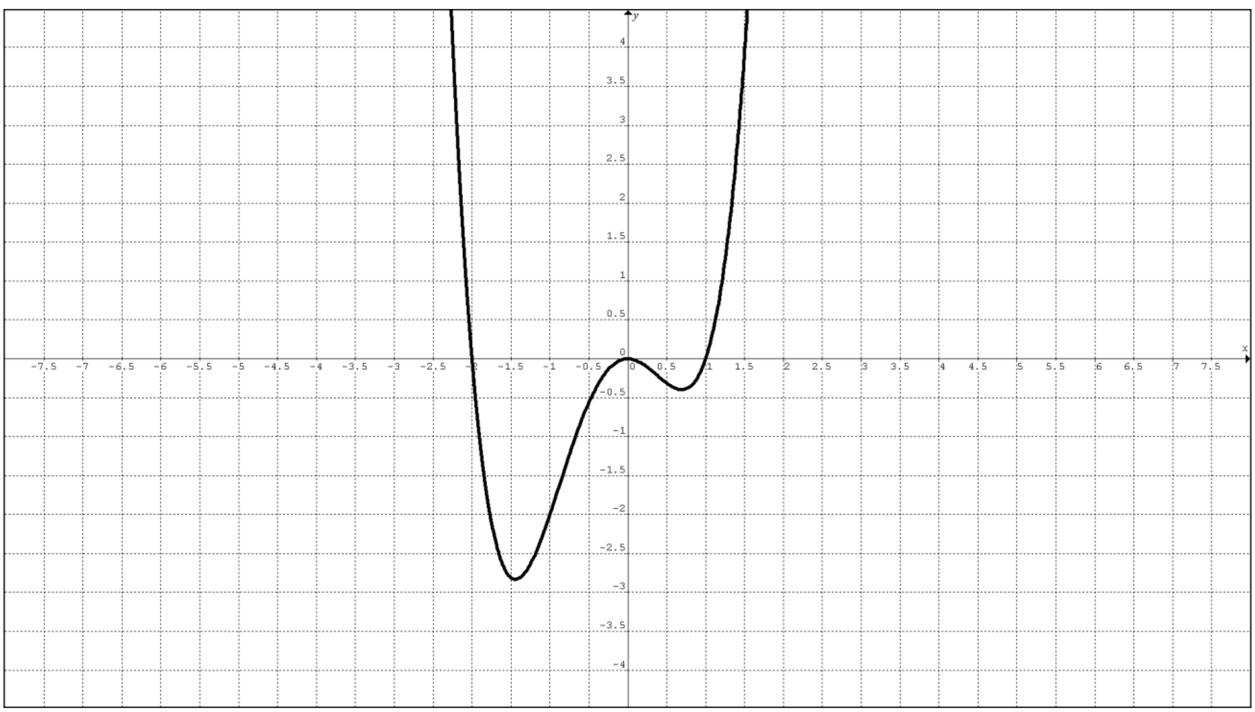
105 Algebra and Trigonometry **Review 4**

Problem 1 Graph the polynomial function $P(x)=(x+2)x^2(x-1)$.

Solution The polynomial is of degree 4 **and therefore it is positive to the left of its smallest real root and to the right of its largest real root. The roots of the polynomial are** $-2, 0,$ and 1. **The sign of the polynomial changes at** -2 **and at** 1, **because they are simple roots, and does not change at** 0 **because** 0 **is a root of multiplicity two. The sign of the polynomial is shown in the table below**

Interval	$(-\infty,-2)$	$(-2,0)$	$(0,1)$	$(1,\infty)$
Sign	**+**	**-**	**-**	**+**

The graph of the polynomial is shown below.



Problem 2 **Consider the polynomial** $P(x)=x^4-3x^2-1$.

