

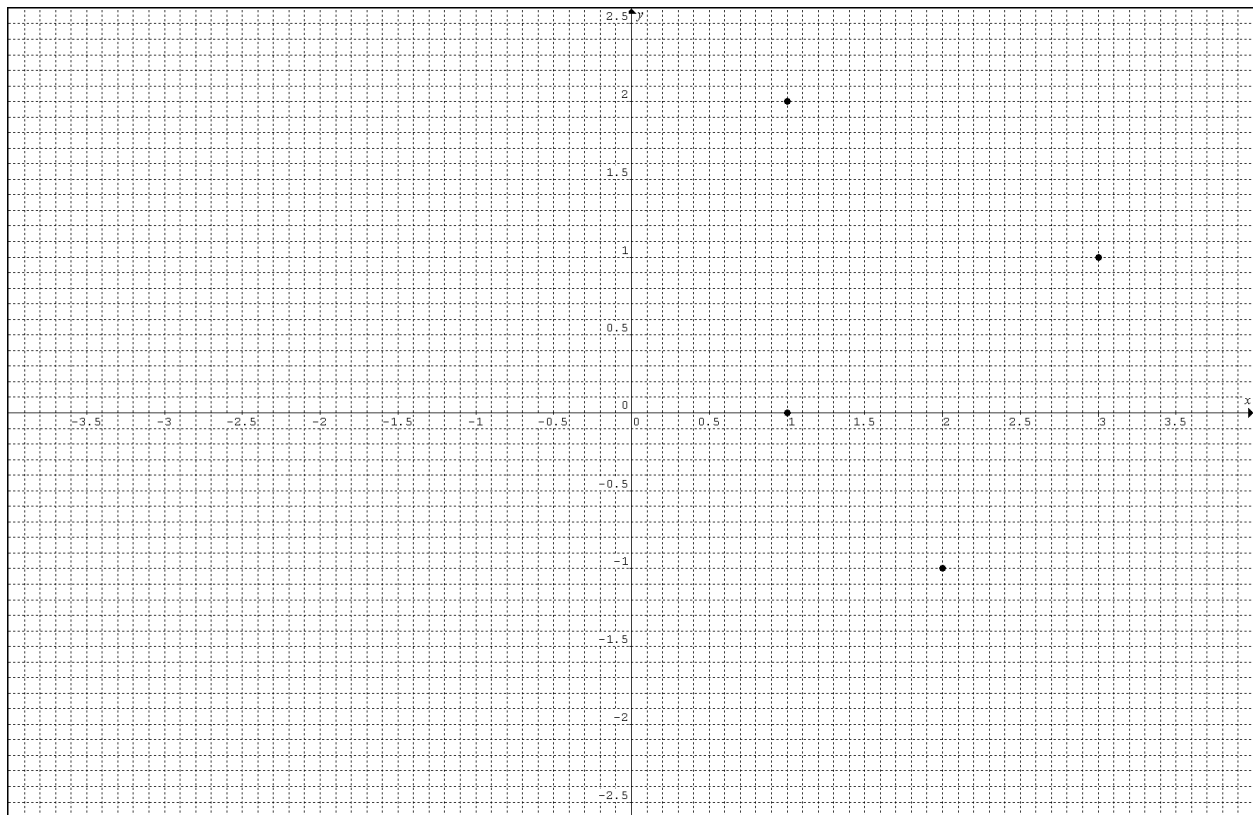
Problem 1

(a) A relation is given as the set of pairs

$$R = \{(1, 0), (2, -1), (3, 1), (1, 2)\}$$

Find the domain and the range of this relation, graph the relation, and decide whether this relation is a function.

Solution. The domain is the set of inputs $\{1, 2, 3\}$. The range is the set of outputs $\{-1, 0, 1, 2\}$. The graph of the relation is shown below.



The relation is not a function because to the same input 1 correspond two different outputs 0 and 2. Also we see that the graph does not pass the vertical line test.

