Problem 1. Consider the quadratic function

$$y = -x^2 + 3x + 1$$

(a) Find the coordinates of the vertex. (2 points)

$$x_v = -\frac{b}{2a} = -\frac{3}{2\times(-1)} = \frac{3}{2}.$$

$$y_v = c - \frac{b^2}{4a} = 1 - \frac{9}{-4} = \frac{13}{4}$$

(b) Find the *x*-intercepts (if any). (2 points)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4\times(-1)\times 1}}{2\times(-1)}.$$

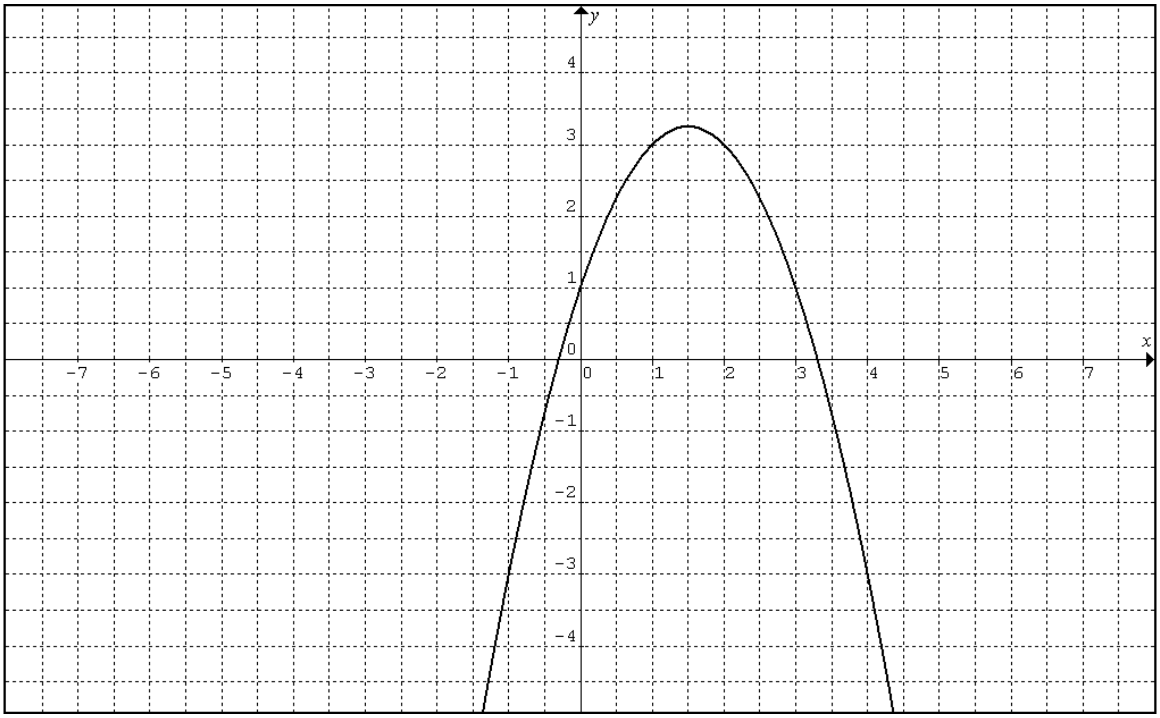
$$x_1 = \frac{3 - \sqrt{13}}{2} \approx -.3,$$

$$x_2 = \frac{3 + \sqrt{13}}{2} \approx 3.3.$$

(c) Find the range of the function. (2 points)

The coefficient a is negative; therefore the range is from the vertex down: $(-\infty, 13/4]$

(d) Graph the function. (4 points)



Problem 2. A rectangular plot of land on the edge of a river is to be enclosed with fence on three sides. Find the dimensions of the rectangular enclosure of the greatest area if the side that goes along the river does not require fencing and the total length of the fence is 200 m. (10 points)