(a) Find the x**-intercepts.**
(b) Find the critical points and the range.
(c) Graph the polynomial

Solution (a) to find the x**-intercepts we have to find the real solutions of the equation** $x^4-3x^2-1=0$. **This equation is of quadratic type and applying the quadratic formula we get**

$$x^2=\frac{-(-3)\pm\sqrt{(-3)^2-4\cdot1\cdot(-1)}}{2a}=\frac{3\pm\sqrt{13}}{2}$$

Sign minus does not provide any real solutions and taking sign plus we obtain

$$x=\pm\sqrt{\frac{3+\sqrt{13}}{2}}\approx\pm1.82$$

(b) To find the range and the position of critical points we have to solve the equation $y=x^4-3x^2-1$ **for** x. **Applying again the quadratic formula we see that**

$$x^2=\frac{3\pm\sqrt{9+4(y+1)}}{2}$$

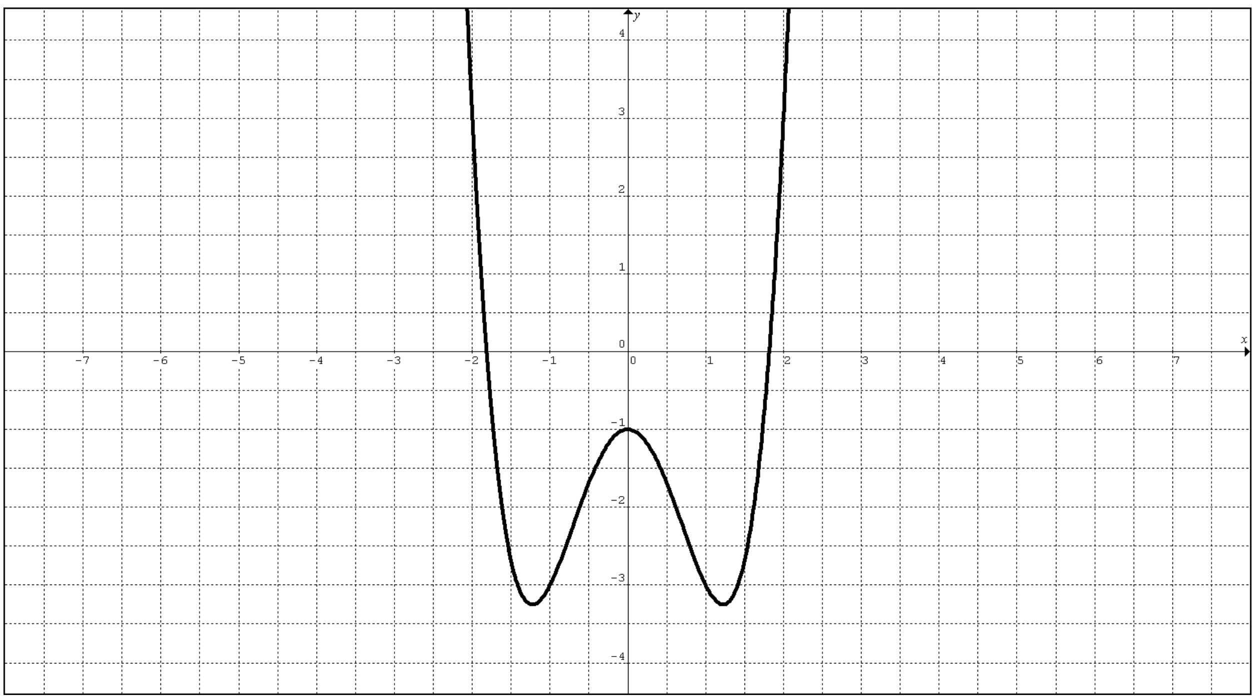
The expression under square root cannot be negative whence $4y+13\geq0$.

Therefore $y\geq-\dfrac{13}{4}$ **and the range of** P **is** $[-13/4,\infty)$.

There are two critical points corresponding to the value $y = -13/4$. For this value of y we obtain $x^2 = \frac{3}{2}$ and $x = \pm\sqrt{3/2} \approx \pm 1.22$. Because polynomial P is an even function there is one more critical point at $(0,-1)$. The list of critical points is

$$(-\sqrt{3/2}, -13/4), (0,-1), (-\sqrt{3/2}, -13/4)$$

(c) The graph of P is shown below.



Problem 3 Consider the linear fraction $R(x)=\dfrac{2x+3}{3x-4}$.

