**161 Precalculus 1**

## Review for the Final Exam

1. Consider two points on the coordinate plane $P$ (-2, 3) and $Q$ (4,-1).

(a) Find the distance between $P$ and $Q$. (2 points)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[4-(-2)]^2 + (-1-3)^2} =$$
$$= \sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$$

(b) Find the coordinates of the midpoint. (2 points)

$$x_m = \frac{x_1 + x_2}{2} = \frac{-2+4}{2} = 1.$$
$$y_m = \frac{y_1 + y_2}{2} = \frac{3+(-1)}{2} = 1.$$

(c) Find the slope of the line through $P$ and $Q$. (2 points)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-3}{4-(-2)} = -\frac{2}{3}.$$

(d) Find the "slope-intercept" equation of this line. (2 points)

First we will find a point-slope equation using, for example, the second point

$$y - y_2 = m(x - x_2),$$
$$y - (-1) = -\frac{2}{3}[x - 4],$$
$$y + 1 = -\frac{2}{3}x + \frac{8}{3}.$$

Finally, $y = -\frac{2}{3}\text{x}+\frac{5}{3}$ is the slope-intercept equation.

2. Consider the quadratic function

$$y = -x^2 + 3x + 1$$

(a) Find the coordinates of the vertex. (2 points)

$$x_v = -\frac{b}{2a} = -\frac{3}{2\times(-1)} = \frac{3}{2}.$$
$$y_v = c - \frac{b^2}{4a} = 1 - \frac{9}{-4} = \frac{13}{4}$$

(b) Find the *x*-intercepts (if any). (2 points)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \times (-1) \times 1}}{2 \times (-1)}.$$

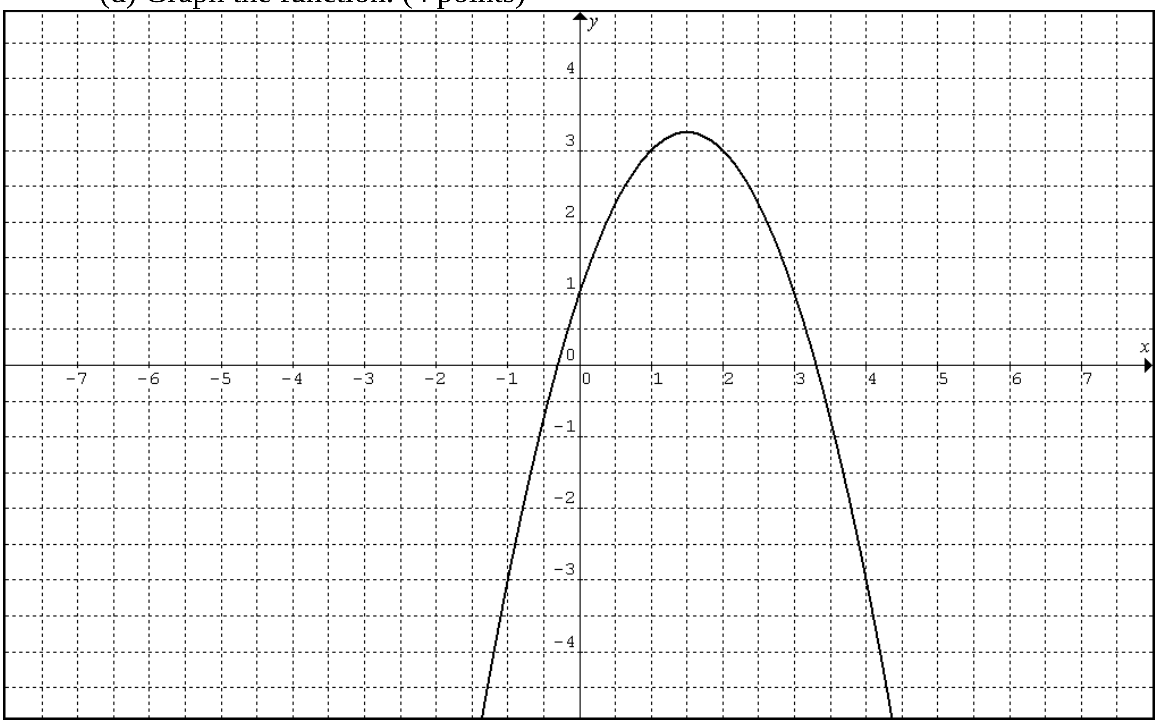
$$x_1 = \frac{3 - \sqrt{13}}{2} \approx -.3,$$

$$x_2 = \frac{3 + \sqrt{13}}{2} \approx 3.3.$$

(c) Find the range of the function. (2 points)

The coefficient *a* is negative, therefore the range is from the vertex down: (-∞, 13/4].

