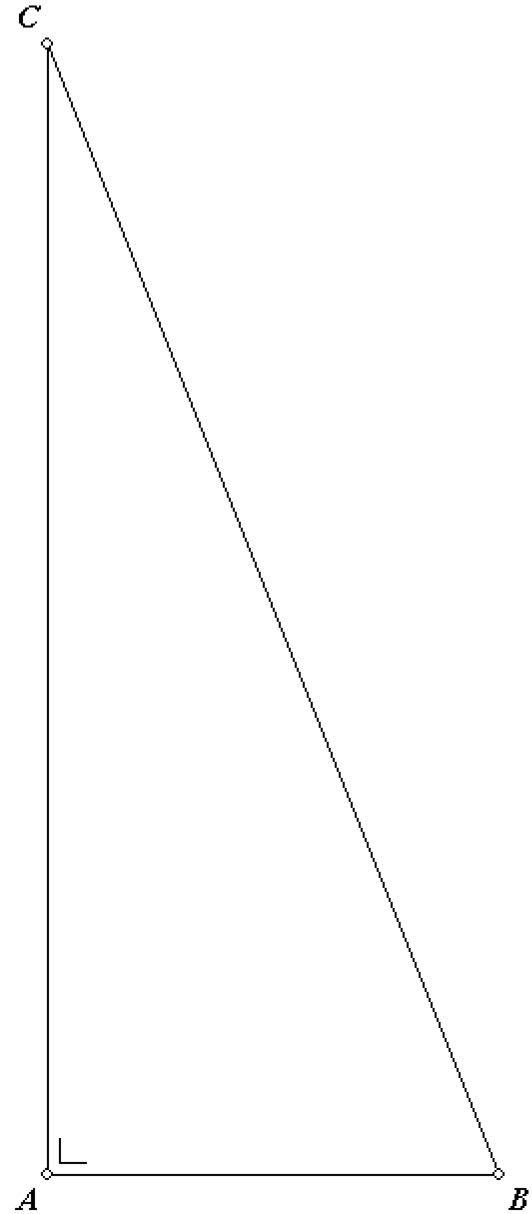



Triangle

On the picture above angle *A* is $37^\circ$, angle *C* is $90^\circ$, and side *BC* is 14. First we can find the third angle $B = 90^\circ - A = 90^\circ - 37^\circ = 53^\circ$. Next we notice that $\tan A = \frac{BC}{AC}$ whence $AC = \frac{BC}{\tan A} = \frac{14}{\tan 37^\circ} \approx 18.58$. Finally, $\sin A = \frac{BC}{AB}$ whence $AB = \frac{BC}{\sin A} = \frac{14}{\sin 37^\circ} \approx 23.26$.