Let *x* be the side along the river and *y* be the perpendicular side. Then

$$x + 2y = 200.$$

Whence

$$x = 200 - 2y.$$

The area of the rectangle is

$$xy = (200 - 2y)y = -2y^2 + 200y.$$

The greatest value of this quadratic function will be at the vertex

$$y = -\frac{200}{2\times(-2)} = 50.$$

Then $x = 200 - 2\times 50 = 100.$

Problem 3. for the polynomial function

$$P(x) = x^2(x^2 - 4)$$

(a) find the x-intercepts and the sign of the function (3 points)
(b) find the critical points (5 points)
(c) graph the function (5 points)

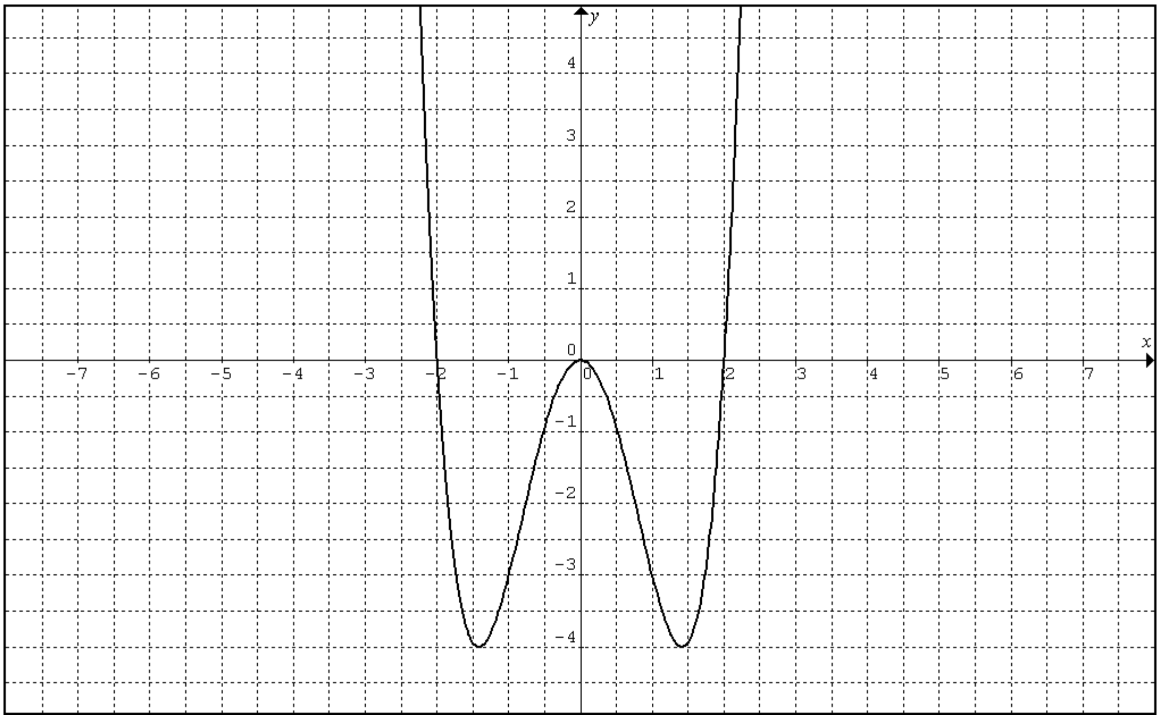
Solution. (a)The x-intercepts are at $-2, 0$, and 2. The function is positive to the left of -2 (because the leading term is x^4. It changes sign at -2 and at 2 (because they are simple roots), but does not change sign at 0 (because it is a root of multiplicity 2).
(b) Because the function is even there is one critical point at $(0,0)$. To find if there are other critical points we will solve the equation $x^4 - 4x^2 - y = 0$ for x. The quadratic formula provides

$$x^2 = \frac{4 \pm \sqrt{16 + 4y}}{2} = 2 \pm \sqrt{4 + y}$$

At each of the remaining critical points we have $y = -4$ whence $x^2 = 2$. The list of critical points is

$$\left\{\left(-\sqrt{2}, -4\right), (0,0), \left(\sqrt{2}, -4\right)\right\}$$

(c) The graph is shown below



Problem 4. for the function

$$f(x) = \sqrt{x-1} + 2$$

(a) Find the domain. (2 points)

