Problem 1 Consider logistic function $y = \frac{5}{2 + 3 \cdot 2^{-x}}$.

(a) Describe the domain and the range of $y$.

(b) Prove that $y$ is one-to-one and find its inverse.

(c) Graph $y$ and its inverse in the same coordinate system.

Solution (a) $y$ is defined for any value of $x$ and therefore its domain is $(-\infty, \infty)$.

To see what is its range notice that $0 < 3 \cdot 2^{-x} < \infty$, whence $2 < 2 + 3 \cdot 2^{-x} < \infty$. Recalling that if $a > b$ then $1/a < 1/b$ we get

$$0 < \frac{5}{2 + 3 \cdot 2^{-x}} < \frac{5}{2}$$

Moreover, because the function $3 \cdot 2^{-x}$ takes all values in the interval $(0, \infty)$, the range of $y$ is exactly the interval $(0, 5/2)$.

(b) Solving the equation $y = \frac{5}{2 + 3 \cdot 2^{-x}}$ for $x$ we obtain

$$2y + 3 \cdot 2^{-x}y = 5,$$

$$3 \cdot 2^{-x}y = 5 - 2y,$$

$$2^{-x} = \frac{5 - 2y}{3y},$$
\[-x = \log_2 \frac{5 - 2y}{3y},\]
\[x = -\log_2 \frac{5 - 2y}{3y} = \log_2 \frac{3y}{5 - 2y}.\]

Therefore the input \(x\) is defined in the unique way as soon as we know the output \(y\), and the inverse function is \(y^{-1}(x) = \log_2 \frac{3x}{5 - 2x}\).

(c) The graphs of \(y\) and \(y^{-1}\) are shown below.
Problem 2 solve the exponential equation $3 \cdot 5^{2x} + 2 \cdot 5^x - 2 = 0$ and approximate its solutions with the accuracy 0.0001.

Solution. It is an equation of quadratic type $3 \cdot 5^x^2 + 2 \cdot 5^x - 2 = 0$.
Applying the quadratic formula we see that

$$5^x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3 \cdot (-2)}}{2a} = \frac{-2 \pm \sqrt{28}}{6} = \frac{-1 \pm \sqrt{7}}{3}.$$ 

Because $5^x$ cannot be negative the only possibility is $5^x = \frac{-1 + \sqrt{7}}{3}$ whence

$$x = \log_5 \frac{-1 + \sqrt{7}}{3} = \frac{\log \frac{-1 + \sqrt{7}}{3}}{\log 5} \approx -0.3731.$$ 

Problem 3 solve the logarithmic equation $\ln(3x + 2) + \ln(2x + 3) = 5$ and approximate its solutions with the accuracy 0.0001.

Solution By the first law of logarithms the equation can be written as

$$\ln(6x^2 + 13x + 6) = 5, \text{ or } 6x^2 + 13x + 6 = e^5.$$ 

Applying the quadratic formula we get

$$x = \frac{-13 \pm \sqrt{13^2 - 4 \cdot 6 \cdot (6 - e^5)}}{12}.$$ 

It is easy to see that we cannot take sign minus in the above expression because then the expressions $3x + 2$ and $2x + 3$ will be negative and their logarithms – undefined. Therefore the equation has one solution

$$x = \frac{-13 + \sqrt{169 + 24(e^5 - 6)}}{12} \approx 3.9076.$$
Problem 4 an artifact is discovered at a certain site. If it has 52% of carbon-14 it originally contained, what is the approximate age of the artifact to the nearest year? (Carbon-14 decays at the rate 0.0125% annually.)

Solution According to the model of radioactive decay the amount of carbon-14 after $t$ years is given by the formula $A(t) = A_0 e^{-kt}$, where $A_0$ is the initial amount of carbon-14. We can find the coefficient $k$ from the conditions of the problem in the following way

$$e^{-k} = \frac{A(1)}{A_0} = \frac{A_0 - 0.000125A_0}{A_0} = .999875,$$

$$k = -\ln(.999875) \approx .000125.$$

Now we can reduce the problem to solving for $t$ the equation

$$e^{-0.000125t} = 0.52,$$

Whence

$$-.000125t = \ln(0.52),$$

$$t = \frac{\ln(0.52)}{-0.000125} \approx 5231.$$
Problem 5 A lake is stocked with 438 fish of a new variety. The size of the lake, the availability of food, and the number of other fish restrict growth in the lake to a limiting value of 2737. The population of fish in the lake after time \( t \), in months, is given by the function

\[
P_t = \frac{2737}{1 + 6.15e^{-0.35t}}
\]

After how many months will the population be 1323?

Solution all we have to do is to solve the equation \( \frac{2737}{1 + 6.15e^{-0.35t}} = 1321 \).

The solution goes as follows

\[
2737 = 1321 + 8124.15e^{-0.35t},
\]

\[
e^{-0.35t} = \frac{2737 - 1321}{8124.15} \approx 0.1743,
\]

\[-0.35t = \ln(0.1743),
\]

\[
t = \frac{\ln(0.1743)}{-0.35} \approx 5
\]