1 Direct Probability

1 Problem A school has 7 men and 5 women on its faculty. What is the probability that women will outnumber men on a randomly selected five-member committee?

2 Problem Of the 120 students in a class, 30 speak Chinese, 50 speak Spanish, 75 speak French, 12 speak Spanish and Chinese, 30 speak Spanish and French, and 15 speak Chinese and French. Seven students speak all three languages. What is the probability that a randomly chosen student speaks none of these languages?

3 Problem Of 120 students in a class, 30 speak Italian, 50 speak Spanish, and 70 speak French. 10 speak Spanish and Italian, 20 speak Spanish and French, and 15 speak Italian and French. Five students speak all three languages. What is the probability that a randomly chosen student from this group speaks none of these languages?

4 Problem A box contains 12 balls, 4 red, 4 white and 4 blue. The balls are drawn one by one, without replacement. What is the probability that all the red balls will be drawn out before any of the white balls?

5 Problem Four fair standard dice are tossed. What is the probability that all four will shew a different number?

6 Problem Ten equally-qualified applicants, 6 men and 4 women, apply for 3 lab technician positions. Unable to justify choosing any of the applicants over the others, the personnel director decides to select 3 at random. What is the probability that one man and two women will be chosen?

7 Problem A box contains 3 red balls, 4 white balls, and 3 blue balls. Balls are drawn from the box one at a time, at random, without replacement. What is the probability that all three red balls will be drawn before any white ball is obtained?

8 Problem Consider the following experiment: A fair coin is flipped until the first heads appears, and the number of flips is recorded. If this experiment is repeated three times, what is the probability that the result (number of flips) is the same all three times?

9 Problem In the world series of foosball, a five-game match is played, and the player who wins the most games is the champion. The probability of Player A winning any given game against player B is constant and equals $\frac{1}{3}$. What is the probability that Player A will be the champion? You may assume that all five games are played, even when a player wins three of the first five games.

10 Problem Twelve cards numbered 1 through 12 are thoroughly shuffled and distributed to three players so that each receives four cards. What is the probability that one of the players receives the three lowest cards (1, 2, and 3)?

11 Problem A student knows how to do 15 out of the 20 core problems for a given chapter. If the TA chooses 3 of the core problems at random for a quiz, what is the probability that the student knows how to do exactly 2 of them?

12 Problem There are 5 College students, 4 Wharton students and 3 Engineering students who are eligible to serve on a certain committee. If the committee is to consist of 2 College students, 2 Wharton students, and 2 Engineering students, how many different committees are possible?

13 Problem Two points are selected at random (independently and uniformly) along a 1-meter-long stick, and the stick is broken at the two selected points. What is the probability that the three pieces so obtained can be assembled into a triangle (i.e., that the length of the longest of the three pieces is less than the sum of the lengths of the other two, i.e., that none of the pieces is more than 0.5 meter in length)?

14 Problem What is the probability that three randomly-selected people were born on different days of the week? (Assume that the chance of someone being born on a given day of the week is $1/7$).

15 Problem An urn has seven red and five green marbles. Five marbles are drawn out of the urn, without replacement. What is the probability that the green marbles outnumber the red ones?

16 Problem A cholera patient lives in a building where his toilet stall has two dispensers (one on the left and another one on the right of the toilet). Initially each roll has 100 sheets of paper. Each time he visits the toilet (which is often, given that he has cholera), he chooses a dispenser at random and uses one sheet (OK, these sheets are very large, but let's continue with the problem. . .). At a certain moment, he first realises that one of the dispensers is empty. What is the probability that the other roll of paper has 25 sheets?

17 Problem Two points (x, y) are chosen at random on a rectangle 5 feet by 3 feet. What is the probability that the two points are within one foot of each other?

18 Problem A number X is chosen at random from the series

$$2, 5, 8, 11 \ldots , 299$$

and another number Y is chosen from the series

$$3, 7, 11, \ldots , 399.$$

What is the probability $\mathbf{P}\left(X = Y\right)$?

19 Problem A hat contains 20 tickets, each with a different number from 1 to 20. If 4 tickets are drawn at random, what is the probability that the largest number is 15 and the smallest number is 9?

20 Problem A box contains four \$10 bills, six \$5 bills, and two \$1 bills. Two bills are taken at random from the box without replacement. What is the probability that both bills will be of the same denomination?

21 Problem An urn contains n black and n white balls. Three balls are chosen from the urn at random and without replacement. What is the value of n if the probability is $\dfrac{1}{12}$ that all three balls are white?

22 Problem An urn contains $3n$ counters: n red, numbered 1 through n, n white, numbered 1 through n, and n blue, numbered 1 through n. Two counters are to be drawn at random without replacement. What is the probability that both counters will be of the same colour or bear the same number?

23 Problem Three distinguishable fair dice are thrown (say, one red, one blue, and one white). The probability that a sum $\mathbf{S}$ of 9 appears is lower than the probability that a sum of 10 appears. Explain why and find these probabilities. (Galileo's Problem)

24 Problem Six cards are drawn without replacement from a standard deck of cards. What is the probability that

❶ three are red and three are black?

