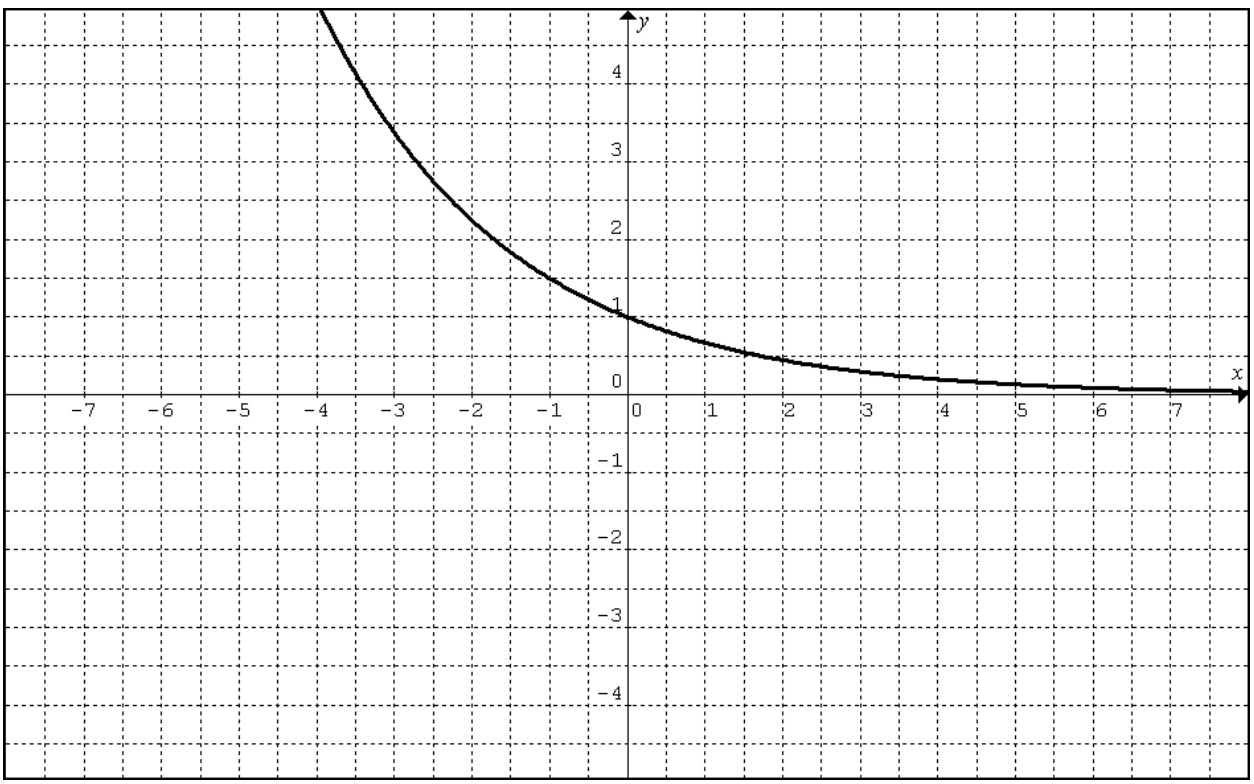
1. Graph the functions $a)\quad y=(2/3)^x \quad b)\quad y=\log_3 x$.

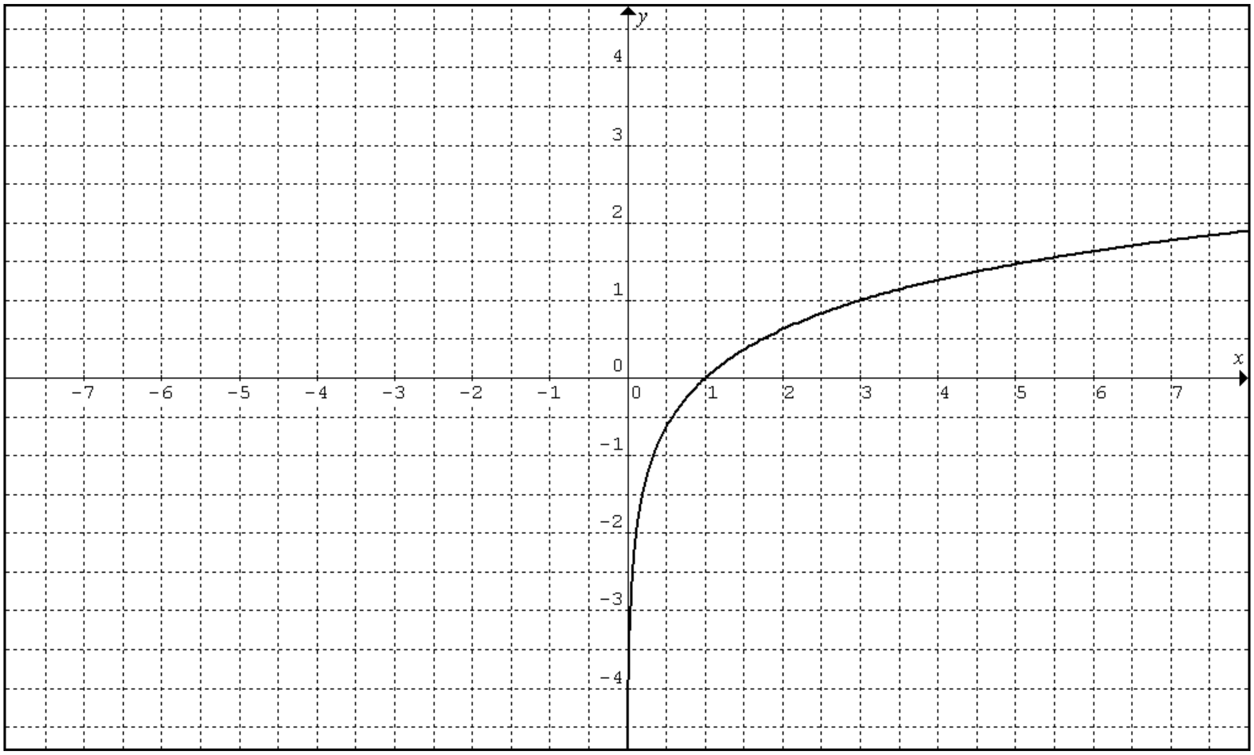
**Solution. (a)** The function $y=\left(\frac{2}{3}\right)^x$ is an exponential function with a base smaller than 1. It means in particular that the function is decreasing and approaching the *x*-axis when *x* moves to the right. Some values of *y* are listed below

