

Solve problems 1 and 2 using the geometric approach.

1. A manufacturer has 240, 370, and 180 pounds of wood, plastic, and steel, respectively. Product A requires 1, 3, and 2 pounds of wood, plastic, and steel per unit, respectively; product B requires 3, 4, and 1 pound per unit, respectively. How many units of each product should be made to maximize gross income in each of the following cases?

(a) Product A sells for \$40 and B for \$60.

(b) Product A sells for \$40 and B for \$50.

Solution. First we will set our problem as a problem of linear programming. Suppose that x units of product A and y units of product B are produced. Then we have to maximize the gross income

(a) $G = 40x + 60y$,

(b) $G = 40x + 50y$,

subject to the following constraints:

(wood) $x + 3y \leq 240$,

(plastic) $3x + 4y \leq 370$,

(steel) $2x + y \leq 180$,

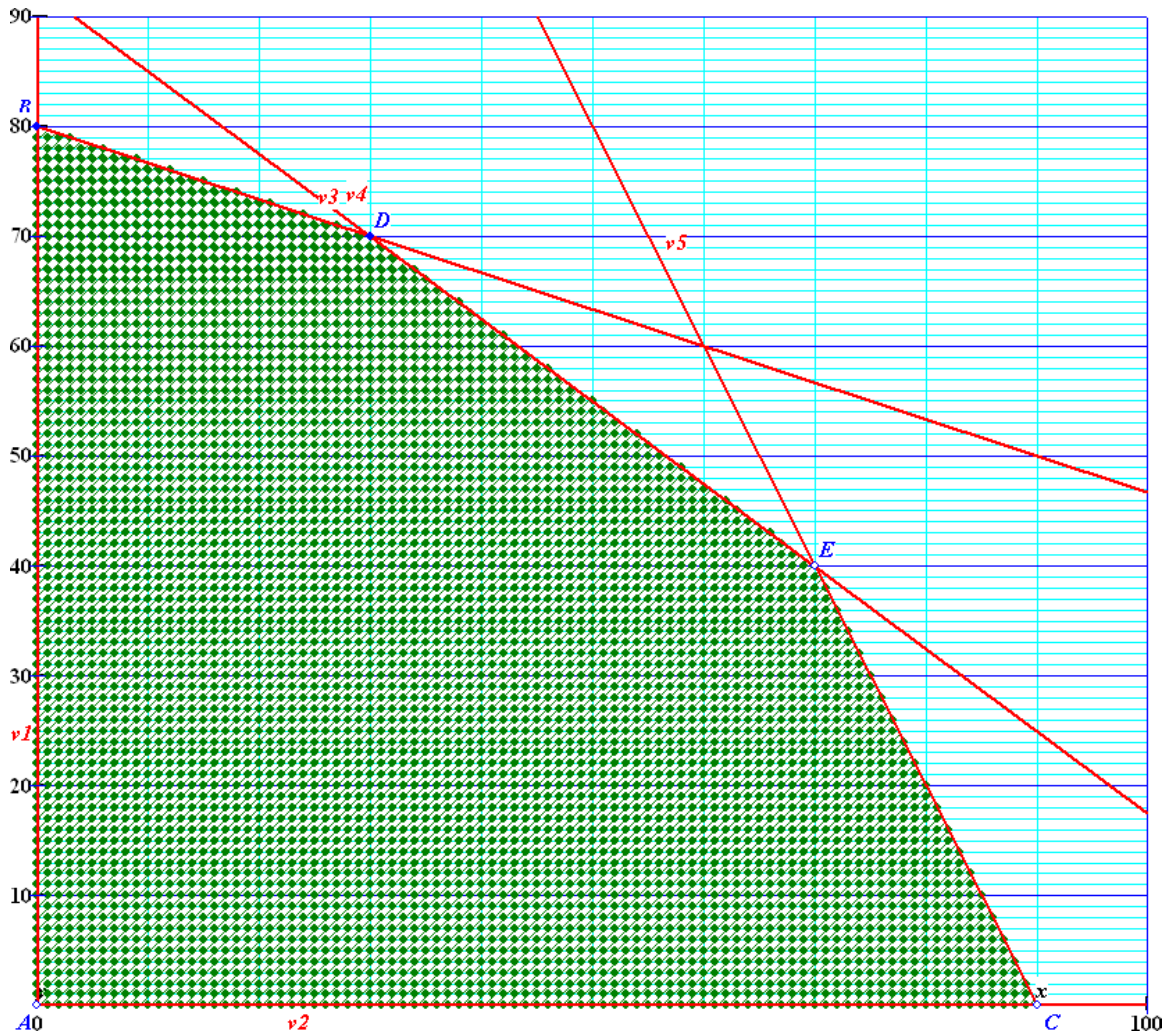
and the standard constraints

$$x \geq 0,$$

$$y \geq 0.$$

We will now graph the above system of inequalities.