(d) Graph the function. (4 points)




3. A rectangular plot of land on the edge of a river is to be enclosed with fence on three sides. Find the dimensions of the rectangular enclosure of the greatest area if the side that goes along the river does not require fencing and the total length of the fence is 200 m. (10 points)

Let *x* be the side along the river and *y* be the perpendicular side. Then

$$x + 2y = 200.$$

whence

$$x = 200 - 2y.$$

The area of the rectangle is

$$xy = (200 - 2y)y = -2y^2 + 200y.$$

The greatest value of this quadratic function will be at the vertex

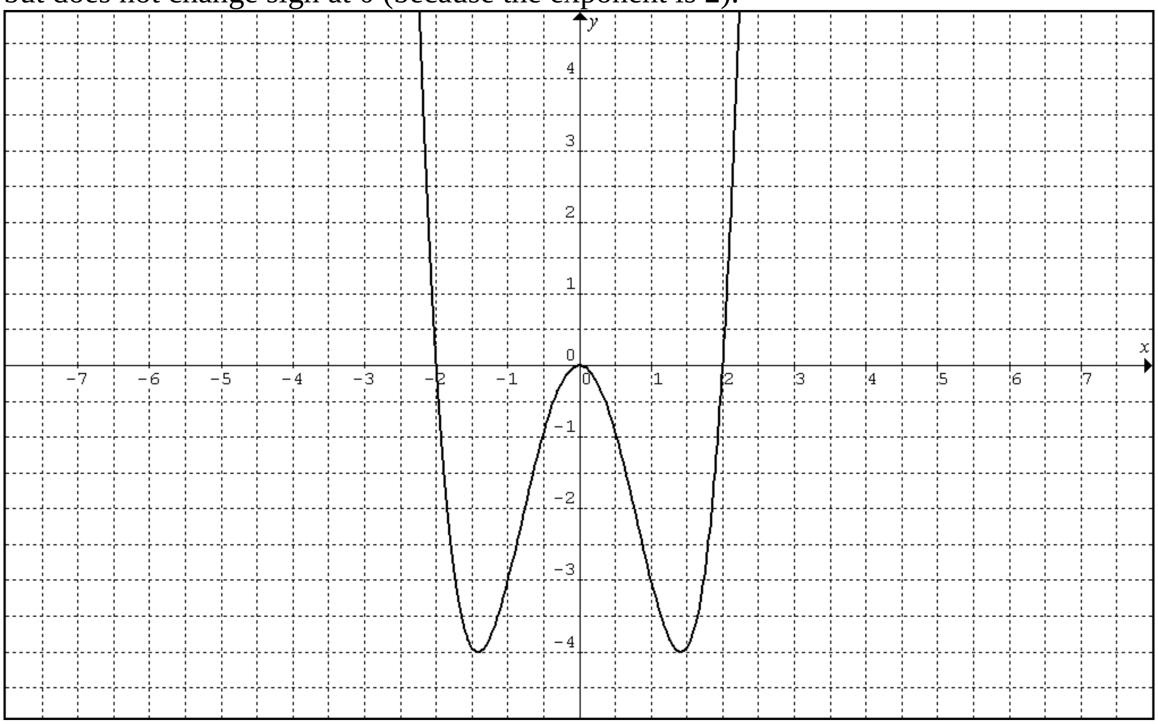
$$y = -\frac{200}{2\times(-2)} = 50.$$

Then $x = 200 - 2\times 50 = 100.$

4. Graph the polynomial function (10 points)

$$P(x) = x^2(x^2-4)$$

The $x$-intercepts are at -2, 0, and 2. The function is positive to the left of -2 (because the leading term is $x^4$). It changes sign at -2 and at 2 (because the exponent in each case is 1), but does not change sign at 0 (because the exponent is 2).




5. Find the rational roots of the polynomial

$$x^3 - 6x^2 + 11x - 6$$

and factor the polynomial completely. (10 points)

Possible rational roots are factors of 6: 1,-1,2,-2,3,-3,6,-6. The solutions cannot be negative (every term becomes negative) so we try only 1,2,3,6. We can see at once that 1 is a root and therefore $x$-1 is a factor. From synthetic (or long) division we see that

$x^3 - 6x^2 + 11x - 6 = (x-1)(x^2-5x+6) = (x-1)(x-2)(x-3)$.

The roots are 1, 2, 3.

6. For the function

$$f(x) = \sqrt{x-1} + 2$$

(a) Find the domain. (3 points)

The function is defined when $x-1\geq 0,$ whence $x\geq 1.$ The domain is the interval $[1,\infty)$

(b) Find the range. (3 points)

All values from 2 and up. The range is the interval $[2,\infty)$.