(a) Find the x **and** y **-intercepts**

(b) Find equations of the horizontal and the vertical asymptote

(c) Graph the function together with its horizontal and vertical asymptotes

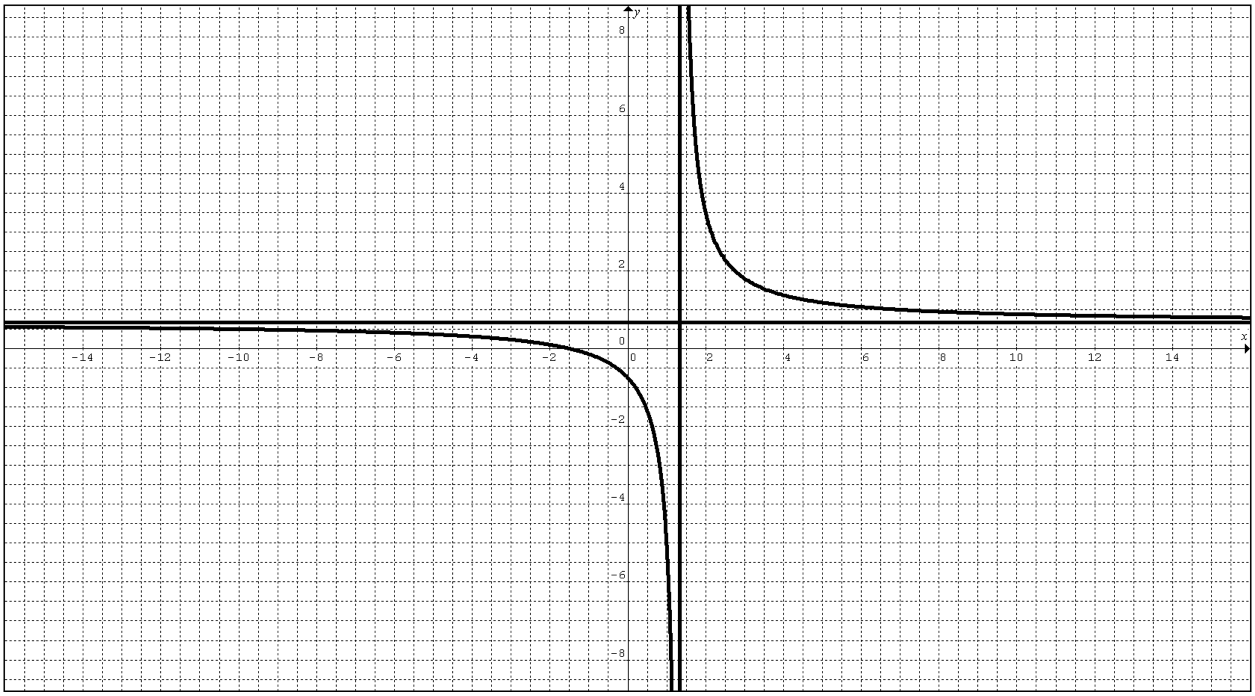
(d) Find an equation of the inverse function and graph it together with its horizontal and vertical asymptotes

Solution (a) to find the x **-intercept we solve the equation** $y=0$ **which is equivalent to** $2x+3=0$ **whence** $x=-3/2$ **and the** x **intercept is** $(-3/2,0)$. **To find the** y **-intercept we just notice that if** $x=0$ **then** $y=-3/4$ **and the** y **-intercept is** $(0,-3/4)$.

(b) Looking at the ratio of leading terms $\dfrac{2x}{3x}=\dfrac{2}{3}$ **we see that an equation of the horizontal asymptote is** $y=\dfrac{2}{3}$.

The function is undefined if $x=4/3$ **whence an equation of the vertical asymptote is** $x=4/3$.