The function is defined when $x - 1 \geq 0$ **whence** $x \geq 1$**. The domain is the interval** $[1, \infty)$

(b) Find the range. (2 points)

Clearly $f(x) \geq 2$ **and** f **takes any value in the interval** $[2, \infty)$ **whence the range of** f **is** $[2, \infty)$

(c) Find the inverse function, its domain and range. (5 points)

We will solve the equation $y = \sqrt{x-1} + 2$ **for** x

$$y-2=\sqrt{x-1},$$
$$(y-2)^2=x-1,$$
$$y^2-4y+4=x-1,$$
$$x=y^2-4y+5.$$

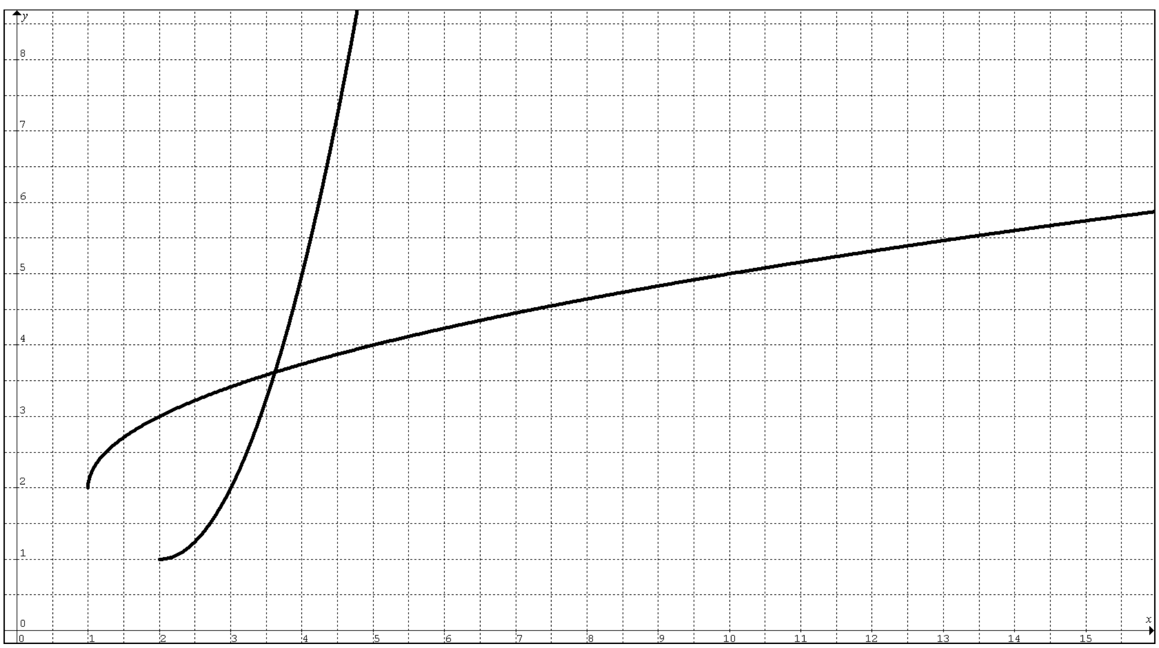
The inverse function is

$$y^{-1}(x)=x^2-4x+5, \quad x\in[2,\infty)$$

The domain of the inverse function is the range of the original function; in our case it is the interval $[2,\infty)$**.**

The range of $y^{-1}(x)$ **is the domain of the original function, i.e. the interval** $[1,\infty)$**.**

(d) Graph the function and its inverse in the same coordinate system. (5 points)



Problem 5. for the rational function

$$f(x)=\frac{x^2-x-1}{x+1}$$

(a) Find equations of the vertical and the slant asymptote. (3 points)

