Problem 1 Consider logistic function $y = \dfrac{5}{2+3\cdot 2^{-x}}$ **.**

(a) Describe the domain and the range of y **.**

(b) Prove that y **is one-to-one and find its inverse.**

(c) Graph y **and its inverse in the same coordinate system.**

Solution (a) y **is defined for any value of** x **and therefore its domain is** $(-\infty,\infty)$ **.**

To see what is its range notice that $0 < 3\cdot 2^{-x} < \infty$ **, whence** $2 < 2+3\cdot 2^{-x} < \infty$ **. Recalling that if** $a > b$ **then** $1/a < 1/b$ **we get**

$$0 < \frac{5}{2+3\cdot 2^{-x}} < \frac{5}{2}$$

Moreover, because the function $3\cdot 2^{-x}$ **takes all values in the interval** $(0,\infty)$ **, the range of** y **is exactly the interval** $(0,5/2)$ **.**

(b) Solving the equation $y = \dfrac{5}{2+3\cdot 2^{-x}}$ **for** x **we obtain**

$$2y + 3\cdot 2^{-x} y = 5,$$

$$3\cdot 2^{-x} y = 5 - 2y,$$

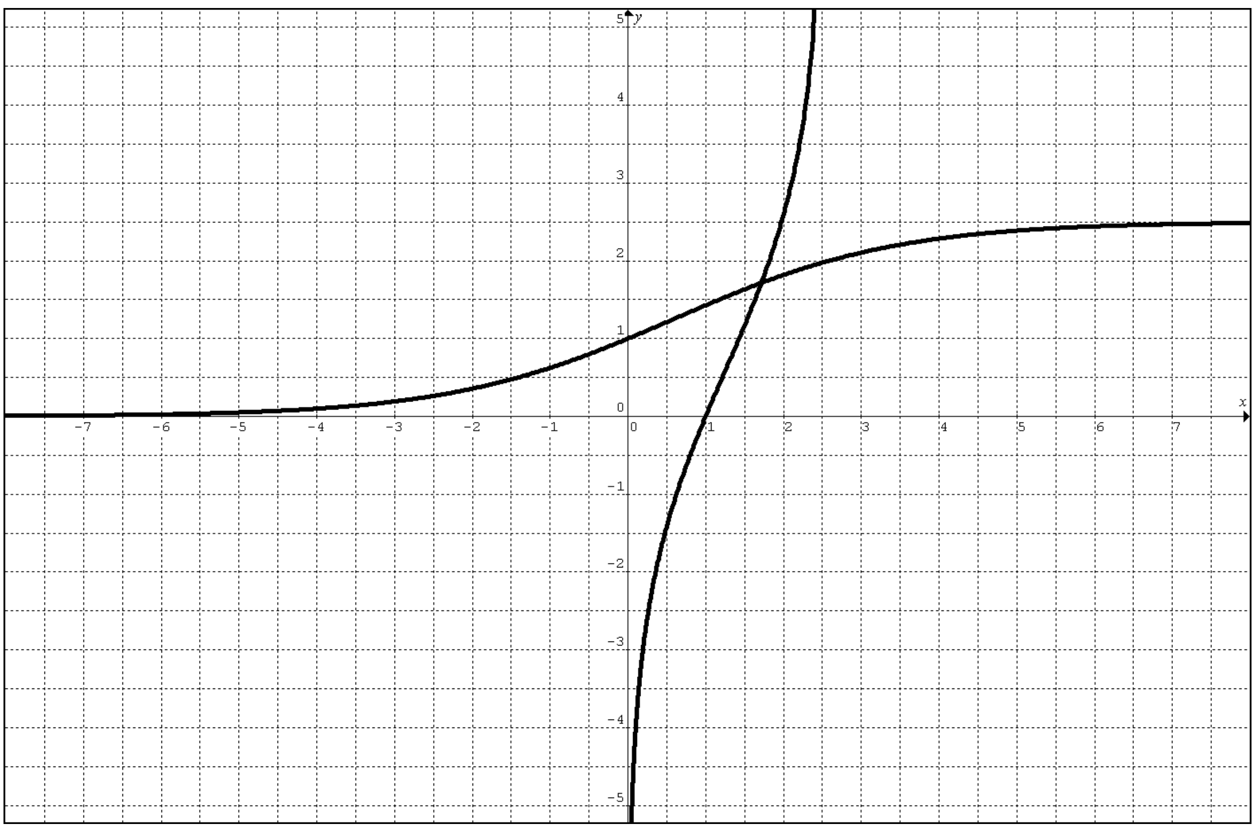
$$2^{-x} = \frac{5-2y}{3y},$$

$$-x = \log_2 \frac{5-2y}{3y},$$

$$x = -\log_2 \frac{5-2y}{3y} = \log_2 \frac{3y}{5-2y}.$$

Therefore the input x is defined in the unique way as soon as we know the output y, and the inverse function is $y^{-1}(x) = \log_2 \frac{3x}{5-2x}$.

(c) The graphs of y and y^{-1} are shown below.



Problem 2 solve the exponential equation $3\cdot 5^{2x}+2\cdot 5^{x}-2=0$ **and approximate its solutions with the accuracy** 0.0001.

Solution. It is an equation of quadratic type $3\cdot\left(5^{x}\right)^{2}+2\cdot 5^{x}-2=0$.

Applying the quadratic formula we see that

$$5^{x}=\frac{-2\pm\sqrt{2^{2}-4\cdot 3(-2)}}{2a}=\frac{-2\pm\sqrt{28}}{6}=\frac{-1\pm\sqrt{7}}{3}.$$

Because 5^{x} **cannot be negative the only possibility is** $5^{x}=\frac{-1+\sqrt{7}}{3}$ **whence**

$$x=\log_{5}\frac{-1+\sqrt{7}}{3}=\frac{\log\frac{-1+\sqrt{7}}{3}}{\log 5}\approx -0.3731.$$

Problem 3 solve the logarithmic equation $\ln(3x+2)+\ln(2x+3)=5$ **and approximate its solutions with the accuracy** 0.0001.