❷ two are queens, two are aces, and two are kings?

❸ four have the same face (number or letter)?

❹ exactly four are from the same suit?

❺ there are no queens?

25 Problem A fair die is tossed twice in succession. Let $\mathbf{A}$ denote the first score and $\mathbf{B}$ the second score. Consider the quadratic equation

$$x^2 + \mathbf{A}x + \mathbf{B} = 0.$$

Find the probability that

❶ the equation has 2 distinct roots.

❷ the equation has a double root.

❸ $x = -3$ be a root of the equation,

❹ $x = 3$ be a root of the equation.

26 Problem Mrs. Flowers plants rosebushes in a row. Eight of the bushes are white and two are red, and she plants them in a random order. What is the probability that she will consecutively plant seven or more white bushes?

27 Problem A calculator has a random number generator button which, when pushed displays a random digit

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

The button is pushed four times. Assuming the numbers generated are independent, what is the probability of obtaining one '0', one '5', and two '9's in any order?

28 Problem If 13 cards are randomly chosen without replacement from an ordinary 52-card deck, what is the probability of obtaining exactly 3 aces?

29 Problem A population consists of 20% zeroes, 40% ones, and 40% twos. A random sample X, Y of size 2 is selected with replacement. Find $\mathbf{P}\left(|X - Y| = 1\right)$.

30 Problem Three fair dice, a red, a white, and a blue one are tossed, obtaining scores R, W, B respectively. What is the probability that $R \leq W \leq B$?

31 Problem From the set $\{1, 2, \ldots, n\}$, k distinct integers are selected at random and arranged in numerical order (lowest to highest). Let $\mathbf{P}\left(i, r, k, n\right)$ denote the probability that integer i is in position r. For example, observe that $\mathbf{P}\left(1, 2, k, n\right) = 0$ and $\mathbf{P}\left(2, 1, 6, 10\right) = 4/15$. Find a general formula for $\mathbf{P}\left(i, r, k, n\right)$.

32 Problem At *Medieval High* there are forty students. Amongst them, fourteen like Mathematics, sixteen like theology, and eleven like alchemy. It is also known that seven like Mathematics and theology, eight like theology and alchemy and five like Mathematics and alchemy. All three subjects are favoured by four students. How many students like neither Mathematics, nor theology, nor alchemy?

33 Problem Consider the set of 5-digit positive integers written in decimal notation.

1. How many are there?
2. How many do not have a 9 in their decimal representation?
3. How many have at least one 9 in their decimal representation?
4. How many have exactly one 9?
5. How many have exactly two 9's?
6. How many have exactly three 9's?
7. How many have exactly four 9's?
8. How many have exactly five 9's?
9. How many have neither an 8 nor a 9 in their decimal representation?
10. How many have neither a 7, nor an 8, nor a 9 in their decimal representation?
11. How many have either a 7, an 8, or a 9 in their decimal representation?

2 Conditional Probability

34 Problem A supermarket buys its eggs from three different chicken ranches. They buy $1/3$ of their eggs from Eggs'R Us, $1/2$ of their eggs from The Yolk Ranch, and $1/6$ of their eggs from Cheap Eggs. The supermarket determines that 1% of the eggs from Eggs'R Us are cracked, 2% of the eggs from the Yolk Ranch are cracked, and 5% of the eggs from Cheap Eggs are cracked. What is the probability that an egg chosen at random is from Cheap Eggs, given that the egg is cracked?

35 Problem 6% of Type A spark plugs are defective, 4% of Type B spark plugs are defective, and 2% of Type C spark plugs are defective. A spark plug is selected at random from a batch of spark plugs containing 50 Type A plugs, 30 Type B plugs, and 20 Type C plugs. The selected plug is found to be defective. What is the probability that the selected plug was of Type A?

36 Problem Four coins A, B, C, D have the following probabilities of landing heads:

$$\mathbf{P}\left(A = H\right) = \frac{1}{5}; \qquad \mathbf{P}\left(B = H\right) = \frac{2}{5}; \qquad \mathbf{P}\left(C = H\right) = \frac{3}{5}; \qquad \mathbf{P}\left(D = H\right) = \frac{4}{5}.$$

A coin is chosen at random and flipped three times. On the first and second flips it lands heads, on the third, tails. Which of the four coins is it the most likely to be?

37 Problem There are three coins in a box. When tossed, one of the coins comes up heads only 30% of the time, one of the coins is fair, and the third comes up heads 80% of the time. A coin is selected at random from the box and tossed three times. If two heads and a tails come up—in this order—what is the probability that the coin was the fair coin?

38 Problem Three fair standard dice are tossed, and the sum is found to be 6. What is the probability that none of the dice landed a 1?