The inequalities show that our region is in the first quadrant and is the intersection of the regions **below** the lines $x + 3y = 240$, $3x + 4y = 370$, and $2x + y = 180$. For the first line the x -intercept is 240 and the y -intercept is $240/3=80$. For the second line the x -intercept is $370/3=123\frac{1}{3}$ and the y -intercept is $370/4=92.5$. For the third line the x -intercept is $180/2=90$ and the y -intercept is 180. A computer generated graph of the region is shown below.



The region has five vertices: A, B, C, D, E . We can see at once that the vertices A, B, C have coordinates $(0, 0)$, $(0, 80)$, and $(90, 0)$, respectively. To find the coordinates of the vertex D we have to solve the system of equations

$$x + 3y = 240$$

$$3x + 4y = 370.$$

We will solve it by elimination; multiplying both parts of the first equation by 3 we get

$$3x + 9y = 720$$

$$3x + 4y = 370.$$

Subtracting the second equation from the first one we have $5y = 350$, whence $y = 70$.

After we plug in this value of y into the first equation we get $x + 210 = 240$ and $x = 30$. So the vertex D is at $(30, 70)$.

We will find the vertex E by solving the system

$$2x + y = 180$$

$$3x + 4y = 370.$$

We multiply both parts of the first equation by 4,

$$8x + 4y = 720$$

$$3x + 4y = 370$$

and subtract the equations. We get $5x = 350$ whence $x = 70$ and from the first equation we see that $y = 40$. So the vertex E is at $(70, 40)$.

The next table shows the values of the objective function G at each vertex in case (a) and in case (b).

Vertex	x	y	(a) $G = 40x + 60y$	(b) $G = 40x + 50y$
A	0	0	0	0
B	0	80	4800	4000
C	30	70	5400	4700
D	70	40	5200	4800
E	90	0	3600	3600

In case (a) we have the best solution when 30 units of product A and 70 units of product B are produced generating the gross income of \$5400.

In case (b) we have the best solution when 70 units of product A and 40 units of product B are produced generating the gross income of \$4800.

2. Suppose that 8, 12, and 9 units of proteins, carbohydrates, and fats, respectively, are the minimum weekly requirements for a person. Food A contains 2, 6, and 1 unit of proteins, carbohydrates, and fats, respectively, per pound; food B contains 1, 1, and 3 units, respectively, per pound. How many pounds of each food type should be made to minimize cost while meeting the requirements in each of the following cases?

(a) Food A costs \$3.40 per pound, and B costs \$1.60 per pound.

(b) Food A costs \$3.00 per pound, and B costs \$1.60 per pound.

Solution. First we will set our problem as a problem of linear programming. Suppose that x pounds of food A and y pounds of food B are made. Then we have to minimize the cost

(a) $C = 3.40x + 1.60y$,

(b) $G = 3.00x + 1.60y$,

subject to the following constraints:

(proteins) $2x + y \geq 8$,

(carbohydrates) $6x + y \geq 12$,

(fats) $x + 3y \geq 9$,

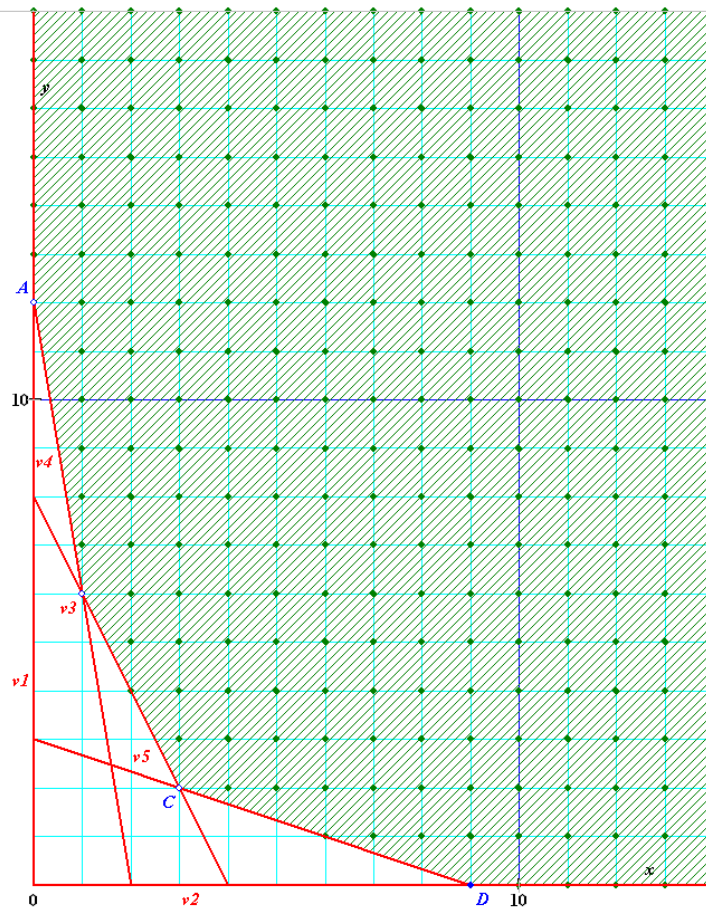
and the standard constraints

$x \geq 0$,

$y \geq 0$.

We will now graph the above system of inequalities.

The inequalities show that our region is in the first quadrant and is the intersection of the regions **above** the lines $2x + y = 8$, $6x + y = 12$, and $x + 3y = 9$. For the first line the x -intercept is $8/2=4$ and the y -intercept is 8. For the second line the x -intercept is $12/6=2$ and the y -intercept is 12. For the third line the x -intercept is 9 and the y -intercept is $9/3=3$. A computer generated graph of the region is shown below.



Our region has four vertices: A, B, C, D . Vertex A has coordinates $(0, 12)$ and vertex D – coordinates $(9, 0)$. To find the coordinates of vertex B we solve the system

$$6x + y = 12$$

$$2x + y = 8.$$

Subtracting the second equation from the first one we get $4x = 4$ whence $x = 1$ and $y = 6$.

To find the coordinates of vertex C we solve the system

$$x + 3y = 9$$

$$2x + y = 8.$$

We multiply both parts of the first equation by 2

$$2x + 6y = 18$$

$$2x + y = 8$$

And subtract the equations. Then $5y = 10$ and $y = 2$. Plugging this value of y for example into the second equation we get $2x + 2 = 8$ and $x = 3$.

The next table shows the values of the objective function C at each vertex in case (a) and in case (b).

Vertex	x	y	$(a) C = 3.40x + 1.60y$	$(b) C = 3.00x + 1.60y$
A	0	12	19.20	19.20
B	1	6	13.00	12.60
C	3	2	13.40	12.20
D	9	0	30.60	27.00

In case (a) we have the best solution when 1 pound of food A and 6 pounds of food B are made at the cost of \$13.00.

In case (b) we have the best solution when 3 pounds of food A and 2 pounds of food B are made at the cost of \$12.20.

Solve problems 3 and 4 using the simplex method.

3. Maximize $z = x_1 + x_2 + 2x_3$ subject to

$$2x_1 + 3x_2 + x_3 \leq 80$$

$$x_1 + x_2 + x_3 \leq 60$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

This is a standard maximization problem of linear programming (see definition on page 220).

We introduce two slack variables $u = 80 - 2x_1 - 3x_2 - x_3$ and $v = 60 - x_1 - x_2 - x_3$. Thus we obtain the following system of equations

$$2x_1 + 3x_2 + x_3 + u = 80$$

$$x_1 + x_2 + x_3 + v = 60$$

$$z - x_1 - x_2 - 2x_3 = 0$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, u \geq 0, v \geq 0$$

We start with the initial solution $x_1 = x_2 = x_3 = 0, z = 0, u = 80, v = 60$. Notice that in this solution the basic variables x_1, x_2, x_3 are passive (equal to 0) and the slack variables u and v are active. The initial tableau below corresponds to this solution.

	x_1	x_2	x_3	u	v	z	
u	2	3	1	1	0	0	80
v	1	1	1	0	1	0	60
z	-1	-1	-2	0	0	1	0

We look for the pivot.