$$y(-2)=\frac{9}{4}=2.25,\ y(-1)=\frac{3}{2}=1.5,\ y(0)=1,\ y(1)=\frac{2}{3}\approx 0.7,\ y(2)=\frac{4}{9}\approx 0.4.$$




**(b)** This is a typical graph of a logarithmic function. The function is defined on $(0,\infty)$, negative on $(0,1)$, and positive on $(1,\infty)$. Some values of the function are listed below:

$\log_3 1 = 0, \log_3 3 = 1, \log_3 9 = \log_3(3^2) = 2,$
$\log_3(1/3) = \log_3(3^{-1}) = -1, \log_3(1/9) = \log_3(3^{-2}) = -2.$




2. Solve equations $a)\quad \log_x(1/32) = 5 \quad b)\quad 3^{2x-1} = 5^{3x+1}$

**Solution (a)** to solve this equation we write $x^{\log_x(1/32)} = x^5$. By the definition of logarithms the left part equals to 1/32 and we have $x^5 = \frac{1}{32} = \left(\frac{1}{2}\right)^5$, whence $x = \frac{1}{2}$.

**(b)** Let us take natural logarithms of both parts of the equation.

$\ln(3^{2x-1}) = \ln(5^{3x+1})$. By properties of logarithms we can write the last equation as $(2x-1)\ln 3 = (3x+1)\ln 5$. After expanding the expressions in both parts we obtain $(2\ln 3)x - \ln 3 = (3\ln 5)x + \ln 5$ and therefore $(2\ln 3 - 3\ln 5)x = \ln 5 + \ln 3$. Finally we have $x = \frac{\ln 5 + \ln 3}{2\ln 3 - 3\ln 5} \approx -.103$.

3. Find the domain $a)\quad f(x)=\log(3x-4)\quad b)\quad f(x)=\sqrt{\log_6(x^2-9)}$

**Solution (a)** $\log(3x-4)$ is defined if $3x-4>0$. It means that $3x>4$ and $x>\dfrac{4}{3}$.

Therefore the domain of $f$ is $\left(\dfrac{4}{3},\infty\right)$.

**(b)** For the square root to be defined we need $\log_6(x^2-9)\geq 0$. This inequality will be satisfied if $x^2-9\geq 1$ or $x^2-10\geq 0$. It can be written as $(x+\sqrt{10})(x-\sqrt{10})\geq 0$. The last inequality holds if either $x\leq -\sqrt{10}$ (both factors in the left part are non-positive) or $x\geq\sqrt{10}$ (both factors are non-negative). The domain is $(-\infty,-\sqrt{10})\cup(\sqrt{10},\infty)$.

4. Simplify

$$a)\quad \log_a(x^3+y^3)-\log_a(x+y)$$
$$b)\quad \log_a(a/\sqrt{x})-\log_a\sqrt{ax}$$

**Solution (a)** Using the properties of logarithms and the formula

$x^3+y^3=(x+y)(x^2-xy+y^2)$ we can

write $\log_a(x^3+y^3)-\log_a(x+y)=\log_a\dfrac{x^3+y^3}{x+y}=\log_a(x^2-xy+y^2)$ .