39 Problem A and B are two events from the same sample space satisfying

$$\mathbf{P}\left(A\right) = \frac{1}{2}; \qquad \mathbf{P}\left(B\right) = \frac{2}{3}; \qquad \mathbf{P}\left(A|B\right) = \frac{1}{4}.$$

Find $\mathbf{P}\left(A^c \cap B^c\right)$.

40 Problem There are three grades of lemons: A, B, and C. 1% of grade A lemons are green; 0.75% of grade B lemons are green; and 0.5% of grade C lemons are green. A shipment with 40% grade A lemons, 40% grade B lemons and 20% grade C lemons is sent to a market. A green lemon was found in the shipment. What is the probability that it is a grade A lemon?

41 Problem On a day when Tom operates the machinery, 70% of its output is high quality. On a day when Sally operates the machinery, 90% of its output is high quality. Tom operates the machinery 3 days out of 5. Three pieces of a random day's output were selected at random and 2 of them were found to be of high quality. What is the probability that Tom operated the machinery that day?

42 Problem An urn contains 4 red marbles and 5 green marbles. A marble is selected at random and its colour noted, then this marble is put back into the urn. If it is red, then 2 more red marbles are put into the urn and if it is green 1 more green marble is put into the urn. A second marble is taken from the urn. Let R_1, R_2 be the events that we select a red marble on the first and second trials respectively, and let G_1, G_2 be the events that we select a green marble on the first and second trials respectively.

❶ Find $\mathbf{P}\left(R_2\right)$.

❷ Find $\mathbf{P}\left(R_2 \cap R_1\right)$.

❸ Find $\mathbf{P}\left(R_1|R_2\right)$.

43 Problem Peter writes to Paul and does not receive an answer. Assuming that one letter in n is lost in the mail, find the probability that Paul received the letter. (Assume that Paul would have answered the letter had he received it.)

44 Problem A deck of cards is shuffled and then divided into two halves of 26 cards each. A card is drawn from one of the halves; it turns out to be an ace. The ace is then placed in the second half-deck. This half is then shuffled, and a card drawn from it. Find the probability that this drawn card is an ace.

45 Problem A simple board game has four fields A, B, C, and D. Once you end up on field A you have won and once you end up on field B you have lost. From fields C and D you move to other fields by flipping a coin. If you are on field C and you throw a head, then you move to field A, otherwise to field D. From field D, you move to field C if you throw a head, and otherwise you mover to field B.

Suppose that you start in field D. What is the probability that you will win (i.e., what is the probability that you will end up on field A)?

3 Expectation

46 Problem A casino game consists of a single toss of a fair die and pays off as follows: if the die comes up with an odd number, the player is paid that number of dollars (i.e., $\$1$ for rolling a 1, $\$3$ for rolling a 3, and $\$5$ for rolling a 5), and if an even number comes up the player is paid nothing. What fee should the casino charge to play the game to make it exactly fair?

47 Problem At a local carnival, Osa pays $\$1$ to play a game in which she chooses a card at random from a standard deck of 52 cards. If she chooses a heart, then she receives $\$2$ (that is, $\$1$ plus her initial bet of $\$1$). If she chooses the Queen of Spades she receives $\$13$. Which of the following is closest to Osa's expected net profit from playing the game?

48 Problem Six different pairs of socks are put in the laundry (12 socks in all, and each sock has only one mate), but only 7 socks come back. What is the expected number of pairs of socks that come back? (*Hint*: This is the sum of the expectations of the number, 0 or 1, of each individual pair that come back.)

49 Problem Five differently-coloured pairs of socks—ten in total—go to the laundry, but only seven make it back. What is the expected number of pairs of socks coming back?

50 Problem There are eight socks in a box, of which four are white and four are black. Socks are drawn one at a time (without replacement) until a pair is produced. What is the expected value of drawings? (Clearly, this number should be between 2 and 3.)

51 Problem If $\mathbf{X}$ denotes the number of 1's when 72 dice are thrown, find $E\mathbf{X}^2$.

52 Problem An urn contains 30 cards: two numbered 1, two numbered 2, $\ldots$, two numbered 15. Ten cards are drawn at random from the urn. What is the expected number of pairs remaining in the urn?

53 Problem A standard deck of cards is turned face up one card at a time. What is the expected number of cards turned up in order to obtain (1) a king, (2) a heart?

54 Problem In the city of Jerez de la Frontera, in Cádiz, Spain, true sherry is made according to a multistage system called *Solera*. Assume that a winemaker has three barrels, A, B, and C. Every year, a third of the wine from barrel C is bottled and replaced by wine from B; then B is topped off with a third of the wine from A; finally A is topped off with new wine. Find the mean of the age of the wine in each barrel, under the assumption that the operation has been going on since time immemorial.

55 Problem Suppose that a class contains 10 boys and 15 girls, and suppose that 8 students are to be selected at random from the class without replacement. Let X denote the number of boys that are selected and let Y denote the number of girls that are selected. Find $E(X - Y)$.