2. Solve the right triangle with the hypotenuse 50 and one side 20.

**Solution**



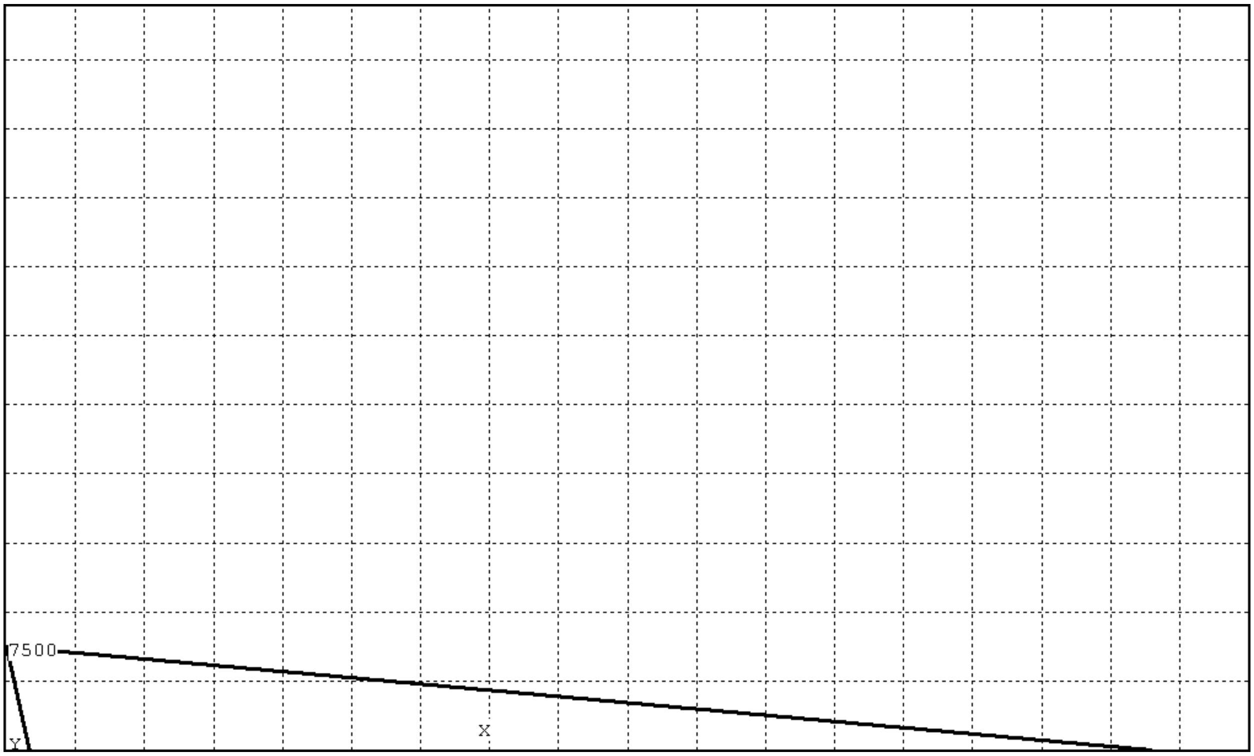

Triangle

x

On the picture above $A = 90^\circ, AB = 20, BC = 50$. Then $\sin C = \frac{AB}{BC} = \frac{20}{50} = 0.4$ and $C = \arcsin 0.4 \approx 23.58^\circ$ whence $B = 90^\circ - \arcsin 0.4 \approx 66.42^\circ$. We have many ways to find *AC*. For example, by the Pythagoras' theorem $AC^2 = BC^2 - AB^2 = 50^2 - 20^2 = 2100$ and $AB = \sqrt{2100} = 10\sqrt{21} \approx 45.83$.

3. From an airplane flying 7500 ft above level ground, one can see two towns directly to the east. The angles of depression to the towns are $5^\circ 10'$ and $77^\circ 30'$. What is the distance between the towns in miles?

