Verify identities.

1. $\sec^2 x \csc^2 x = \sec^2 x + \csc^2 x$.

Solution

$$\sec^2 x + \csc^2 x = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x \sin^2 x} =$$
$$= \frac{1}{\cos^2 x \sin^2 x} = \frac{1}{\cos^2 x} \cdot \frac{1}{\sin^2 x} = \sec^2 x \csc^2 x.$$

2. $\sin 2x = \frac{2\cot x}{\csc^2 x}$.

Solution $\frac{2\cot x}{\csc^2 x} = 2\cot x \sin^2 x = 2\frac{\cos x}{\sin x}\sin^2 x = 2\cos x \sin x = \sin 2x.$

3. $\frac{\sin 4x + \sin 6x}{\cos 4x - \cos 6x} = \cot x$.

Solution We use the sum to products formulas

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$,

and

$\cos A - \cos B = 2\sin\frac{B-A}{2}\sin\frac{A+B}{2}$.

Therefore $\frac{\sin 4x + \sin 6x}{\cos 4x - \cos 6x} = \frac{2\sin 5x \cos(-x)}{2\sin 5x \sin x} = \frac{\cos(-x)}{\sin x} = \frac{\cos x}{\sin x} = \cot x.$

4. Simplify $(\sin x + \cos x)^2 - \sin 2x$.

Solution

$$(\sin x + \cos x)^2 - \sin 2x = \sin^2 x + 2\sin x \cos x + \cos^2 x - 2\sin x \cos x =$$
$$= \sin^2 x + \cos^2 x = 1$$

5. Simplify (write as a single term) $6\sin 2x + 8\cos 2x$.

Solution we use the formula $A\sin\theta \pm B\cos\theta = \sqrt{A^2 + B^2}\sin(\theta \pm \arctan\frac{B}{A})$. According to this formula $6\sin 2x + 8\cos 2x = \sqrt{6^2 + 8^2}\sin(2x + \arctan\frac{8}{6}) = 10\sin(2x + \arctan\frac{4}{3}).$

In problems 6 -10 find **all** the solutions of the equations below and indicate the solutions in interval $[0, 2\pi)$.

6. $\tan 3x = -\sqrt{3}$.

Solution $3x = \arctan(-\sqrt{3}) + n\pi = -\arctan\sqrt{3} + n\pi = -\frac{\pi}{3} + n\pi,$ where n is any integer, $n \in \mathbf{Z}$. Therefore $x = -\frac{\pi}{9} + n\frac{\pi}{3}, \quad n \in \mathbf{Z}$.

We get solutions in $[0, 2\pi)$ for $n = 1, 2, 3, 4, 5, 6$. The corresponding values of x are $x = \frac{2\pi}{9}, \frac{5\pi}{9}, \frac{8\pi}{9}, \frac{11\pi}{9}, \frac{14\pi}{9}, \frac{17\pi}{9}$.

7. $2\sin^2 x + 3\sin x + 1 = 0$.

Solution the left part is a quadratic trinomial for $\sin x$. If we factor it we will get $(\sin x + 1)(2\sin x + 1) = 0$, whence either $\sin x = -1$ or $\sin x = -1/2$. In the first case the solutions are

$$x = -\frac{\pi}{2} + 2n\pi, \quad n \in \mathbf{Z},$$

In the second one $x = (-1)^n \arcsin\left(-\frac{1}{2}\right) + n\pi = (-1)^{n+1}\frac{\pi}{6} + n\pi, \quad n \in \mathbf{Z}$.

Solutions in the interval $[0, 2\pi)$ are $\frac{7\pi}{6}, \frac{3\pi}{2},$ and $\frac{11\pi}{6}$.

8. $\sec^2 x + 2\tan^2 x = 3$.

Solution by the Pythagorean identity $\sec^2 x = \tan^2 x + 1$ we can rewrite our equation as $3\tan^2 x + 1 = 3$ whence $\tan^2 x = \frac{2}{3}$ and $\tan x = \pm\sqrt{\frac{2}{3}}$; therefore

$$x = \pm\arctan\left(\frac{2}{3}\right) + n\pi, \quad n \in \mathbf{Z}.$$

There are four solutions in the interval $[0, 2\pi)$:

$\arctan\left(\frac{2}{3}\right), \arctan\left(\frac{2}{3}\right) + \pi, -\arctan\left(\frac{2}{3}\right) + \pi, -\arctan\left(\frac{2}{3}\right) + 2\pi$.

9. $4\sin x \tan x - \tan x + 20\sin x - 5 = 0$.

Solution we will factor the left part by grouping:

$\tan x(4\sin x-1)+5(4\sin x-1)=0$ or $(4\sin x-1)(\tan+5)=0$ whence either $\sin x=\frac{1}{4}$ or $\tan x+5=0$. Solutions of the first equation are $x=(-1)^n\arcsin\left(\frac{1}{4}\right)+n\pi,\quad n\in\mathbf{Z}$,

and of the second one - $x=\arctan(-5)+n\pi=-\arctan 5+n\pi,\quad n\in\mathbf{Z}.$

Solutions in the interval $[0,2\pi)$ are $\arcsin\left(\frac{1}{4}\right),\pi-\arcsin\left(\frac{1}{4}\right),\pi-\arctan 5, 2\pi-\arctan 5$.

10. $\cos 6x-\cos 2x=-\sin 4x$.

Solution again using the formula $\cos A-\cos B=2\sin\frac{B-A}{2}\sin\frac{A+B}{2}$ we get $2\sin(-4x)\sin 2x=-\sin 4x$ or $\sin 4x-2\sin 4x\sin 2x=0$, or $\sin 4x(1-2\sin 2x)=0$.

Either $\sin 4x=0$, $4x=n\pi$, and

$$x=n\frac{\pi}{4},n\in\mathbf{Z},$$

or $\sin 2x=\frac{1}{2}$, $2x=(-1)^n\arcsin\left(\frac{1}{2}\right)+n\pi=(-1)^n\frac{\pi}{6}+n\pi$ and

$$x=(-1)^n\frac{\pi}{12}+n\frac{\pi}{2},\quad n\in\mathbf{Z}.$$

Solutions in the interval $[0,2\pi)$ are $\frac{\pi}{4},\frac{\pi}{2}-\frac{\pi}{12},\frac{\pi}{2},\frac{3\pi}{4},\pi,\pi+\frac{\pi}{12},\frac{5\pi}{4},\frac{3\pi}{2}-\frac{\pi}{12}..$