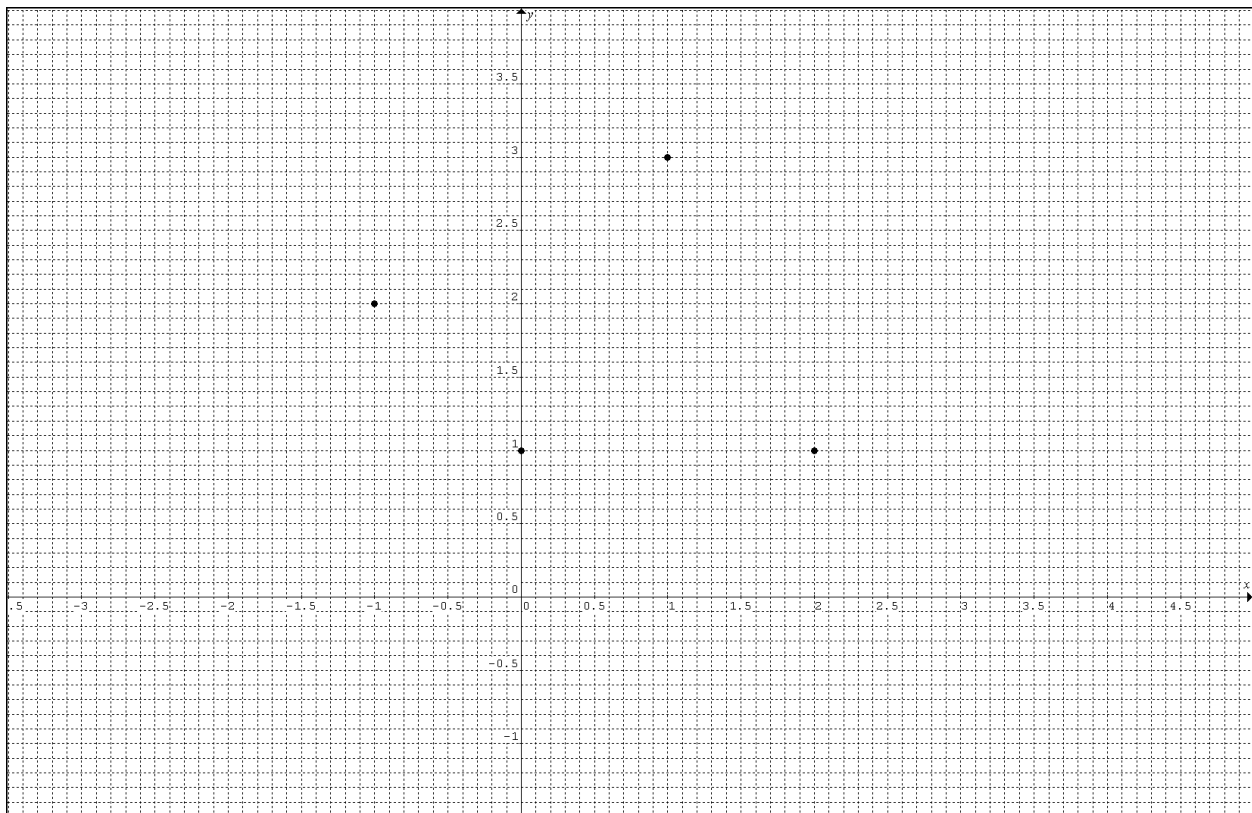
(b) Describe the relation inverse to the relation from Problem 1. Is this relation a function? Graph it.

Solution We get the inverse relation if we interchange the input and the output in each pair.

$$R^{-1} = \{(0,1), (-1,2), (1,3), (2,1)\}.$$

This relation is a function: to each input corresponds only one output.

The graph below passes the vertical line test.