**Solution;** look at the picture below




We see two right triangles with horizontal sides *X* and *Y*. The distance between the towns is $D = X - Y$. From the smaller triangle we see that $\frac{7500}{Y} = \tan 77^\circ 30' = \tan 77.5^\circ \approx 4.5107$ and $Y = \frac{7500}{4.5107} \approx 1663\, ft$. Similarly from the larger triangle we obtain $\frac{7500}{X} = \tan 5^\circ 10' \approx \tan 5.17^\circ \approx 0.0905$ and $X = \frac{7500}{0.0905} \approx 82892\, ft$. Therefore $D = 82892 - 1663 = 81229\, ft$. Recalling that there are 5280 feet in one mile we get the distance in miles $D = \frac{81229}{5280} \approx 15.4\, mi$.

4. Solve the triangle $ABC$ if $B = 30^\circ, C = 45^\circ,$ and $AC = 6$.

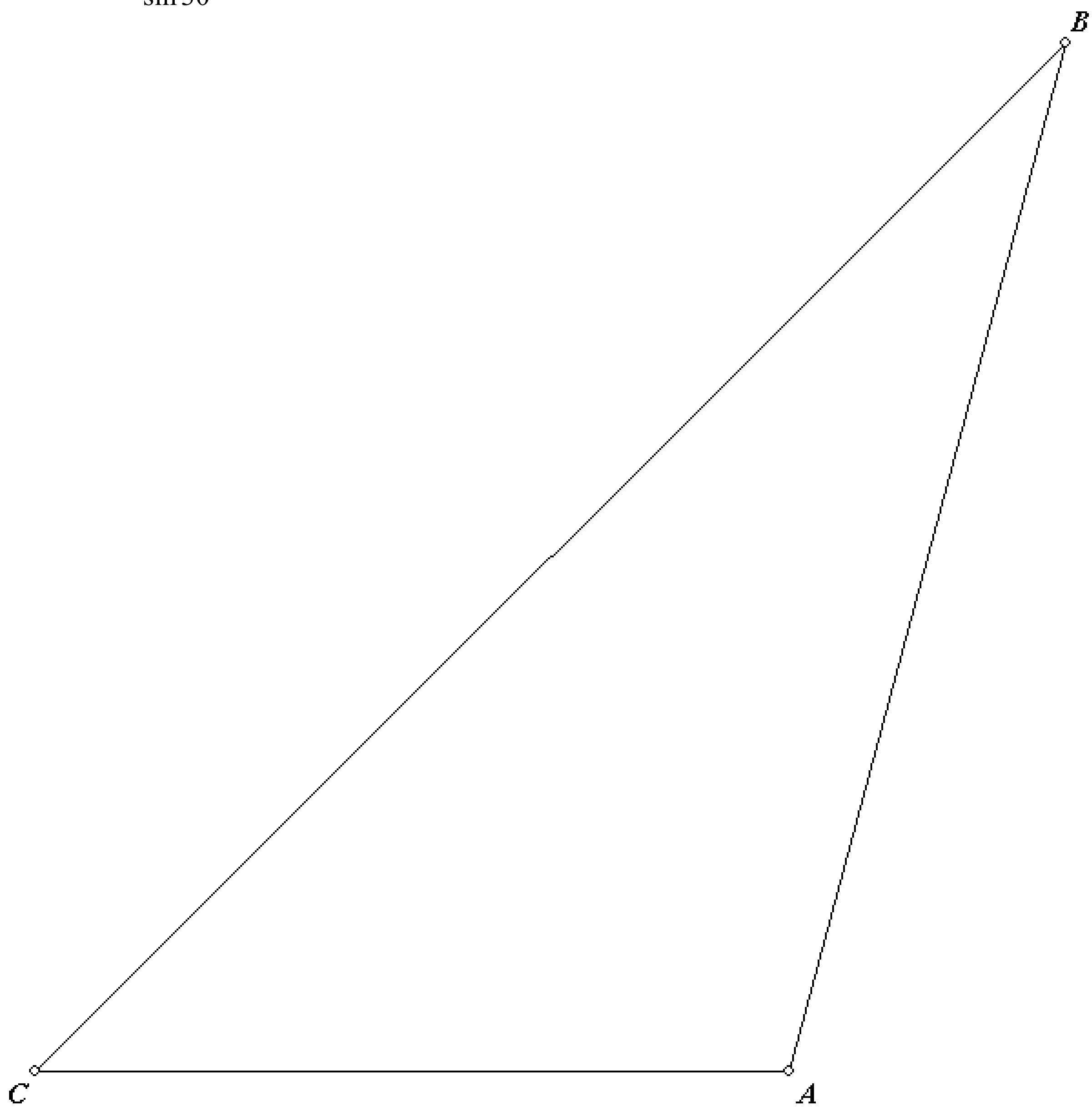
**Solution**; the picture of the triangle is shown below. First notice that $A = 180^\circ - B - C = 180^\circ - 30^\circ - 40^\circ = 105^\circ$. Next we apply the law of Sines:

$$\frac{\sin A}{BC} = \frac{\sin B}{AC} = \frac{\sin C}{AB}$$

whence

$$\frac{\sin 105^\circ}{BC} = \frac{\sin 30^\circ}{6} = \frac{\sin 45^\circ}{AB}.$$

From the last equation we obtain that $BC = \frac{6\sin 105^\circ}{\sin 30^\circ} \approx 11.59$ and that $AB = \frac{6\sin 45^\circ}{\sin 30^\circ} \approx 8.49$.




5. Solve the triangle $ABC$ if $A = 35^\circ$ , $BC = 6,$ and $AC = 8.$

**Solution;** first of all let us see whether the problem has no solution, one solution, or two solutions. Because $AC\sin A = 8\sin 35^\circ \approx 4.6$ we have the inequality $AC\sin A < BC < AC$, which means that the problem has two solutions.

Next, applying the law of Sines we see that

$$\frac{\sin 35^\circ}{6} = \frac{\sin B}{8} = \frac{\sin C}{AB}.$$

From the proportion above we can find that

$$\frac{\sin 35^\circ}{6} = \frac{\sin B}{8} = \frac{\sin C}{AB} \quad \sin B = \frac{8\sin 35^\circ}{6} \approx 0.7648.$$

In our first solution we take

$$B \approx \arcsin(0.7648) \approx 49.89^\circ.$$

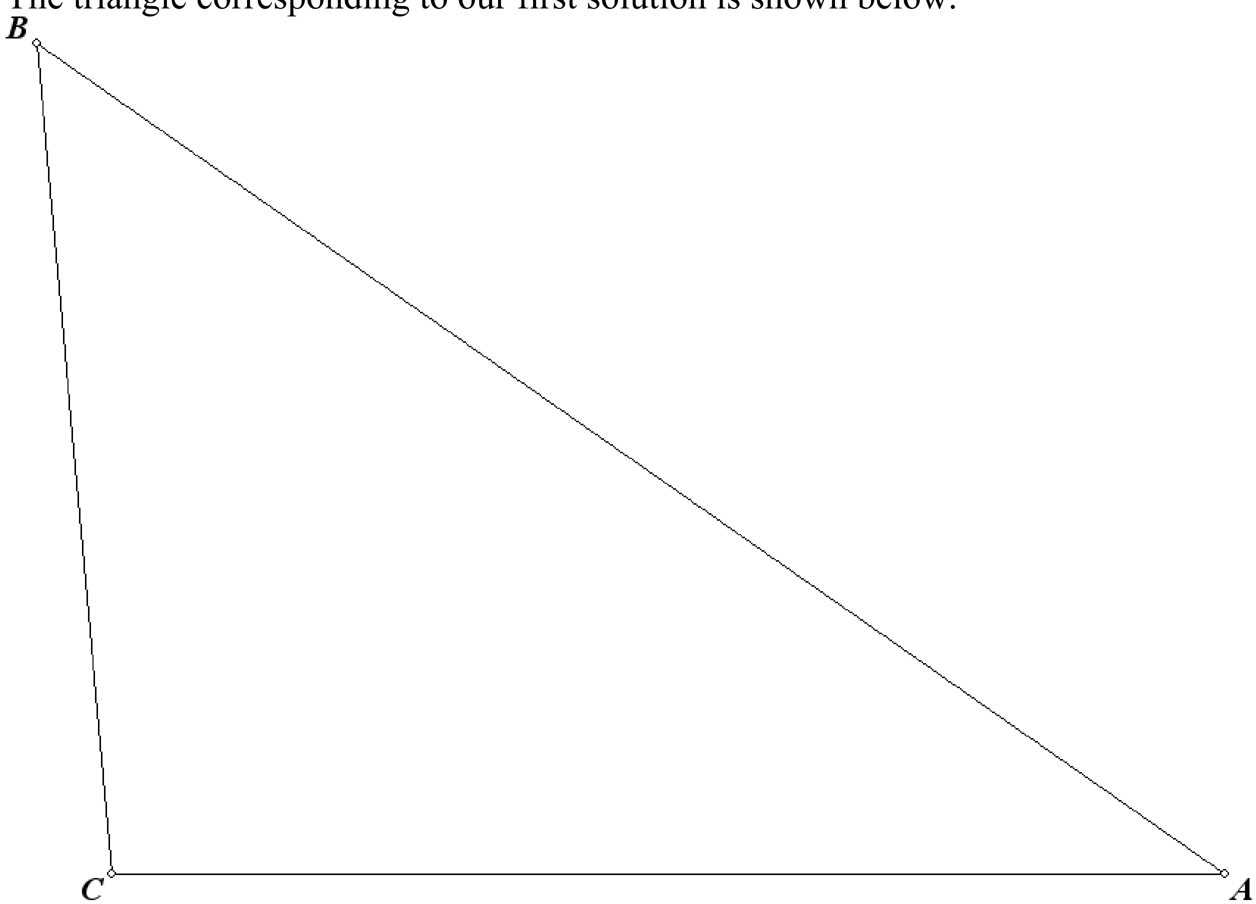
Therefore

$$C \approx 180^\circ - 35^\circ - 49.89^\circ = 95.11^\circ.$$

And finally,

$$AB = \frac{6\sin C}{\sin 35^\circ} = \frac{6\sin 95.11^\circ}{\sin 35^\circ} \approx 10.42.$$

The triangle corresponding to our first solution is shown below.




To get the second solution we take

$$B = 180^\circ - \arcsin(0.7648) \approx 130.11^\circ.$$

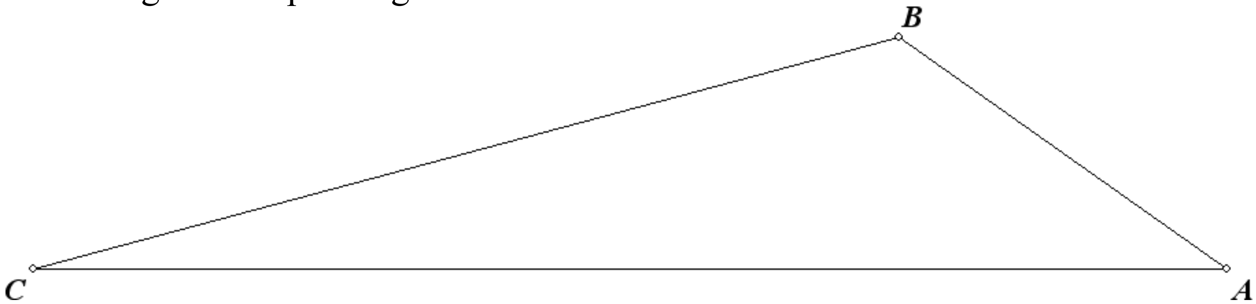
Then

$$C = 180^\circ - 35^\circ - 130.11^\circ = 14.89^\circ.$$

And

$$AB = \frac{6\sin C}{\sin 35^\circ} = \frac{6\sin 14.89^\circ}{\sin 35^\circ} \approx 2.69\,.$$