56 Problem Suppose that one word is selected at random from the sentence THE GIRL PUT ON HER BEAUTIFUL RED HAT. If X denotes the number of letters in the word that is selected, what is the value of $\mathrm{var}(X)$?

57 Problem Three men, A, B, and C shoot at a target. Suppose that A shoots three times and the probability that he will hit the target on any given shot is $1/8$. B shoots five times and the probability that he will hit the target is $1/4$, and C shoots twice and the probability that he will hit the target on any given shot is $1/2$. What is the expected number of times that the target will be hit?

58 Problem Suppose that a player starts with a fortune of \$8. A fair coin is tossed three times. If the coin comes up heads, the player's fortune is doubled, otherwise it is halved. What is the player's expected fortune?

59 Problem Eccentric Mrs. A has a house with one front door and one back door. She places two pairs of walking shoes at each door (eight shoes total). For each walk, she chooses one door at random, puts on a pair of shoes, returns after a walk to a randomly selected door, and takes off the shoes at that door. Find the expected number of walks until Mrs. A discovers that there are no shoes available at the door she has chosen for her next walk. (Hint: Condition, as you did in the solution of the sherry problem.)

60 Problem Let A, B, C be the outcomes of three distinguishable fair dice and consider the system

$$Ax - By = C; \qquad x - 2y = 3.$$

Find the following probabilities

1. that the system has no solution.
2. that the system has infinitely many solutions.
3. that the system has exactly one solution.
4. that the system has the unique solution $x = 3, y = 0$.

4 Normal Random Variable

61 Problem The faces of a die are numbered $1, 1, 2, 2, 3, 3$, but it is otherwise balanced. The die is tossed 600 times, and the scores are added up. Approximate the probability that the sum is between 1200 and 1230.

62 Problem The probability that a coin lands on heads is $2/5$. The coin is flipped 150 times. Find: (1) the expected number of heads; (2) the variance on the number of heads; (3) an estimate, using normal random variables, for the probability that there be between 70 and 90 heads.

5 Markov Chains

63 Problem A regional planning board knows that although the population of their region remains stable, each year 30% of the people in the eastern portion move to the west, and 40% of the people in the western section move to the east. What is the fraction of the people in the region who, in the long run, live in the west?

64 Problem Using data collected for a particular region over many years, an insurance company has ascertained that 20% of the drivers involved in an automobile accident one year are also involved in an accident the following year, while only 10% of the drivers not involved in an accident one year are involved in an accident the following year. Use these percentages as approximate empirical probabilities to find the probability that (in the long run) a driver chosen at random will be involved in an accident during any given year.

65 Problem Three people, A, B, C, are playing catch. The probabilities each will throw the other are $\mathbf{P}(A \to B) = \frac{1}{2}$, $\mathbf{P}(A \to C) = \frac{1}{2}$, $\mathbf{P}(B \to A) = \frac{1}{4}$, $\mathbf{P}(B \to C) = \frac{3}{4}$, $\mathbf{P}(C \to A) = \frac{1}{2}$, and $\mathbf{P}(C \to B) = \frac{1}{2}$. What is the probability that A will have the ball in the long run?

6 Some Answers

1 The experiment is choosing five people from amongst 12, and so the sample space has size $\binom{12}{5} = 792$. The women will outnumber the men if there are (a) 3 women and 2 men; (b) 4 women and 1 man; or (c) 5 women. The numbers of successes is thus

$$\binom{5}{3}\binom{7}{2} + \binom{5}{4}\binom{7}{1} + \binom{5}{5}\binom{7}{0} = 246.$$

The probability sought is thus $\frac{246}{792} = \frac{41}{132}$.

2 We use inclusion-exclusion, where C, F, S, respectively, denote the sets of Chinese, French and Spanish speakers. We have

$$\begin{aligned} \mathrm{card}\,(C \cup F \cup S) &= \mathrm{card}\,(C) + \mathrm{card}\,(F) + \mathrm{card}\,(S) \\ &\quad -\mathrm{card}\,(C \cap F) - \mathrm{card}\,(F \cap S) - \mathrm{card}\,(S \cap C) \\ &\quad +\mathrm{card}\,(C \cap F \cap S) \\ &= 30 + 50 + 75 - 15 - 30 - 12 + 7 \\ &= 105, \end{aligned}$$

students speak at least one language, hence $120 - 105 = 15$ students speak none of the languages. The probability sought is $\frac{15}{120} = \frac{1}{8}$.

3 By Inclusion-Exclusion, the number of students speaking at least one of these languages is

$$30 + 50 + 70 - (10 + 20 + 15) + 5 = 110,$$

and so 10 do not speak any of these languages. The required probability is thus $\frac{10}{120} = \frac{1}{12}$.

4 A successful alignment has a configuration of the form $RRRRWWWW$. There are $\frac{8!}{4!4!} = 70$ of these permutations, of which only one is successful. So the probability sought is $\frac{1}{70}$.

