

Problem 1 Perform computations and present the result as a complex number in the standard form $a + bi$.

$$\frac{(3 + 2i)(4 - 3i)}{(6 + 5i)(5 - 6i)}$$

Solution

$$\frac{(3 + 2i)(4 - 3i)}{(6 + 5i)(5 - 6i)} = \frac{12 - 9i + 8i - 6i^2}{30 - 36i + 25i - 30i^2}.$$

Recalling that $i^2 = -1$ we see that the above expression is equal to $\frac{18 - i}{60 - 11i}$.

Finally we perform the division

$$\frac{18 - i}{60 - 11i} = \frac{(18 - i)(60 + 11i)}{(60 - 11i)(60 + 11i)} = \frac{1091 + 138i}{60^2 + 11^2} = \frac{1091}{3721} + \frac{138}{3721}i.$$

Problem 2 Find all the solutions of the equation $x^6 - x^3 + 6 = 0$.

Solution The left part of the equation is a trinomial of quadratic type and it can be factored as

$$(x^3 - 3)(x^3 + 2)$$

Therefore we have to solve two equations of degree 3 $x^3 = 3$ and $x^3 = -2$.

Recall that all the solutions of the equation $x^3 = a$ where a is a real number can be obtained by multiplying all the solutions of the equation $x^3 = 1$ by $\sqrt[3]{a}$.

As we discussed in class the solutions of the last equation are

$$1, \quad \frac{-1 + \sqrt{3}i}{2}, \quad \frac{-1 - \sqrt{3}i}{2}.$$

Therefore all six solutions of our original equation can be obtained as

$$x_1 = \sqrt[3]{3}, \quad x_2 = \sqrt[3]{3} \frac{-1 + \sqrt{3}i}{2}, \quad x_3 = \sqrt[3]{3} \frac{-1 - \sqrt{3}i}{2},$$
$$x_4 = -\sqrt[3]{2}, \quad x_5 = -\sqrt[3]{2} \frac{-1 + \sqrt{3}i}{2}, \quad x_6 = -\sqrt[3]{2} \frac{-1 - \sqrt{3}i}{2}.$$

Problem 3 Solve the radical equation $\sqrt{x} + \sqrt{x+1} = \sqrt{x+2}$.

Solution. Squaring both parts of the equation we get

$$x + 2\sqrt{x(x+1)} + x + 1 = x + 2.$$

Or equivalently

$$2\sqrt{x(x+1)} = 1 - x.$$

Squaring both parts again we obtain

$$4x(x+1) = x^2 - 2x + 1,$$

which is equivalent to the quadratic equation

$$3x^2 + 6x - 1 = 0$$

By quadratic formula

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 3 \cdot (-1)}}{2 \cdot 3} = \frac{-6 \pm \sqrt{48}}{6} = -1 \pm \frac{2\sqrt{3}}{3}. \text{ Clearly, the}$$

negative value of x is not a solution of the original equation, but the positive

value is (e.g. we can use our calculators to see that $-1 + (2/3)\sqrt{3} \approx 0.1547$

and that $\sqrt{0.1547} + \sqrt{1.1547} \approx \sqrt{2.1547}$). Finally

$$x = -1 + \frac{2\sqrt{3}}{3}.$$

Problem 4 Using your calculator and the method of halving the interval locate the positive solution of the equation $x^3 + x^2 - 1 = 0$ with accuracy 0.01.

Solution Let $f(x) = x^3 + x^2 - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$.

Therefore the solution is in the interval $[0, 1]$. The successive steps are shown in the table below.

Interval	Midpoint	Sign of the value of f at the midpoint
$[0, 1]$	$1/2$	-
$[1/2, 1]$	$3/4$	-
$[3/4, 1]$	$7/8$	+
$[3/4, 7/8]$	$13/16$	+
$[3/4, 13/16]$	$25/32$	+
$[3/4, 25/32]$	$49/64$	+
$[3/4, 49/64]$	$97/128$	+
$[3/4, 97/128]$		

The length of the last interval is $1/128 < 0.01$. We have located the positive solution of our equation in the interval

$$3/4, 97/128 = [0.75, 0.7578125]$$

Problem 5 Using the rational roots test, the factor theorem, and the synthetic division find all the solutions of the polynomial equation and factor the polynomial completely.

$$6x^4 - x^3 - 5x^2 + 11x - 6 = 0$$

Solution. By the rational roots test if there are rational solutions of the equation above then their numerators can be only the numbers $\pm 1, \pm 2, \pm 3, \pm 6$, and their denominators can be only from the list $1, 2, 3, 4, 6$. Therefore the list of all possible rational roots is

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm 1/2, \pm 3/2, \pm 1/3, \pm 2/3, \pm 1/4, \pm 3/4, \pm 1/6.$$

Plugging in the numbers from the list into the original equation (either by using the synthetic division or a calculator) we successively eliminate the numbers $\pm 1, \pm 2, \pm 3, \pm 6, \pm 1/2, 3/2$. But $-3/2$ is a solution of the original equation. Indeed the synthetic division goes as follows.

-3/2	6	-1	-5	11	-6
		-9	15	-15	6
	6	-10	10	-4	0

Whence $-3/2$ is indeed a root of the original equation and

$$\begin{aligned} 6x^4 - x^3 - 5x^2 + 11x - 6 &= \\ &= (x + 3/2)(6x^3 - 10x^2 + 10x - 4) = (2x + 3)(3x^3 - 5x^2 + 5x - 2) \end{aligned}$$

We have reduced the problem to solving the cubic equation

$$3x^3 - 5x^2 + 5x - 2 = 0.$$

The list of possible rational roots for this equation is much shorter:

$\pm 1/3, \pm 2/3$. We easily eliminate $\pm 1/3$ but when we come to $2/3$ we hit another root.

2/3	3	-5	5	-2
		2	-2	2
	3	-3	3	0

Therefore

$$3x^3 - 5x^2 + 5x - 2 = (x - 2/3)(3x^2 - 3x + 3) = \\ = (3x - 2)(x^2 - x + 1).$$

It remains to solve the quadratic equation $x^2 - x + 1 = 0$. By the quadratic

formula $x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{1 \pm \sqrt{3}i}{2}$. Finally, the complete list
of solutions of the original equation is

$$-\frac{3}{2}, \frac{2}{3}, \frac{1 + \sqrt{3}i}{2}, \frac{1 - \sqrt{3}i}{2}$$

The polynomial can be completely factored as

$$6x^4 - x^3 - 5x^2 + 11x - 6 = (3x + 2)(2x - 3) \left(x - \frac{1 + \sqrt{3}i}{2} \right) \left(x - \frac{1 - \sqrt{3}i}{2} \right)$$