The denominator is equal to 0 **if** $x=1$ **whence the vertical asymptote has equation** $x=1$**.**

To find the slant asymptote we apply the synthetic division

-1	**1**	**-1**	**-1**
		-1	**2**
	1	**-2**	**1**

The quotient is $x-2$ **whence an equation of the slant asymptote is**

$$y=x-2$$

(b) Find the *x* and *y* intercepts, if any, and the sign of the function (3 points)

The y**-intercept clearly is** $(0,-1)$**. We find the** x**-intercepts by solving the quadratic equation** $x^2-x-1=0$**. The** x**-intercepts are**

$$\left\{\left(\frac{1-\sqrt{5}}{2},0\right),\left(\frac{1+\sqrt{5}}{2},0\right)\right\}$$

The sign of the function is shown in the table below

Inter -val	$(-\infty,-1)$	$\left(-1,\frac{1-\sqrt{5}}{2}\right)$	$\left(\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right)$	$\left(\frac{1+\sqrt{5}}{2},\infty\right)$
Sign	**minus**	**plus**	**minus**	**plus**

(c) Find the critical points, if any, and the range of the function.

We proceed with solving the equation $y=\dfrac{x^2-x-1}{x+1}$

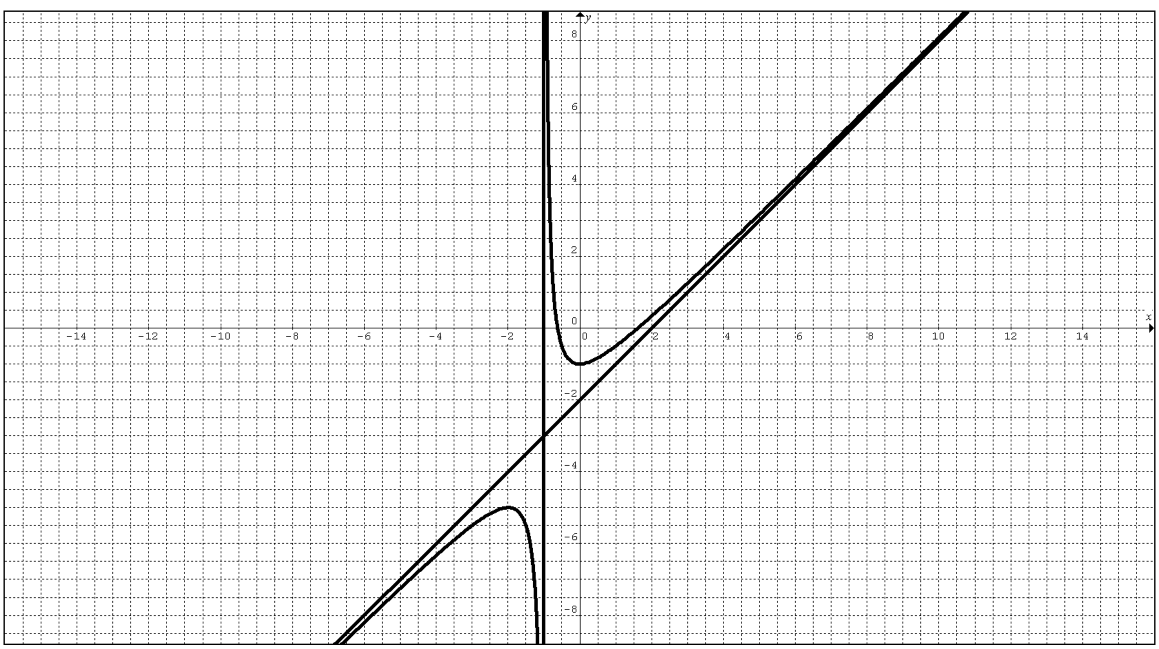
for x **.**

$$yx + y = x^2 - x - 1,$$
$$x^2 - (1+y)x - (1+y) = 0,$$
$$x = \frac{1 + y \pm \sqrt{(1+y)^2 + 4(1+y)}}{2},$$
$$x = \frac{1 + y \pm \sqrt{(y+1)(y+5)}}{2}$$

The equation $(y+5)(y+1) = 0$ provides two critical points. The first one is $(-2,-5)$ and the second $(-1,0)$. The range of the function is

$$(-\infty,-5] \cup [-1,\infty)$$

(d) Graph the function together with its vertical and slant asymptotes.