Aliter: The number of alignments of the balls is $\frac{12!}{(4!)^3}$. The number of alignments in which the red balls are drawn before the white balls are the number of configurations of the following type:

$$x_1Rx_2Rx_3Rx_4Rx_5Wx_6Wx_7Wx_8Wx_9$$

where the x_i can be occupied from 0 to 4 blue balls. The number of such configurations is the number of solutions of

$$x_1 + \cdots + x_9 = 4$$

in non-negative integers, which is $\binom{4+9-1}{9-1} = \binom{12}{8} = \frac{12!}{4!8!}$. The probability sought is thus

$$\frac{12!}{4!8!} \cdot \frac{(4!)^3}{12!} = \frac{1}{70}.$$

5 This is plainly

$$\frac{6 \cdot 5 \cdot 4 \cdot 3}{6^4} = \frac{5}{18}.$$

6 The experiment consists of choosing 3 people out of 10, and so the sample space size is $\binom{10}{3} = 120$. Success occurs when one man and two women chosen, which can be done in $\binom{6}{1}\binom{4}{2} = 36$ ways. The probability sought is $\frac{36}{120} = \frac{3}{10}$.

7 Any successful permutation will have the form $RRRWWWW$ with the location of the blues irrelevant. There is only one permutation (the above) of the $\frac{7!}{4!3!} = 35$ of these letters which is successful. Hence the probability sought is $\frac{1}{35}$.

Aliter: The experiment consists in permuting the letters $RRRWWWWBBB$ and hence the sample space size is $\frac{10!}{3!4!3!}$. In order to obtain success, we must have an arrangement of the form

$$x_1Rx_2Rx_3Rx_4Wx_5Wx_6Wx_7Wx_8,$$

where the x_i may have from 0 to 3 blue balls. The number of such arrangements is the number of non-negative integral solutions to $x_1 + x_2 + \cdots + x_8 = 3$, namely $\binom{8+3-1}{8-1} = \binom{10}{7} = \frac{10!}{7!3!}$. Hence the probability sought is

$$\frac{\frac{10!}{7!3!}}{\frac{10!}{3!4!3!}} = \frac{3!4!}{7!} = \frac{1}{35}.$$

8 Let $\mathbf{X}_i$ be the random variable counting the number of times until heads appears for times $i = 1, 2, 3$. Observe that $\mathbf{P}(\mathbf{X}_i = n) = \frac{1}{2^n}$ (in fact, $\mathbf{X}_i$ is geometric with $p = \frac{1}{2}$). Hence the desired probability is

$$\sum_{n=1}^{\infty} \mathbf{P}(\mathbf{X}_1 = n)\,\mathbf{P}(\mathbf{X}_2 = n)\,\mathbf{P}(\mathbf{X}_3 = n) = \sum_{n=1}^{\infty} \frac{1}{8^n} = \frac{\frac{1}{8}}{1 - \frac{1}{8}} = \frac{1}{7}.$$

9 This is plainly

$$\binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 + \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1 + \binom{5}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^0 = \frac{17}{81}.$$

10 The experiment consists in choosing three positions to be occupied by the three cards, this can be done in $\binom{12}{3}$ ways. Success is accomplished by selecting one of the players, in $\binom{3}{1}$ and three of his cards, (in $\binom{4}{3}$) ways, to be the three lowest cards. The probability required is thus $\frac{\binom{3}{1}\binom{4}{3}}{\binom{12}{3}} = \frac{3}{55}$.

11 The TA chooses 3 problems in $\binom{20}{3} = 1140$ ways. Success means $\binom{15}{2}\binom{5}{1} = 525$ ways of choosing exactly two correct answers. The probability sought is thus $\dfrac{525}{1140} = \dfrac{35}{76}$.

12 This is clearly $\binom{5}{2}\binom{4}{2}\binom{3}{2} = 180$.

13 Consider a stick l meters long. Let x, y, and $l - x - y$ be the lengths of the three parts of the rod. If these parts are to form a triangle, then the triangle inequality must be satisfied, that is, the sum of any two sides of the triangle must be greater than the third. So we simultaneously must have

$$x + y > l - x - y \implies x + y > \frac{l}{2},$$

$$x + l - x - y > y \implies y < \frac{l}{2},$$

$$y + l - x - y > x \implies x < \frac{l}{2}.$$

Since trivially $0 \leq x + y \leq l$, what we are asking is for the ratio of the area of the region

$$\mathcal{A} = \{(x, y) : 0 < x < \frac{l}{2}, 0 < y < \frac{l}{2}, x + y > \frac{l}{2}\}$$

to that of the triangle with vertices at $(0, 0)$, $(l, 0)$ and $(0, l)$. This is depicted in figure 1. The desired probability is thus

$$\frac{\frac{l^2}{8}}{\frac{l^2}{2}} = \frac{1}{4}.$$

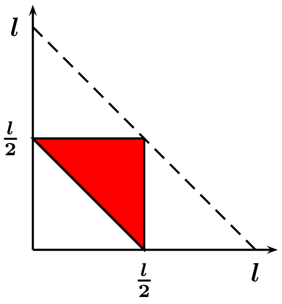

Figure 1: Example 13

14 The sample space consists of all vectors $D_1D_2D_3$ where D_i is a day of the week, hence the sample space size is $7^3 = 343$. Success consists in getting a vector with all the D_i different, and there are $7 \cdot 6 \cdot 5 = 210$ of these. The desired probability is thus $\dfrac{210}{343} = \dfrac{30}{49}$.