The smallest number in the z -row is -2 in the x_3 -column so the pivot will be in this column. Next we look at the positive ratios of numbers in the last column to the right and the numbers in the pivot column. The ratio in the u -row is $80/1=80$ and the ratio in the v -row is $60/1=60$. Therefore the smallest ratio is in the v -row and the pivot is on the intersection of the v -row and the x_3 -column. It means that in the next solution the slack variable v will be passive (equal to 0) and the basic variable x_3 will be active.

The pivot is in the bold print in the table above. It is equal to 1 so we can start at once to eliminate the elements above and below the pivot. We do it by performing the operations $R_1 - R_2$ and $R_3 + 2R_2$. The result is the next tableau

	x_1	x_2	x_3	u	v	z	
u	1	2	0	1	-1	0	20
x_3	1	1	1	0	1	0	60
z	1	1	0	0	2	1	120

Because all the elements in the z -row are non-negative we got the best solution. In this solution only one of the basic variables - x_3 is active, and the solution is

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 60$$

$$z = 120.$$

4. A candy store plans to make three mixtures, I, II, and III, out of three different candies, A, B, and C. Mixture I sells for \$1 per pound and contains 30% of candy A, 10% of candy B, and 60% of candy C. Mixture II sells for \$2.50 per pound and contains 50% of candy A, 30% of candy B, and 20% of candy C. Mixture III sells for \$3 per pound and contains 60% of candy A, 30% of candy B, and 10% of candy C. If the owner has available 900 pounds of candy A, 750 pounds of candy B, and 425 pounds of candy C, how many pounds of each mixture should be made up in order to maximize revenue?

First we have to write our problem as a standard maximization problem of linear programming. Suppose x pounds of mixture I, y pounds of mixture II, and z pounds of mixture III are made up. We have to maximize the objective function (revenue)

$$R = x + 2.5y + 3z$$

subject to the following constraints

$$(\text{Candy A}) \quad 0.3x + 0.5y + 0.6z \leq 900$$

$$(\text{Candy B}) \quad 0.1x + 0.3y + 0.3z \leq 750$$

$$(\text{Candy C}) \quad 0.6x + 0.2y + 0.1z \leq 425$$

and, of course, the standard constraints

$$x \geq 0, y \geq 0, z \geq 0.$$

We introduce three slack variables, u , v , and w and write the following system of equations

$$0.3x + 0.5y + 0.6z + u = 900$$

$$0.1x + 0.3y + 0.3z + v = 750$$

$$0.6x + 0.2y + 0.1z + w = 425$$

$$R - x - 2.5y - 3z = 0$$

From this system we construct the initial tableau below

	x	y	z	u	v	w	R	
u	0.3	0.5	0.6	1	0	0	0	900
v	0.1	0.3	0.3	0	1	0	0	750
w	0.6	0.2	0.1	0	0	1	0	425
R	-1	-2.5	-3	0	0	0	1	0

The smallest value in the R -row – negative 3, is in the z -column. The corresponding ratios are $425/0.1=4250$, $750/0.3=2500$, $900/0.6=1500$. The smallest ratio corresponds to the u -row and the pivot is 0.6. We have to make the pivot equal to 1 so we perform the operation $R_1 \div 0.6$. The result is

	x	y	z	u	v	w	R	
u	0.5	5/6	1	5/3	0	0	0	1500
v	0.1	0.3	0.3	0	1	0	0	750
w	0.6	0.2	0.1	0	0	1	0	425
R	-1	-2.5	-3	0	0	0	1	0

Next we eliminate the elements below the pivot with the help of the operations $R_2 - 0.3R_1$, $R_3 - 0.1R_1$, and $R_4 + 3R_1$. Keep in mind that the slack variable u becomes passive and its place is taken by the basic variable z .

	x	y	z	u	v	w	R	
z	0.5	5/6	1	5/3	0	0	0	1500
v	-0.05	0.05	0	-0.5	1	0	0	300
w	0.55	7/60	0	-1/6	0	1	0	275
R	0.5	0	0	0	0	0	1	4500

We have no negative numbers in the R -row; therefore we got the best solution:

$$x = 0$$

$$y = 0$$

$$z = 1500$$

$$R = 4500.$$

Use duality to solve Problem 5.