Problem 6. Consider the following logistic function

$$y = \frac{10}{2+3e^{-x}}$$

(a) Find the range of the function.(2points)

For very large negative values of x the expression e^{-x} takes very large positive values, whence y takes values close to 0. On the other hand, if x is a large positive number, then e^{-x} is close to 0 and the function takes values close to 5. The range therefore is $(0,5)$.

(b) Find the inverse function. (3points)

$$2y + 3ye^{-x} = 10,$$

$$e^{-x} = \frac{10-2y}{3y},$$

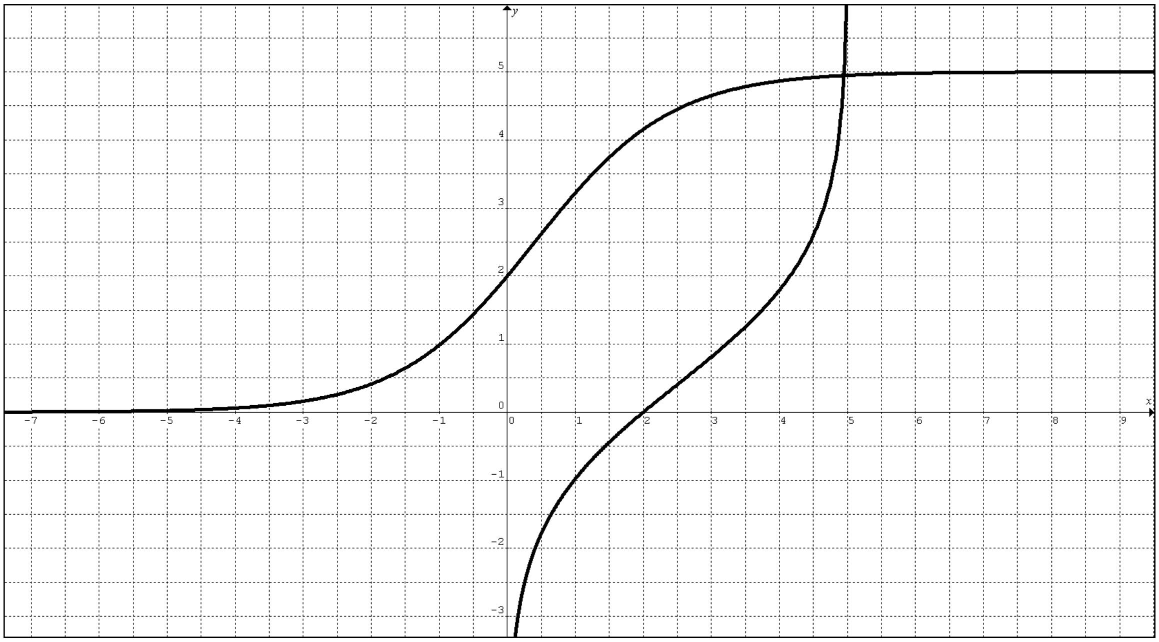
$$-x = \ln\left(\frac{10-2y}{3y}\right),$$

$$x = \ln\left(\frac{3y}{10-2y}\right).$$

Therefore

$$y^{-1}(x) = \ln\left(\frac{3x}{10-2x}\right), \quad 0 < x < 5.$$

(c) Graph the function together with its inverse



Problem 7. The half-life of radon $\left({}^{222}\mathrm{Rn}\right)$ is 3.82 days. If the initial amount of radon is 10g after what time in days it will remain 2g of radon? Round to the nearest hundredth of a day. (5 points)