15 This is plainly

$$\frac{\binom{5}{3}\binom{7}{2} + \binom{5}{4}\binom{7}{1} + \binom{5}{5}\binom{7}{0}}{\binom{12}{5}} = \frac{41}{132}.$$

16 For the patient to notice for the first time that the left dispenser is empty, he must have pulled out 100 sheets from the left, 75 from the right, and on the 101st attempt on the left he finds that there is no sheet. So we have a configuration like

$$\underbrace{\ldots L \ldots R \ldots}_{100\ L\text{'s and } 75\ R\text{'s}}\ L,$$

where all the L's, except for the one on the last position, can be in any order, and all the R's can be in any order. This happens with probability $\binom{175}{75}\left(\frac{1}{2}\right)^{75}\left(\frac{1}{2}\right)^{100} \cdot \frac{1}{2} = \binom{175}{75}\left(\frac{1}{2}\right)^{176}$. The same probability can be obtained for the right dispenser and hence the probability sought is $2\binom{175}{75}\left(\frac{1}{2}\right)^{176} = \binom{175}{75}\left(\frac{1}{2}\right)^{175}$.

17 We want $\mathbf{P}\left(|x-y|<1\right) = \mathbf{P}\left(-1+x<y<1+x\right)$. This is the area shaded in figure 2. The area of the rectangle is $3\cdot 5 = 15$, of the white triangle $\frac{1}{2}\cdot(2)(2) = 2$, and of the white trapezoid $\frac{1}{2}\cdot(1+4)(3) = \frac{15}{2}$. The desired probability is thus

$$\frac{15-2-\frac{15}{2}}{15} = \frac{11}{30}.$$

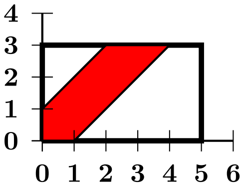

Figure 2: Problem 17.

18 There are 100 terms in each of the arithmetic progressions. Hence we may choose X in 100 ways and Y in 100 ways. The size of the sample space for this experiment is thus $100\cdot 100 = 10000$. Now we note that 11 is the smallest number that belongs to both progressions. Since the first progression has common difference 3 and the second progression has common difference 4, and since the least common multiple of 3 and 14 is 12, the progressions have in common numbers of the form

$$11+12k.$$

We need the largest integer k with

$$11+12k \leq 299 \implies k = 24.$$

Therefore, the 25 numbers

$$11 = 11+12\cdot 0, 23 = 11+12\cdot 1, 35 = 11+12\cdot 2, \ldots, 299 = 11+12\cdot 24$$

belong to both progressions and the probability sought is

$$\frac{25}{10000} = \frac{1}{400}.$$

19 For this to happen, we choose the ticket numbered 9, the one numbered 15 and the other two tickets must be chosen from amongst the five tickets numbered $10, 11, 12, 13, 14$. The probability sought is thus

$$\frac{\binom{5}{2}}{\binom{20}{4}} = \frac{10}{4845} = \frac{2}{969}.$$

20 There are $4+6+2 = 12$ bills. The experiment can be performed in $\binom{12}{2} = 66$ ways. To be successful we must choose either 2 tens (in $\binom{4}{2} = 6$ ways), **or** 2 fives (in $\binom{6}{2} = 15$ ways), **or** 2 ones (in $\binom{2}{2} = 1$ way). The probability sought is thus

$$\frac{\binom{4}{2}+\binom{6}{2}+\binom{2}{2}}{\binom{12}{2}} = \frac{6+15+1}{66} = \frac{1}{3}.$$

21 We have

$$\begin{aligned}
\frac{\binom{n}{3}}{\binom{2n}{3}} = \frac{1}{12} \quad &\implies \quad \frac{n(n-1)(n-2)}{2n(2n-1)(2n-2)} = \frac{1}{12} \\
&\implies \quad \frac{n-2}{4(2n-1)} = \frac{1}{12} \\
&\implies \quad 3(n-2) = 2n-1 \\
&\implies \quad n = 5.
\end{aligned}$$

22 This is plainly

$$\frac{\binom{3}{1}\binom{n}{2}+\binom{3}{2}\binom{n}{1}}{\binom{3n}{2}} = \frac{3n(n-1)+6n}{3n(3n-1)} = \frac{n+1}{3n-1}.$$

23 The sample space has size $6^3 = 216$. A simple count yields 25 ways of obtaining a 9 and 27 of getting a 10. Hence $\mathbf{P}(\mathbf{S} = 9) = \frac{25}{216} \approx 0.1157$, and $\mathbf{P}(\mathbf{S} = 10) = \dfrac{27}{216} = \frac{1}{8} = 0.125$