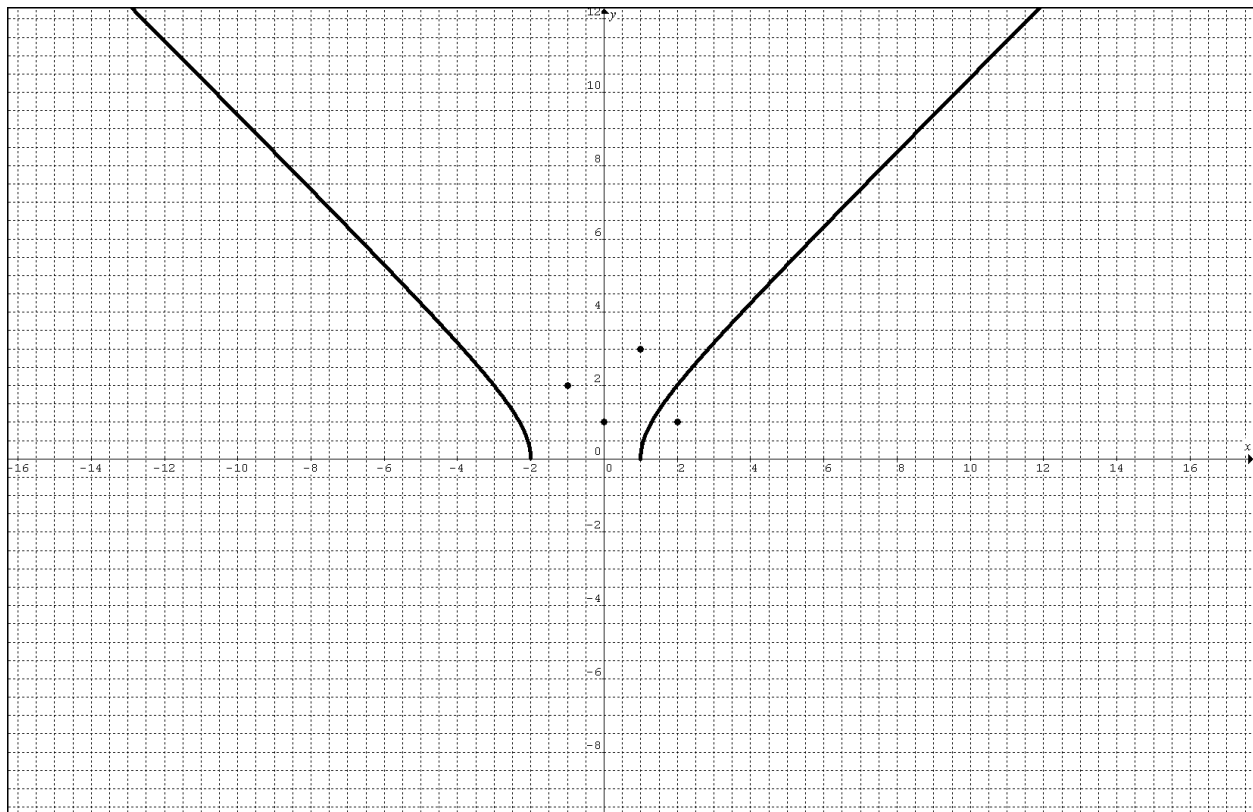


Problem 2 Consider the following two functions

$$f(x) = \sqrt{x}, g(x) = x^2 + x - 2$$

Describe the composition $f \circ g$. Find the domain and the range of this composition. Graph the composition.

Solution $f(g(x)) = \sqrt{g(x)} = \sqrt{x^2 + x - 2}$. To find the domain of this function notice that the expression under the square root cannot be negative; thus we have to solve the inequality $x^2 + x - 2 \geq 0$. To solve it we factor the left part and write the equation as $(x-1)(x+2) \geq 0$. The left part will be nonnegative either on the interval $(-\infty, -2]$ or on the interval $[1, \infty)$. Therefore the domain is $(-\infty, -2] \cup [1, \infty)$. On this domain the quadratic function g takes all nonnegative values whence the composition $f \circ g = \sqrt{g}$ also takes all nonnegative values and its range is $[0, \infty)$.



Problem 3 Consider the composition $g \circ f$ where g and f are functions from Problem 2. Describe the composition; find its domain and its range; graph the composition function.