Solution By the first law of logarithms the equation can be written as

$\ln(6x^{2}+13x+6)=5$**, or** $6x^{2}+13x+6=e^{5}$**. Applying the quadratic formula we get** $x=\frac{-13\pm\sqrt{13^{2}-4\cdot 6\cdot(6-e^{5})}}{12}$**. It is easy to see that we cannot take sign minus in the above expression because then the expressions** $3x+2$ **and** $2x+3$ **will be negative and their logarithms – undefined. Therefore the equation has one solution**

$$x=\frac{-13+\sqrt{169+24(e^{5}-6)}}{12}\approx 3.9076$$

Problem 4 an artifact is discovered at a certain site. If it has 52% of carbon-14 it originally contained, what is the approximate age of the artifact to the nearest year? (Carbon-14 decays at the rate 0.0125% annually.)

Solution According to the model of radioactive decay the amount of carbon-14 after t years is given by the formula $A(t) = A_0 e^{-kt}$, where A_0 is the initial amount of carbon-14. We can find the coefficient k from the conditions of the problem in the following way

$$e^{-k} = \frac{A(1)}{A_0} = \frac{A_0 - 0.000125A_0}{A_0} = .999875,$$
$$k = -\ln(.999875) \approx .000125.$$

Now we can reduce the problem to solving for t the equation

$$e^{-0.000125t} = 0.52,$$

Whence

$$-.000125t = \ln(0.52),$$
$$t = \frac{\ln(0.52)}{-.000125} \approx 5231$$

Problem 5 A lake is stocked with 438 fish of a new variety. The size of the lake, the availability of food, and the number of other fish restrict growth in the lake to a limiting value of 2737. The population of fish in the lake after time t, in months, is given by the function

$$\mathrm{P}\;\;\mathrm{t}\;\; = \frac{2737}{1+6.15e^{-0.35t}}$$

After how many months will the population be 1323?

Solution all we have to do is to solve the equation $\dfrac{2737}{1+6.15e^{-0.35t}} = 1321$.

The solution goes as follows

$$\begin{aligned}
2737 &= 1321 + 8124.15e^{-0.35t}, \\
e^{-0.35t} &= \frac{2737-1321}{8124.15} \approx 0.1743, \\
-0.35t &= \ln(0.1743), \\
t &= \frac{\ln(0.1743)}{-0.35} \approx 5
\end{aligned}$$