24 The required probabilities are plainly $\dfrac{\binom{26}{3}^2}{\binom{52}{6}}$; $\dfrac{\binom{4}{2}^3}{\binom{52}{6}}$; $\dfrac{\binom{13}{1}\binom{4}{4}\binom{48}{2}}{\binom{52}{6}}$; $\dfrac{\binom{4}{1}\binom{13}{4}\binom{39}{2}}{\binom{52}{6}}$; $\dfrac{\binom{48}{6}}{\binom{52}{6}}$

25 To have 2 distinct roots we need the discriminant $\mathbf{A}^2 - 4\mathbf{B} > 0$. Since $1 \leq \mathbf{A} \leq 6$ and $1 \leq \mathbf{B} \leq 6$ this occurs for the 17 ordered pairs $(\mathbf{A}, \mathbf{B})$: $(3,1)$, $(3,2)$, $(4,1)$, $(4,2)$, $(4,3)$, $(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$, $(5,5)$, $(5,6)$, $(6,1)$, $(6,2)$, $(6,3)$, $(6,4)$, $(6,5)$, $(6,6)$, so the desired probability is $\dfrac{17}{36}$.

To have a double root we need $\mathbf{A}^2 - 4\mathbf{B} = 0$. This occurs when for the 2 ordered pairs $(\mathbf{A}, \mathbf{B})$: $(2,1)$ and $(4,4)$. Hence the desired probability is $\dfrac{2}{36} = \dfrac{1}{18}$.

If $x = -3$ is a root, then $(-3)^2 - 3\mathbf{A} + \mathbf{B} = 0$, that is $9 + \mathbf{B} = 3\mathbf{A}$. This occurs for the 2 ordered pairs $(\mathbf{A}, \mathbf{B})$: $(4,3)$ and $(5,6)$. Hence the desired probability is $\dfrac{2}{36} = \dfrac{1}{18}$.

If $x = 3$ were a root, then $3^2 + 3\mathbf{A} + \mathbf{B} = 0$, which is impossible since the sum on the sinistral side is strictly positive and hence never 0. The desired probability is thus 0.

26 The sample space is the number of permutations of 10 objects of two types: 8 of type W (for *white*) and 2 of type R (for *red*). There are $\dfrac{10!}{8!2!} = 45$ such permutations. Now, to count the successful permutations, observe that we need a configuration of the form

$$X_1RX_2RX_3.$$

If one of the $X_i = 7W$ then another one must be $1W$ and the third must be $0W$, so there are $3! = 6$ configurations of this type. Similarly, if one of the $X_i = 8W$, the other two must be $0W$ and again there are $\dfrac{3!}{2!} = 3$ configurations of this type. The desired probability is hence $\dfrac{9}{45} = \dfrac{1}{5}$.

27 A particular configuration with one '0', one '5', and two '9's has probability $(\frac{1}{10})^1(\frac{1}{10})^1(\frac{1}{10})^2 = \dfrac{1}{10000}$ of occurring. Since there are $\dfrac{4!}{2!} = 12$ such configurations, the desired probability is thus $\dfrac{12}{10000} = \dfrac{3}{2500}$.

28 This is plainly $\dfrac{\binom{4}{3}\binom{48}{10}}{\binom{52}{13}} = \dfrac{858}{20825}$.

29 We have

$$\begin{aligned}
\mathbf{P}(|X-Y| = 1) &= \mathbf{P}(X - Y = 1) + \mathbf{P}(Y - X = 1) \\
&= 2\mathbf{P}(X - Y = 1) \\
&= 2(\mathbf{P}(X = 1 \cap Y = 0) + \mathbf{P}(X = 2 \cap Y = 1)) \\
&= 2(\mathbf{P}(X = 1)\,\mathbf{P}(Y = 0) + \mathbf{P}(X = 2)\,\mathbf{P}(Y = 1)) \\
&= 2((.4)(.2) + (.4)(.4)) \\
&= .48,
\end{aligned}$$

since the sampling with replacement gives independence.

30 Each of the dice may land in 6 ways and hence the size of the sample space for this experiment is $6^3 = 216$. Notice that there is a one to one correspondence between vectors

$$(R, W, B), \quad 1 \leq R \leq W \leq B \leq 6$$

and vectors

$$(R', W', B'), \quad 1 \leq R' < W' < B' \leq 8.$$

This can be seen by putting $R' = R, W' = W + 1$, and $B' = B + 2$. Thus the number of vectors (R', W', B') with $1 \leq R' < W' < B' \leq 8$ is $\binom{8}{3} = 56.$ The probability sought is thus

$$\frac{56}{216} = \frac{7}{27}.$$

31 The $r-1$ integers before i must be taken from the set $\{1,2,\ldots,i-1\}$ and the $k-r$ after i must be taken from the set $\{i+1,i+2,\ldots,n\}$. Hence $\mathbf{P}\left(i,r,k,n\right)=\dfrac{\binom{i-1}{r-1}\binom{n-i}{k-r}}{\binom{n}{k}}$.