The equation of radioactive decay is

$$A(t) = A(0)e^{-kt},$$

where the constant k and the half-live T are connected as $kT = \ln 2$. In our case $k = \dfrac{\ln 2}{3.82} \approx 0.1815$. It remains to solve the equation

$$\begin{aligned}
&10e^{-0.1815t} = 2,\\
&e^{-0.1815t} = 0.2,\\
&-0.1815t = \ln 0.2,\\
&t = -\frac{\ln 0.2}{0.1815} \approx 8.87
\end{aligned}$$

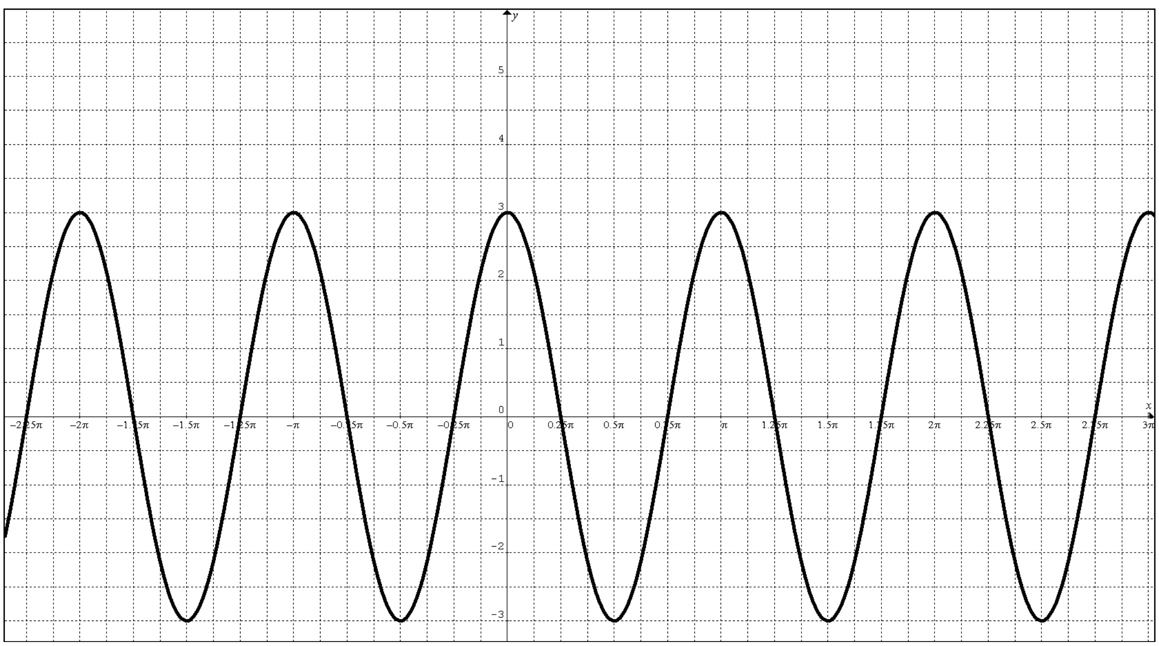
Problem 8 Consider the function $y = 3\sin(2x + \pi/2)$.
(a) Find the amplitude, the period, and the shift (3 points)

The amplitude is 3.

The period is $\dfrac{2\pi}{2} = \pi$.

The shift is $-\pi/2 \div 2 = -\dfrac{\pi}{4}$.