(c) Find the inverse function, its domain and range. (4 points)

$$y=\sqrt{x-1}+2,\ \sqrt{x-1}=y-2,\ x-1=(y-2)^2,$$

$$x=(y-2)^2+1=y^2-4y+5.$$

The inverse $g(x)=x^2-4x+5.$

The domain of the inverse function is the range of the original function' in our case the interval $[2,\infty)$.

The range equals to the domain of the original function, $[1,\infty)$.

7. For the rational function

$$f(x)=\frac{x-1}{x^2-4}$$

(a) Find the horizontal or slant asymptote, if any. (3 points)

The ratio of leading terms is $x/x^2=1/x$ and it is close to 0 when $x$ is a large positive or a large negative number. Therefore the graph has the horizontal asymptote $y$=0 (the $x$-axis).

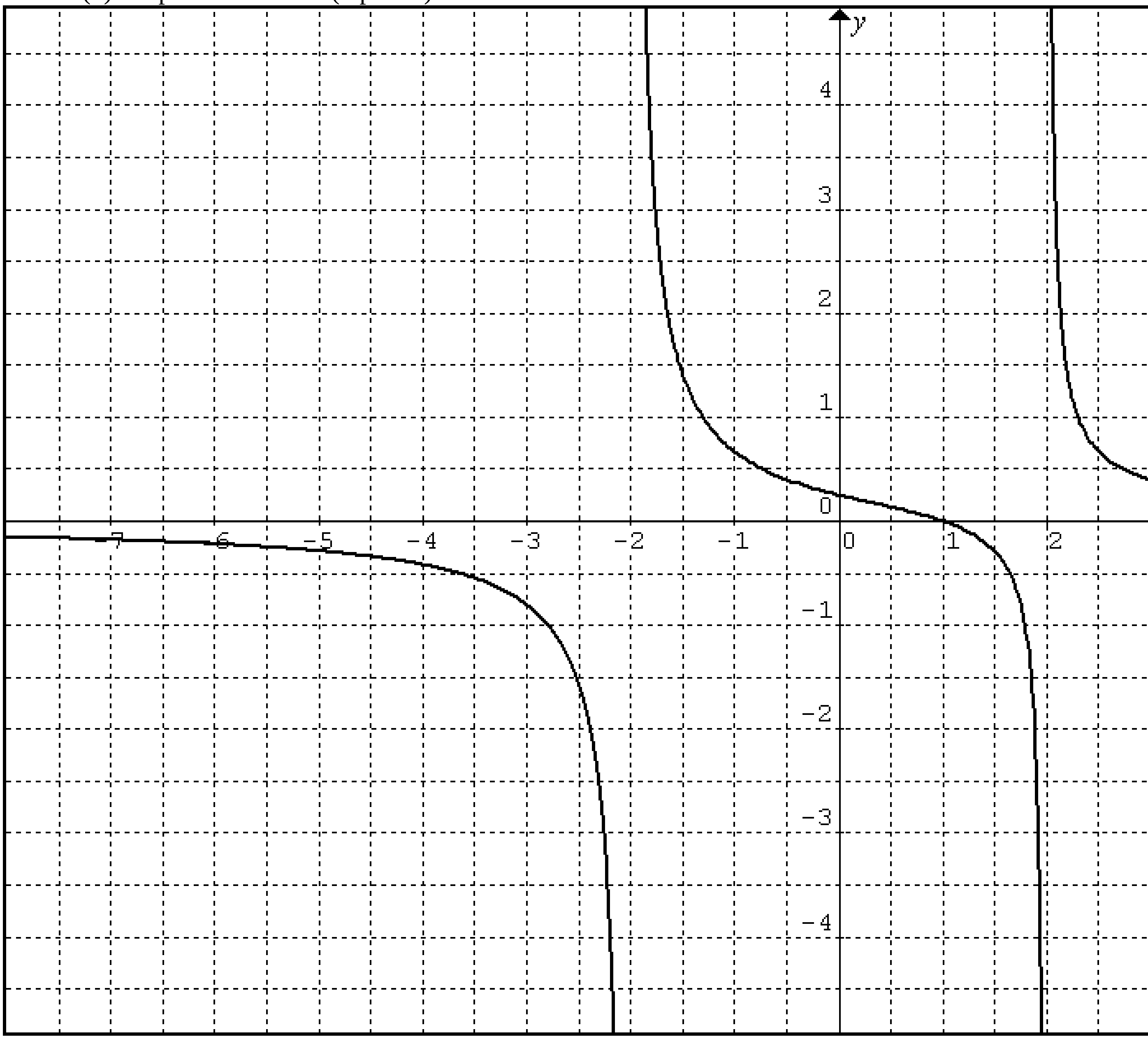
(b) Find the vertical asymptotes, if any. (3 points)

The denominator is 0 when $x$=2 or $x$=-2. The vertical asymptotes go through points 2 and -2 on the $x$-axis.