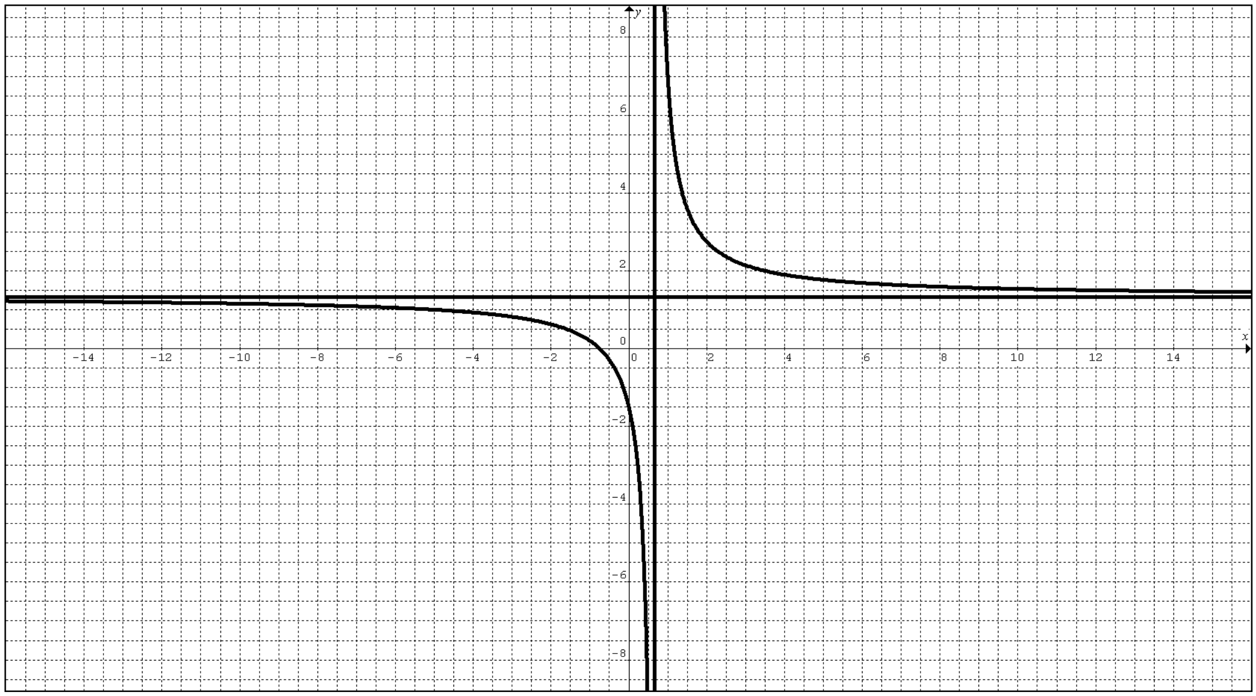
(c) The graph is shown below



(d) Solving the equation $y = \dfrac{2x+3}{3x-4}$ **for** x **we get** $3xy - 4y = 2x + 3$ **whence** $(3y-2)x = 4y+3$ **and** $x = \dfrac{4y+3}{3y-2}$**. The inverse function is**

$$R^{-1}(x) = \frac{4x+3}{3x-2}$$

The x **and** y **-intercepts are, respectively,** $(-3/4, 0)$ **and** $(0, -3/2)$**. The horizontal asymptote is** $y = 4/3$**, the vertical asymptote is** $x = 2/3$**. The graph is shown below.**


y
x
8
6
4
2
0
-2
-4
-6
-8
-14
-12
-10
-8
-6
-4
-2
0
2
4
6
8
10
12
14

Problem 4 Consider the rational function $R(x) = \dfrac{x^2 + x + 1}{x + 1}$.

(a) **Find the** x **and the** y**-intercepts, if any**

(b) **Find an equation of the vertical and the slant asymptote**

(c) **Find the critical points, if any, and the range of the function**

(d) **Graph the function together with its vertical and slant asymptotes**

Solution (a) the equation $x^2 + x + 1 = 0$ **has no real solutions whence there are no** x**-intercepts. The** y**-intercept i** $(0,1)$.

(b) The vertical asymptote is clearly $x = -1$. **To find an equation of the slant asymptote we will divide** $x^2 + x + 1$ **by** $x + 1$ **using e.g. the synthetic division**

-1	**1**	**1**	**1**
		-1	**0**
	1	**0**	**1**

The quotient is x **whence an equation of the slant asymptote is**

$$y = x$$

(c)To find the range and the critical points (if they exist) we have to solve the equation $y = \dfrac{x^2 + x + 1}{x + 1}$ **for** x. **This equation is equivalent to the following quadratic equation** $x^2 + (1 - y)x + (1 - y) = 0$. **The quadratic formula provides**