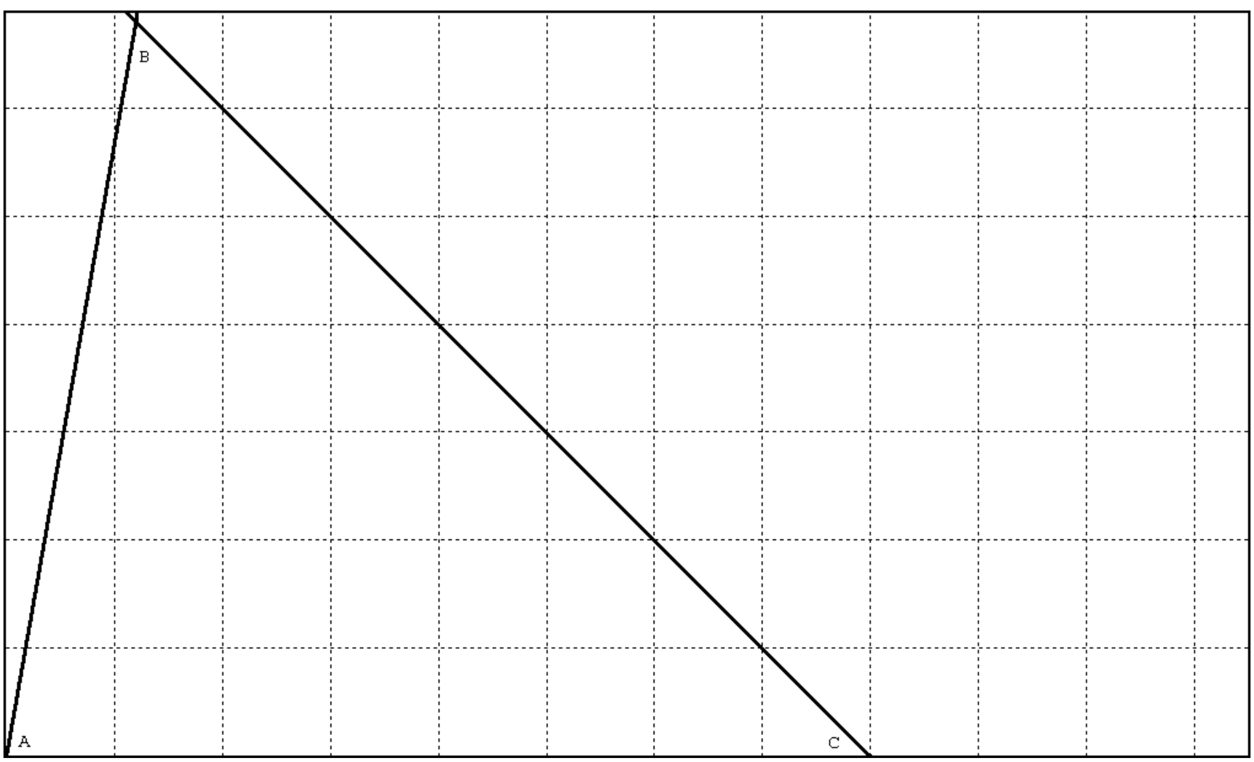
The triangle corresponding to the second solution is shown below.




6. A pole leans away from the sun at an angle of $10^0$ from the vertical. When the angle of elevation of the sun is $45^0$, the pole casts a shadow 52 ft long on level ground. How long is the pole?

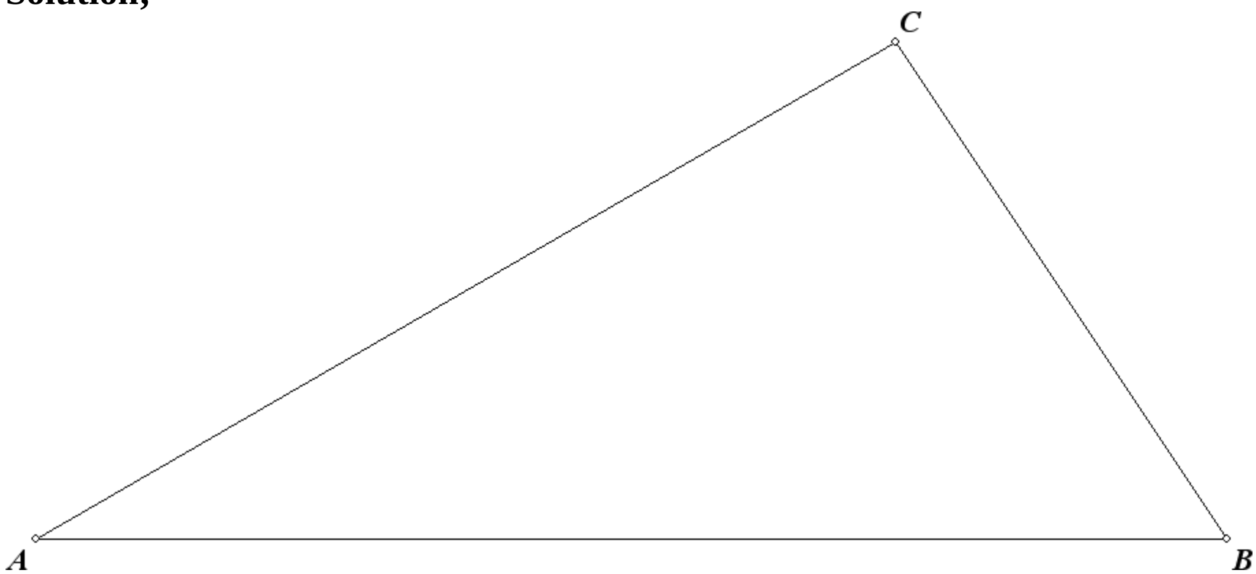
**Solution;** the picture below illustrates the problem. In the triangle *ABC* angle *C* is $45^\circ$, angle *A* is $90^\circ$-$10^\circ$=$80^\circ$, and side *AC* is 52 *ft*. The length of the pole equals to side *AB*. Angle *C* is equal to $180^\circ$-$80^\circ$-$45^\circ$=$55^\circ$. By the law of Sines we have

$\frac{\sin 45^\circ}{AB} = \frac{\sin 55^\circ}{52}$ whence $AB = \frac{52\sin 45^\circ}{\sin 55^\circ} \approx 44.9\,ft$ .