(b) Graph the function. (5 points)



(c) Find the standard domain where the function is one-to-one and find the inverse function. (5points)

The domain is defined by the inequality $-\dfrac{\pi}{2} \le 2x + \dfrac{\pi}{2} \le \dfrac{\pi}{2}$ **whence**

$$-\frac{\pi}{2} \le x \le 0$$

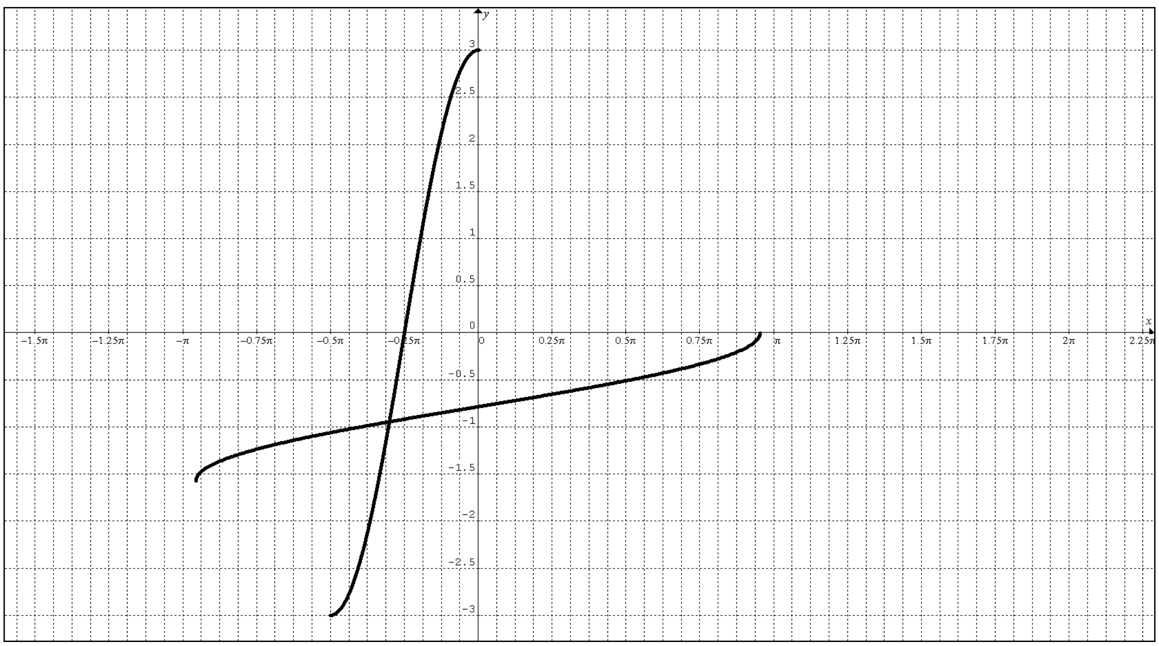
To find the inverse function we write

$$\sin(2x+\pi/2)=y/3,$$
$$2x+\pi/2=\arcsin(y/3),$$
$$x=\frac{1}{2}\arcsin\left(\frac{y}{3}\right)-\frac{\pi}{4}.$$

Whence

$$y^{-1}(x)=\frac{1}{2}\arcsin\left(\frac{x}{3}\right)-\frac{\pi}{4}.$$

(d) Graph the function with restricted domain and the inverse function in the same coordinate system (5 points)



Problem 9 verify the identity. (5 points)

$$\frac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)}=\frac{1+\cos x}{3-\cos x}$$

We will work with the left part of the identity and apply the power reduction formula

$$\sin^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{2}$$

Then the left part becomes

$$\frac{1-\frac{1-\cos x}{2}}{1+\frac{1-\cos x}{2}}=\frac{2-(1-\cos x)}{2+(1-\cos x)}=\frac{1+\cos x}{3-\cos x}$$

Problem 10 Find all the solutions of the equation
$2\cos^2 x+3\cos x=-1$. **(5points)**

Writing the equation as $2\cos^2 x+3\cos x+1=0$ **and factoring the left part we get**

$$(2\cos x+1)(\cos x+1)=0$$

The problem now is reduced to solving two basic equations

$$\cos x=-1$$

which has solutions

$$x=(2n+1)\pi,\quad n\in Z$$

and

$$\cos x=-\frac{1}{2}$$

with the solutions

$$x=\pm\arccos\left(-\frac{1}{2}\right)+2n\pi=\pm\frac{2\pi}{3}+2n\pi,\quad n\in Z.$$