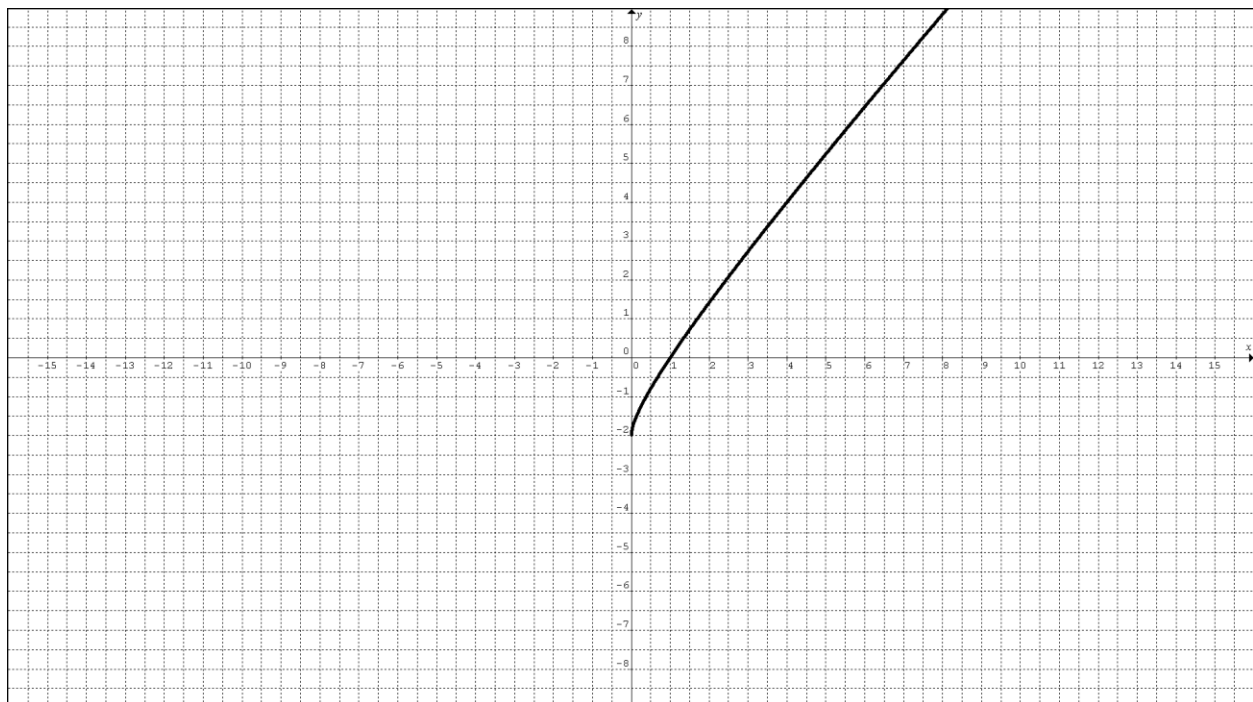
Solution $g(f(x)) = (\sqrt{x})^2 + \sqrt{x} - 2 = x + \sqrt{x} - 2$. Clearly this expression is defined if and only if $x \geq 0$ and therefore the domain of the composition is $[0, \infty)$. The question about the range can be reduced to the following: what values does the quadratic trinomial $u^2 + u - 1$ take if $u \geq 0$? To answer this question notice that the vertex of this quadratic trinomial is at

$u_v = -\frac{b}{2a} = -\frac{1}{2}$. To the right of the vertex the quadratic function is

increasing because $a > 0$ and therefore the range of the composition is

$$[g(f(0), \infty) = [-2, \infty).$$

The graph of the composition is shown below



Problem 4 Consider the function $f(x) = \sqrt{x^3 + 1}$. Find its domain and its range. Prove that the function f is one-to-one and find its inverse f^{-1} .

Solution The function f is defined if $x^3 + 1 \geq 0$. Factoring the left part of this inequality we write it as $(x+1)(x^2 - x + 1) \geq 0$. Because

$$x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0 \text{ the last inequality is equivalent to } x + 1 \geq 0$$

or $x \geq -1$. Thus the domain of f is $[-1, \infty)$. The range of f is clearly $[0, \infty)$.

To prove that f is one-to-one and to find its inverse let $y = \sqrt{x^3 + 1}$. Notice that $y \geq 0$. By squaring both parts we obtain $y^2 = x^3 + 1$, and $x = \sqrt[3]{y^2 - 1}$. Thus, if the output y is given the input x is defined in the unique way which proves that f is one-to-one. Finally we get an expression for f^{-1} by interchanging the input and the output.

$$f^{-1}(x) = \sqrt[3]{x^2 - 1}, x \geq 0.$$

Problem 5 Graph the functions f and f^{-1} from Problem 4 in the same coordinate system.

Solution