7. Solve the triangle $ABC$ if $AB = 12,\ AC = 10,$ and $A = 30^\circ$

**Solution;**




We apply the law of Cosines.

$$BC^2 = AB^2 + AC^2 - 2AB \cdot BC \cos A = 12^2 + 10^2 - 2 \cdot 10 \cdot 12 \cos 30^\circ \approx 36.15 .$$

Whence

$$BC \approx \sqrt{36.15} \approx 6.01 .$$

Next we apply the law of Cosines again to find angle $B$,

$$\cos B = \frac{AB^2 + BC^2 - AC^2}{2AB \cdot BC} \approx \frac{12^2 + 6.01^2 - 10^2}{2 \cdot 12 \cdot 6.01} \approx .5555 .$$

Therefore

$$B \approx \arccos .5555 \approx 56.26^\circ .$$

Finally, $C = 180^\circ - A - B \approx 180^\circ - 30^\circ - 56.26^\circ = 93.74^\circ$ .

8. Solve the triangle $ABC$ if $BC = 16,\ AC = 13,$ and $AB = 19.$

**Solution;** by the law of Cosines

$$\cos A = \frac{AB^2 + AC^2 - BC^2}{2AB \cdot AC} = \frac{19^2 + 13^2 - 16^2}{2 \cdot 19 \cdot 13} \approx 0.5547 .$$

Whence

$$A \approx \arccos(0.5547) \approx 56.31^\circ .$$

Similarly

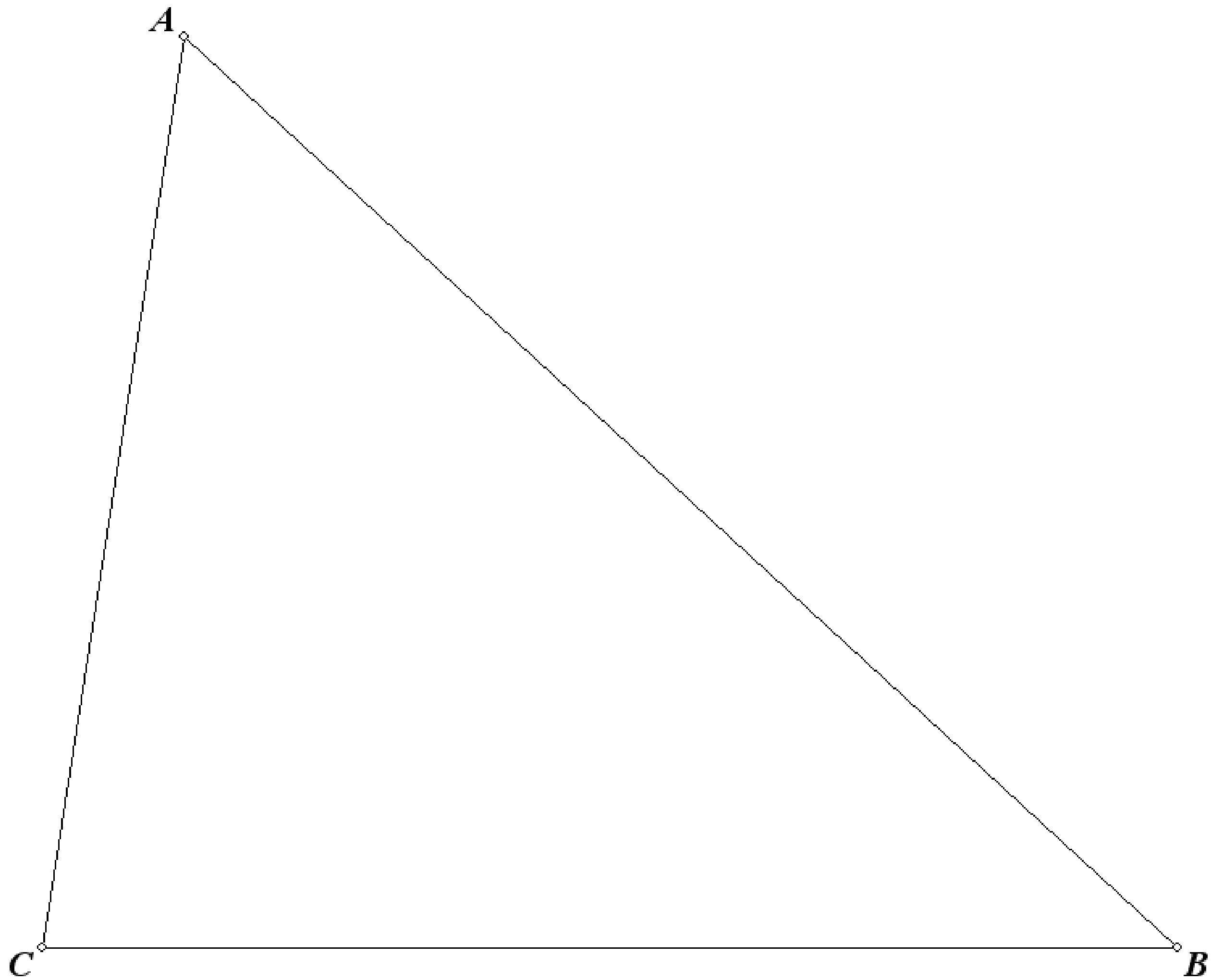
$$\cos B = \frac{AB^2 + BC^2 - AC^2}{2AB \cdot BC} = \frac{19^2 + 16^2 - 13^2}{2 \cdot 19 \cdot 16} \approx 0.7368$$