(c) Find the $x$ and $y$ intercepts, if any. (3 points)

The $x$-intercept: $x$=1. The $y$-intercept (when $x$=0) is ¼.

(d) Graph the function. (5 points)



8. For the functions

$$f(x) = \sqrt{x} \text{ and } g(x) = x^2 + 1$$

(a) Find the compositions (4 points)

$$f(g(x)) \text{ and } g(f(x))$$

$f(g(x)) = \sqrt{x^2 + 1}, \quad g(f(x)) = (\sqrt{x})^2 + 1 = x + 1.$

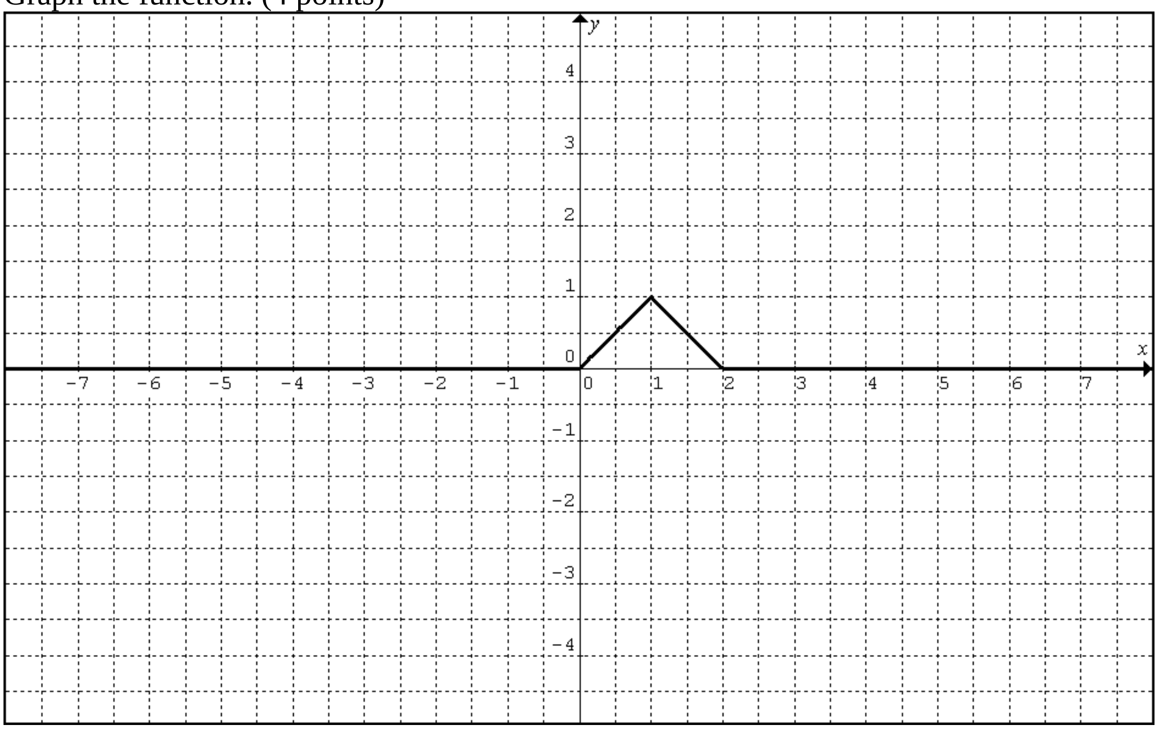
(b) Find the domain of each composition function. (8 points)

1. (-∞, ∞). 2. [0, ∞).