**(b)** First notice that for expression to make sense we must have $x>0$ whence $a>0$ and $\sqrt{ax}=\sqrt{a}\sqrt{x}$. Next, using properties of logarithms we write

$$\log_a(a/\sqrt{x})-\log_a\sqrt{ax}=\log_a a-\log_a\sqrt{x}-\log_a\sqrt{a}-\log_a\sqrt{x}=$$
$$=1-\frac{1}{2}\log_a x-\frac{1}{2}-\frac{1}{2}\log_a x=\frac{1}{2}-\log_a x.$$

5. Solve $a)\quad 4^{2\log_4 x} = 7, \quad b)\quad (x+3)\log_a a^{x^2} = x$

**Solution (a)** by the definition of logarithms we have $4^{\log_4 x} = x$ whence

$4^{2\log_4 x} = \left(4^{\log_4 x}\right)^2 = x^2$ and our equation becomes $x^2 = 7$. Only the positive solution makes sense (logarithm of a negative number is undefined) and therefore $x = \sqrt{7}$.

**(b)** By the property of logarithms $a^{\log_a x^2} = x^2$ whence $(x+3)x^2 = x$ and $(x+3)x^2 - x = 0$.

Factoring out $x$ we get $x[(x+3)x-1] = 0$. One solution is $x = 0$ , we obtain two more solutions from the quadratic equation $x^2 + 3x - 1 = 0$. Solving by quadratic formula we get $x = \dfrac{-3 \pm \sqrt{3^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1} = \dfrac{-3 \pm \sqrt{13}}{2}$. All three solutions make sense and satisfy the original equation.

6. Solve $3^{x^2+4x} = \dfrac{1}{27}$

**Solution** $3^{x^2+4x} = \dfrac{1}{27} = \dfrac{1}{3^3} = 3^{-3}$ whence $x^2 + 4x = -3$, $x^2 + 4x + 3 = 0$, $(x+1)(x+3) = 0$, and we have two solutions $x = -1, x = -3$.

7. Solve $e^{2x} + 1 = 5e^x$.

**Solution** because $e^{2x} = (e^x)^2$ we have a quadratic equation for $e^x$: $(e^x)^2 - 5e^x + 1 = 0$.

Solving for $e^x$ by quadratic formula we get $e^x = \dfrac{5 \pm \sqrt{21}}{2}$. Both solutions are positive and therefore make sense (recall that $e^x > 0$). Taking natural logarithms of both parts we get two solutions of the original equation

$$x=\ln\frac{5-\sqrt{21}}{2}\approx -1.57 \text{ and } x=\ln\frac{5+\sqrt{21}}{2}\approx 1.57$$

.

8. Solve $a)\quad \log(\sqrt{x})=\sqrt{\log x}\quad b)\quad \log(2x-1)-\log(x-2)=1$ **Solution**

**(a)** because $\log(\sqrt{x})=\frac{1}{2}\log x$ our equation can be written as

$\frac{1}{2}\log x-\sqrt{\log x}=0$ or $\log x-2\sqrt{\log x}=0$. If we take $u=\sqrt{\log x}$ we get a quadratic

equation for *u*: $u^2-2u=0$ or $u(u-2)=0$. It has two solutions $u=0$ and $u=2$.

In the first case $\sqrt{\log x}=0$ whence $\log x=0$. In the second case

$\sqrt{\log x}=2$ whence $\log x=4$. Respectively we have two solutions for *x*:

$x=10^0=1$ and $x=10^4=10000$.

**(b)** By the properties of logarithms $\log\frac{2x-1}{x-2}=1$ whence

$$\frac{2x-1}{x-2}=10^1=10,\quad 2x-1=10x-20,\quad -8x=-19,\quad x=\frac{19}{8}=2.375.$$

9. The population of Dallas was 680,000 in 1960. In 1969 it was 815,000. Find the corresponding exponential model of the population growth and estimate the population in 2010.

**Solution** Let $t = 0$ correspond to year 1960, then $t = 9$ corresponds to year 1969, and $t = 50$ to year 2010. The exponential model of population growth can be written as

$$P(t) = P(0)e^{kt}.$$

We know that $P(0) = 680000$ and that $P(9) = 815000$. From it we can find that $e^{9k} = \frac{815000}{680000} = \frac{163}{136}$, $9k = \ln\frac{163}{136}$, and $k = \frac{1}{9}\ln\frac{163}{136} \approx .0201$. Therefore our prediction for year 2010 is $P(50) = 680000e^{50k} \approx 680000e^{50\cdot 0.0201} \approx 1858000$.

10. The half-life of Polonium is 3 min. After 28 min. how much of a 410 g sample remains radioactive?

**Solution** The equation of radioactive decay is

$$Q(t) = Q(0)e^{kt},$$

where in our case *t* is time in minutes and *Q* (0) = 410. Because half-life of Polonium is 3 min. we have

$Q(3) = \frac{1}{2}Q(0)$ whence $e^{3k} = \frac{1}{2}$, $3k = \ln\frac{1}{2} = -\ln 2$, and $k = -\frac{\ln 2}{3} \approx -0.2310$.

Finally, $Q(28) = Q(0)e^{28k} \approx 410e^{28\cdot(-.2310)} \approx .636g = 636mg$