and

$$B \approx \arccos(0.7368) \approx 42.54^\circ .$$

Finally,

$$C = 180^\circ - A - B \approx 180^\circ - 56.31^\circ - 42.54^\circ = 81.15^\circ$$




9. A hill is inclined $10^0$ to the horizontal. A 26-ft pole stands on the top of the hill. How long a rope will it take to reach from the top of the pole to a point 32 ft downhill from the base of the pole?

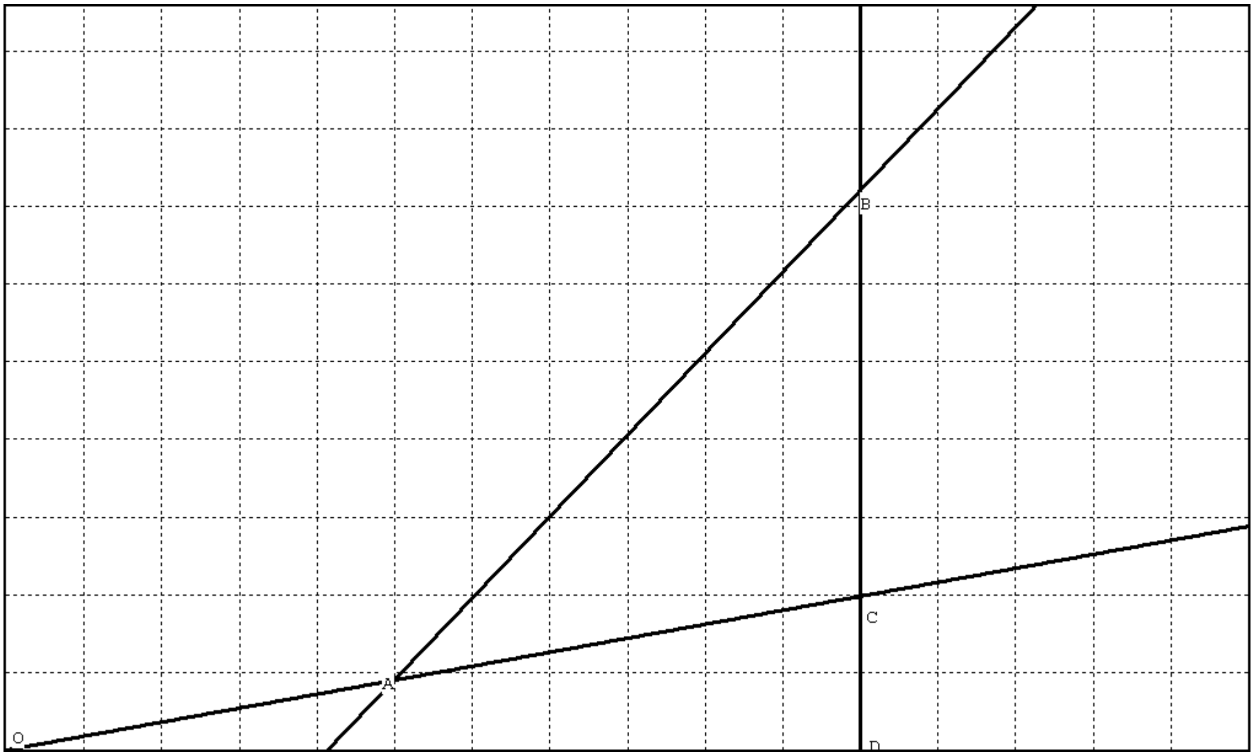
**Solution;** look at the picture below. The line *OC* represents the slope of the hill, so the angle $\angle COD$ equals to $10^\circ$. The side *AC* is 32 *ft* and the side *BC* is 26 *ft*. Te length of the rope is equal to the side *AB*.