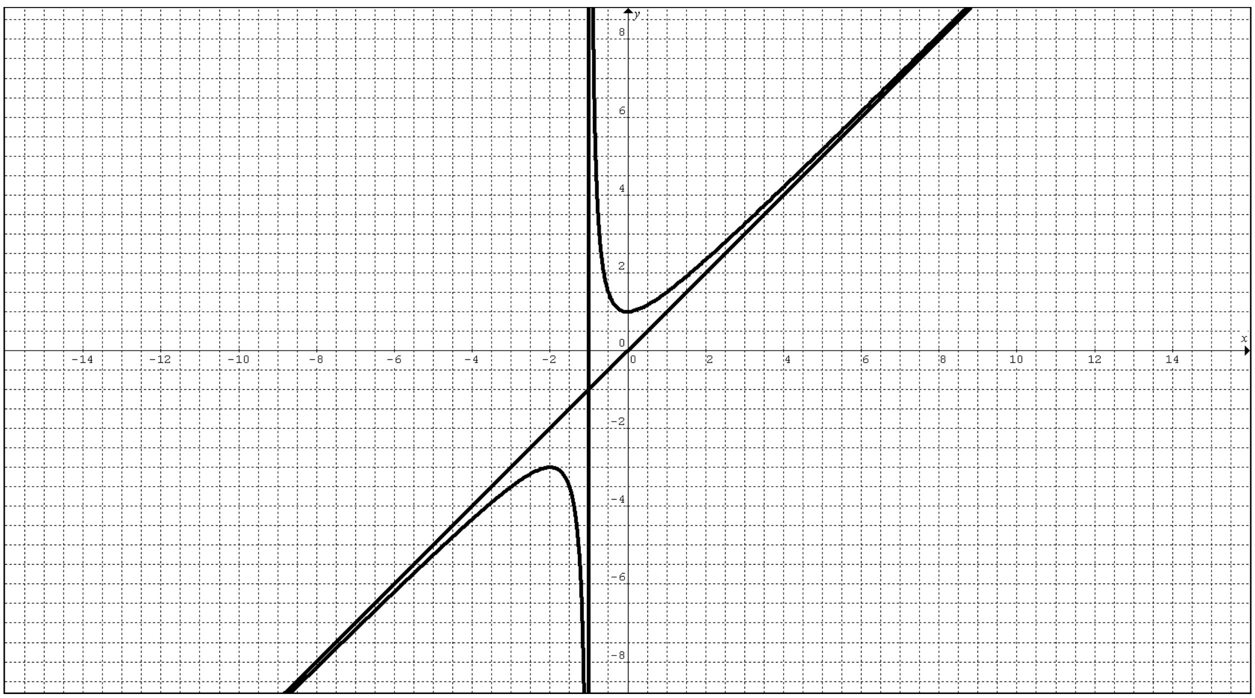
$$x = \frac{y - 1 \pm \sqrt{(y-1)^2 + 4(y-1)}}{2} = \frac{y - 1 \pm \sqrt{(y-1)(y+3)}}{2}$$

The expression is defined if and only if $(y-1)(y+3) \geq 0$ **which happens when either** $y \geq 1$ **or** $y \leq -3$. **The range of** R **is therefore**

$$(-\infty, -3] \cup [1, \infty)$$

To find the critical points we plug in into the expression for x **the values** $y = 1$ **and** $y = -3$ **getting** $x = 0$ **and** $x = -2$**, respectively. The critical points are located at** $(-2, -3)$ **and at** $(0, 1)$**.**

(d) The graph of R **and its asymptotes is shown below**



Problem 5 Consider the function $R = \dfrac{x^2 - 4}{x^2 - 1}$

(a) Find the x**-intercepts**

(b) Find the vertical asymptotes

(c) Find the horizontal or slant asymptote, if any

(d) Find the sign of the function

(e) Graph the function and its asymptotes

Solution (a) the x**-intercepts are** $(-2,0)$ **and** $(2,0)$

(b) the vertical asymptotes are $x=-1$ **and** $x=1$

(c) the degree of the numerator equals the degree of the denominator; the ratio of the leading terms is 1**; therefore there is the horizontal asymptote with the equation** $y=1$

(d) we see from (c) that the sign of R **is positive far right and far left. Because** $R(x) = \dfrac{(x+2)(x-2)}{(x+1)(x-1)}$ **every root of the numerator and of the denominator is simple and the sign of the function changes at points** $-2, -1, 1$ and 2.

Interval	$(-\infty,-2)$	$(-2,-1)$	$(-1,1)$	$(1,2)$	$(2,\infty)$
Sign of R	+	-	+	-	+

(f) The graph is shown below