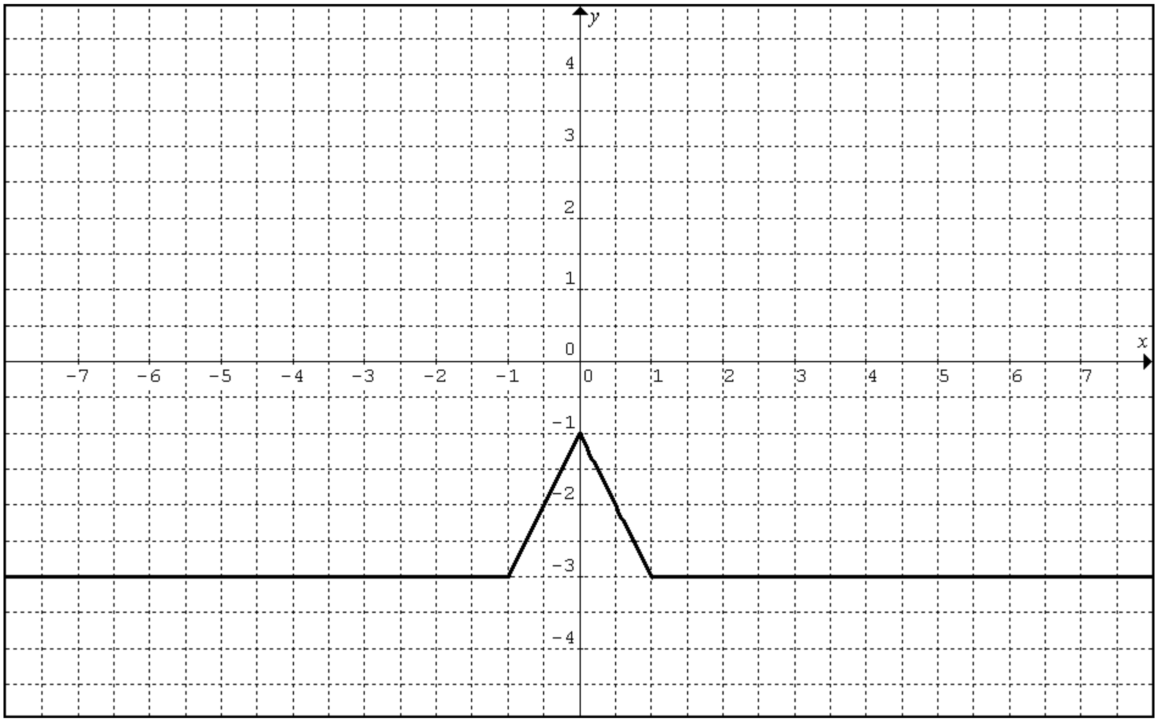
9. Function *f* is defined in the following way

$$f(x) = \begin{cases} 0, x \le 0 \\ x, 0 \le x \le 1 \\ 2 - x, 1 \le x \le 2 \\ 0, x \ge 2 \end{cases}$$

(a) Graph the function. (4 points)



(b) Graph the function $g(x) = 2f(x+1) - 3$. (4 points)




(c) How are the two graphs related? (2 points)

We obtain the second graph if we move the first one by one unit to the left, stretch it twice along the *y*-axis, and move by three units down.

10. Consider the following functions.

$$f(x) = 3x^4 + x^2 + 1,$$
$$g(x) = x^5 - x^3 + 3x,$$
$$h(x) = x^4 - x^3$$

(a) Which of these functions are even? (2 points)

$f(-x) = f(x),\ f$ is an even function.

(b) Which are odd? (2 points)

$g(-x) = -g(x),\ g$ is an odd function.

(c) What kind of symmetry do their graphs have? (2 points)

Graph of *f* is symmetric about the *y*-axis.

Graph of *g* is symmetric about the origin.

11. Consider the following equation of a conic. $-8x^2 + y^2 - 2x + 5y = 0$.

a. Identify the conic (2 points). The conic is a hyperbola (or a degenerate conic) because the coefficients by $x^2$ and $y^2$ have opposite signs.

b. Write an equation of the conic in the standard form (5 points).

$$-8x^2 - 2x = -8(x^2 + \frac{1}{4}x) = -8[(x + \frac{1}{8})^2 - \frac{1}{64}] = -8(x + \frac{1}{8})^2 + \frac{1}{8};$$

$$y^2 + 5y = (y + \frac{5}{2})^2 - \frac{25}{4}.$$

After we plug these expressions into original equation it becomes

$(y + 5/2)^2 - 8(x + 1/8)^2 = 49/8$, or

$$\frac{(y+5/2)^2}{49/8} - \frac{(x+1/8)^2}{49/64} = 1.$$

c. Find the center, the vertices and the foci (5 points). The center is at (-1/8, -5/2). The axis of the hyperbola is vertical, $a = 7/8$, $b = 7/\sqrt{8} \approx 2.5$, and $f = \sqrt{\frac{49}{8} + \frac{49}{64}} = \frac{21}{8}$. The vertices are at $(-1/8, -5/2 - 7/\sqrt{8})$ and $(-1/8, -5/2 - 7/\sqrt{8})$. The foci are at (-1/8, - 41/8) and at (-1/8, 1/8).

d. Graph the conic (5 points).