The angle $\angle OCD$ equals to $90^\circ$ - $\angle COD$ = $90^\circ$ - $10^\circ$=$80^\circ$. Therefore the angle $\angle ACB$ equals to $180^\circ$-$80^\circ$=$100^\circ$. By the law of Cosines we have

$$AB^2 = AC^2 + BC^2 - 2AC \cdot BC \cos 100^\circ = 32^2 + 26^2 - 2 \cdot 32 \cdot 26 \cos 100^\circ \approx 1988.95 .$$

Therefore

$$AB \approx \sqrt{1988.95} \approx 44.60\, ft .$$




**Review of extra credit problems (will be discussed only if time allows).**

10. Convert to polar notation and then multiply $(4+4i\sqrt{3})(2\sqrt{3}+2i)$.

**Solution;** to convert a complex number $x+yi$ to its polar form $r(\cos\theta + i\sin\theta)$ we use the formulas

$$r=\sqrt{x^2+y^2},$$

$$\theta=\begin{cases}\arctan\frac{b}{a} \text{ if } a>0,\\ \arctan\frac{b}{a}+\pi \text{ if } a<0,\\ \frac{\pi}{2} \text{ if } a=0, b>0,\\ -\frac{\pi}{2} \text{ if } a=0, b<0,\\ \text{undefined if } a=b=0.\end{cases}$$

In particular, for $z=x+yi=4+4i\sqrt{3}$ we have $r=\sqrt{4^2+(4\sqrt{3})^2}=\sqrt{64}=8$, and $\theta=\arctan\frac{4\sqrt{3}}{4}=\arctan\sqrt{3}=\frac{\pi}{3}$, whence $4+4i\sqrt{3}=8(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$. Similarly,

$$2\sqrt{3}+2i=\sqrt{(2\sqrt{3})^2+2^2}(\cos(\arctan\frac{2}{2\sqrt{3}})+i\sin(\arctan\frac{2}{2\sqrt{3}}))=4(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}).$$

Next we use the formula

$$r_1(\cos\theta_1+i\sin\theta_1)\cdot r_2(\cos\theta_2+i\sin\theta_2)=r_1r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))$$

to get

$$(4+4i\sqrt{3})(2\sqrt{3}+2i)=32(\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right))=$$

$$=32(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})=32i.$$

11. Convert to polar notation and then divide $\frac{\sqrt{3}+i}{2\sqrt{3}-2i}$.

**Solution;** first we notice that

$$\sqrt{3}+i=\sqrt{(\sqrt{3})^2+1^2}(\cos(\arctan\frac{1}{\sqrt{3}})+i\sin(\arctan\frac{1}{\sqrt{3}}))=2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$$

and that

$$2\sqrt{3}-2i=\sqrt{(2\sqrt{3})^2+(-2)^2}(\cos(\arctan\frac{-2}{2\sqrt{3}})+i\sin(\arctan\frac{-2}{2\sqrt{3}}))=4(\cos(-\frac{\pi}{6})+i\sin(-\frac{\pi}{6})).$$

Next we apply the formula

$$\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}=\frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))$$

to get

$$\frac{\sqrt{3}+i}{2\sqrt{3}-2i}=\frac{1}{2}(\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right))=\frac{1}{2}(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})=\frac{1}{2}i.$$

12. Find $(\sqrt{3}-i)^5$.

**Solution;** first we convert $(\sqrt{3}-i)$ to the polar form

$(\sqrt{3}-i)=2(\cos(-\frac{\pi}{6})+i\sin(-\frac{\pi}{6}))$ and then we use de Moivre’s formula

$$[r(\cos\theta+i\sin\theta)]^n=r^n(\cos n\theta+i\sin n\theta).$$

Therefore

$$(\sqrt{3}-i)^5=2^5(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right))=32(\cos\left(\frac{5\pi}{6}\right)-i\sin\left(\frac{5\pi}{6}\right))=$$

$$=32(-\frac{\sqrt{3}}{2}-\frac{1}{2}i)=-16(\sqrt{3}+i).$$

13. Solve $x^6 = -64$.

**Solution;** first we convert the right part to the polar form $-64 = 64(\cos\pi + i\sin\pi)$.
Next recall how we solve the equation $x^n = r(\cos\theta + i\sin\theta)$. The formula is

$$x = \sqrt[n]{r}\left(\cos\frac{\theta + 2m\pi}{n} + i\sin\frac{\theta + 2m\pi}{n}\right), \quad m = 0, 1, \ldots, n-1.$$

In our case we have

$$x = \sqrt[6]{64}\left(\cos\frac{\pi + 2m\pi}{6} + i\sin\frac{\pi + 2m\pi}{6}\right), \quad m = 0, 1, \ldots, 5.$$

We get the following six solutions of our equation.

$$x_0 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i,$$

$$x_1 = 2\left(\cos\frac{\pi + 2\pi}{6} + i\sin\frac{\pi + 2\pi}{6}\right) = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i,$$

$$x_2 = 2\left(\cos\frac{\pi + 4\pi}{6} + i\sin\frac{\pi + 4\pi}{6}\right) = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i,$$

$$x_3 = 2\left(\cos\frac{\pi + 6\pi}{6} + i\sin\frac{\pi + 6\pi}{6}\right) = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -\sqrt{3} - i,$$

$$x_4 = 2\left(\cos\frac{\pi + 8\pi}{6} + i\sin\frac{\pi + 4\pi}{6}\right) = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -2i,$$

$$x_5 = 2\left(\cos\frac{\pi + 10\pi}{6} + i\sin\frac{\pi + 10\pi}{6}\right) = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{3} - i.$$

Notice that we have three pairs of conjugate complex solutions.