5. Minimize $z = 8x_1 + 6x_2 + 7x_3$ subject to

$$x_1 + 2x_2 + x_3 \geq 1$$

$$-x_1 + x_2 + x_3 \geq 1$$

$$2x_1 - 3x_2 + x_3 \geq -3$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

This is a standard minimization problem of linear programming (see definition on page 249). Using the method described in our book we will formulate and solve the **dual** problem which is a standard maximization problem. The following table contains all the information about original problem. In the first three rows we see the coefficients in the constraints and their right parts, in the last row – coefficients of the objective function.

1	2	1	1
-1	1	2	1
2	-3	1	-3
8	6	7	

Reflecting the table above about the main diagonal we get the table for the dual problem.

1	-1	2	8
2	1	-3	6
1	2	1	7
1	1	-3	

We will write now the dual problem.

Maximize $z' = y_1 + y_2 - 3y_3$ subject to

$$y_1 - y_2 + 2y_3 \leq 8$$

$$2y_1 + y_2 - 3y_3 \leq 6$$

$$y_1 + y_2 + y_3 \leq 7$$

$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$$

We will solve the dual problem using the simplex method.

We introduce the slack variables (which we name as the variables in the original problem x_1, x_2 , and x_3) and write the system

$$y_1 - y_2 + 2y_3 + x_1 = 8$$

$$2y_1 + y_2 - 3y_3 + x_2 = 6$$

$$y_1 + y_2 + y_3 + x_3 = 7$$

$$z' - y_1 - y_2 + 3y_3 = 0$$

The initial tableau is

	y_1	y_2	y_3	x_1	x_2	x_3	z'	
x_1	1	-1	2	1	0	0	0	8
x_2	2	1	-3	0	1	0	0	6
x_3	1	1	1	0	0	1	0	7
z'	-1	-1	3	0	0	0	1	0

We see that the smallest elements of z' -row are in y_1 and y_2 -columns. The smallest positive ratio of the elements in last column to the right and the elements in these two columns is $6/2=3$. Hence the pivot is on the intersection of y_1 -column and v -row.

We make the pivot equal to one by performing the operation $R_2 \div 2$.

	y_1	y_2	y_3	x_1	x_2	x_3	z'	
x_1	1	-1	2	1	0	0	0	8
x_2	1	1/2	-3/2	0	1/2	0	0	3
x_3	1	1	1	0	0	1	0	7
z'	-1	-1	3	0	0	0	1	0

Next we eliminate elements above and below the pivot by performing the operations $R_1 - R_2, R_3 - R_2, R_4 + R_2$. The slack variable v becomes passive and the basic variable y_1 becomes active.

	y_1	y_2	y_3	x_1	x_2	x_3	z'	
x_1	0	-3/2	7/2	1	-1/2	0	0	5
y_1	1	1/2	-3/2	0	1/2	0	0	3
x_3	0	1/2	5/2	0	-1/2	1	0	4
z'	0	-1/2	3/2	0	1/2	0	1	3

There is a negative number in the z' -row which means that we did not reach the best solution yet. Let us look for the new pivot. The only negative number in the z' -row is in y_2 -column. The corresponding positive ratios are $3 \div (1/2) = 6$ and $4 \div (1/2) = 8$.

Therefore the pivot is on the intersection of the y_2 -column and the y_1 -row. First let us make the pivot equal to 1 using the operation $R_2 \times 2$.

	y_1	y_2	y_3	x_1	x_2	x_3	z'	
x_1	0	-3/2	7/2	1	-1/2	0	0	5
y_1	2	1	-3	0	1	0	0	6
x_3	0	1/2	5/2	0	-1/2	1	0	4
z'	0	-1/2	3/2	0	1/2	0	1	3

Next we eliminate the elements above and below the pivot. The corresponding operations are $R_1 + (3/2)R_2$, $R_3 - (1/2)R_2$, $R_4 + (1/2)R_2$. The variable y_1 becomes passive and y_2 becomes active. The resulting tableau is

	y_1	y_2	y_3	x_1	x_2	x_3	z'	
x_1	3	0	-1	1	1	0	0	14
y_2	2	1	-3	0	1	0	0	6
x_3	-1	0	4	0	-1	1	0	3
z'	1	0	0	0	1	0	1	6

We got the solution of the dual problem and we can read the values of the variables x_1 , x_2 , and x_3 of the original problem from the last tableau in the z' -row in the u , v , and w -columns. Also the value of z equals to the value of z' . We get

$$x_1 = 0$$

$$x_2 = 1$$

$$x_3 = 0$$

$$z = 6.$$