105 Algebra and Trigonometry **Review 7**

Problem 1. Verify the identity

$$\frac{\sin x+\cos x}{\sin x-\cos x}=\frac{1+2\sin x\cos x}{2\sin^2 x-1} \quad (?)$$

Solution. To verify that two fractions are identical it is enough to verify that the cross-products are identical, i.e. that

$$2\sin^3 x-\sin x+2\sin^2 x\cos x-\cos x=$$
$$=\sin x-\cos x+2\sin^2 x\cos x-2\sin x\cos^2 x \quad (?)$$

After canceling out identical terms in the left and the right parts we get

$$2\sin^3 x-\sin x=\sin x-2\sin x\cos^2 x \quad (?)$$

Dividing both parts by $\sin x$ **provides**

$$2\sin^2 x-1=1-2\cos^2 x \quad (?)$$

Finally, moving $-2\cos^2 x$ **to the left and** -1 **to the right we get**

$$2\sin^2 x+2\cos^2 x=2,$$

which is obviously equivalent to the first Pythagorean identity.

Problem 2. Verify the identity

$$\cot^2 \frac{u}{2} = \frac{\csc u + \cot u}{\csc u - \cot u} \quad (?)$$

Solution. We will work with the right part of the alleged identity

$$\frac{\csc u + \cot u}{\csc u - \cot u} = \frac{1/\sin u + \cos u/\sin u}{1/\sin u - \cos u/\sin u}$$

Multiplying both the numerator and the denominator of the right part by $\sin u$ **we get the expression** $\frac{1+\cos u}{1-\cos u}$**. Finally, recalling that by power reduction formulas** $1+\cos u = 2\cos^2 \frac{u}{2}$ **and** $1-\cos u = 2\sin^2 \frac{u}{2}$ **we obtain the desired identity.**

In problems 3 – 5

(a) Describe all the solutions of the given trigonometric equation.

(b) List the solutions in the interval $[0, 2\pi)$ (if any) and approximate them with the accuracy 0.0001.

Problem 3 $\sec^2 x - 5\sec x + 1 = 0.$

Solution. (a) We have here an equation of the quadratic type. Applying the quadratic formula we obtain

$$\sec x = \frac{5 \pm \sqrt{21}}{2}$$

The sign minus provides a positive value smaller than 1 which is not in the range of $\sec x$; therefore we only have to consider the possibility

$$\sec x = \frac{5 + \sqrt{21}}{2}.$$

Or, equivalently

$$\cos x = \frac{2}{5+\sqrt{21}} = \frac{2\left(5-\sqrt{21}\right)}{\left(5+\sqrt{21}\right)\left(5-\sqrt{21}\right)} =$$

$$= \frac{2\left(5-\sqrt{21}\right)}{4} = \frac{5-\sqrt{21}}{2}.$$

All the solutions of the last equation can be described as

$$x = \pm\arccos\left(\frac{5-\sqrt{21}}{2}\right) + 2n\pi, \quad n \in Z$$

(b) There are two solutions in the interval $[0, 2\pi)$**:**

$$x = \arccos\left(\frac{5-\sqrt{21}}{2}\right) \approx 1.3605.$$

And

$$x = 2\pi - \arccos\left(\frac{5-\sqrt{21}}{2}\right) \approx 4.923$$

Problem 4 $2\sin x - 3\cos x = 1.$

Solution (a) We transform the left part of the equation according to the formula

$$A\sin x \pm B\cos x = \sqrt{A^2+B^2}\sin\left(x \pm \arctan\left(B/A\right)\right).$$

The equation now can be written as

$$\sqrt{13}\sin(x-\arctan(3/2)) = 1$$

Or

$$\sin(x-\arctan(3/2)) = 1/\sqrt{13} = \sqrt{13}/13.$$

From here

$$x-\arctan(3/2) = (-1)^n \arcsin\left(\sqrt{13}/13\right) + n\pi, \quad n \in Z$$

And finally

$$x = (-1)^n \arcsin\left(\sqrt{13}/13\right) + \arctan(3/2) + n\pi, \quad n \in Z$$

(b) We have two solutions in the interval $[0, 2\pi)$ corresponding to the values $n = 0$ and $n = 1$.

$$x = \arcsin\left(\sqrt{13}/13\right) + \arctan(3/2) \approx 1.2638$$

And

$$x = -\arcsin\left(\sqrt{13}/13\right) + \arctan(3/2) + \pi \approx 3.8434.$$

Problem 5 $\cos 3x - \cos 7x = \sqrt{2}\sin 2x.$

Solution (a) We will transform the left part according to the sum-to-product formula

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

Then the equation becomes

$$2\sin 5x \sin 2x = \sqrt{2}\sin 2x$$

Or

$$\sin 2x(2\sin 5x - \sqrt{2}) = 0$$

If $\sin 2x = 0$ **then** $2x = n\pi, \quad n \in Z$**, or**

$$x = n\frac{\pi}{2}, \quad n \in Z \quad (*)$$

If $\sin 5x = \sqrt{2}/2$ **then**

$5x = (-1)^n \arcsin\left(\sqrt{2}/2\right) + n\pi = (-1)^n (\pi/4) + n\pi, \quad n \in Z$**,**

Or

$$x = (-1)^n (\pi/20) + n(\pi/5), \quad n \in Z \quad (**)$$

Formulas (*) and (**) together define all the solutions of the equation

(b) It is easy to see the solutions provided by (*) in the interval $[0, 2\pi)$**are**

$$0, \pi/2, \pi, 3\pi/2$$

And the solutions provided by (**) in the same interval are

$$\pi/20, 3\pi/20, 9\pi/20, 11\pi/20, 17\pi/20, 19\pi/20, 25\pi/20, 27\pi/20,$$
$$33\pi/20, 35\pi/20.$$