32 Let A be the set of students liking Mathematics, B the set of students liking theology, and C be the set of students liking alchemy. We are given that

$$\text{card}\left(A\right)=14, \text{card}\left(B\right)=16,$$

$$\text{card}\left(C\right)=11, \text{card}\left(A\cap B\right)=7, \text{card}\left(B\cap C\right)=8, \text{card}\left(A\cap C\right)=5,$$

and

$$\text{card}\left(A\cap B\cap C\right)=4.$$

By the Principle of Inclusion-Exclusion,

$$\begin{aligned}\text{card}\left(A^c\cap B^c\cap C^c\right) &= 40-\text{card}\left(A\right)-\text{card}\left(B\right)-\text{card}\left(C\right)\\ &\quad +\text{card}\left(A\cap B\right)+\text{card}\left(A\cap C\right)+\text{card}\left(B\cap C\right)\\ &\quad -\text{card}\left(A\cap B\cap C\right).\end{aligned}$$

Substituting the numerical values of these cardinalities

$$40-14-16-11+7+5+8-4=15.$$

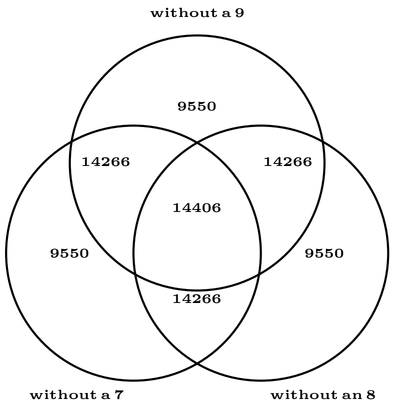

Figure 3: Example 33.

33

1. There are 9 possible choices for the first digit and 10 possible choices for the remaining digits. The number of choices is thus $9\cdot 10^4=90000$.
2. There are 8 possible choices for the first digit and 9 possible choices for the remaining digits. The number of choices is thus $8\cdot 9^4=52488$.
3. The difference $90000-52488=37512.$
4. We condition on the first digit. If the first digit is a 9 then the other four remaining digits must be different from 9, giving $9^4=6561$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{1}=4$ ways of choosing were the 9 will be, and we have 9^3 ways of filling the 3 remaining spots. Thus in this case there are $8\cdot 4\cdot 9^3=23328$ such numbers. In total there are $6561+23328=29889$ five-digit positive integers with exactly one 9 in their decimal representation.
5. We condition on the first digit. If the first digit is a 9 then one of the remaining four must be a 9, and the choice of place can be accomplished in $\binom{4}{1}=4$ ways. The other three remaining digits must be different from 9, giving $4\cdot 9^3=2916$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{2}=6$ ways of choosing were the two 9's will be, and we have 9^2 ways of filling the two remaining spots. Thus in this case there are $8\cdot 6\cdot 9^2=3888$ such numbers. Altogether there are $2916+3888=6804$ five-digit positive integers with exactly two 9's in their decimal representation.
6. Again we condition on the first digit. If the first digit is a 9 then two of the remaining four must be 9's, and the choice of place can be accomplished in $\binom{4}{2}=6$ ways.

The other two remaining digits must be different from $\mathbf{9}$, giving $\mathbf{6 \cdot 9^2 = 486}$ such numbers. If the first digit is not a $\mathbf{9}$, then there are $\mathbf{8}$ choices for this first digit. Also, we have $\binom{4}{3} = \mathbf{4}$ ways of choosing were the three $\mathbf{9}$'s will be, and we have $\mathbf{9}$ ways of filling the remaining spot. Thus in this case there are $\mathbf{8 \cdot 4 \cdot 9 = 288}$ such numbers. Altogether there are $\mathbf{486 + 288 = 774}$ five-digit positive integers with exactly three $\mathbf{9}$'s in their decimal representation.

7. If the first digit is a $\mathbf{9}$ then three of the remaining four must be $\mathbf{9}$'s, and the choice of place can be accomplished in $\binom{4}{3} = \mathbf{4}$ ways. The other remaining digit must be different from $\mathbf{9}$, giving $\mathbf{4 \cdot 9 = 36}$ such numbers. If the first digit is not a $\mathbf{9}$, then there are $\mathbf{8}$ choices for this first digit. Also, we have $\binom{4}{4} = \mathbf{4}$ ways of choosing were the four $\mathbf{9}$'s will be, thus filling all the spots. Thus in this case there are $\mathbf{8 \cdot 1 = 8}$ such numbers. Altogether there are $\mathbf{36 + 8 = 44}$ five-digit positive integers with exactly three $\mathbf{9}$'s in their decimal representation.
8. There is obviously only $\mathbf{1}$ such positive integer. **Remark:**Observe that $\mathbf{37512 = 29889 + 6804 + 774 + 44 + 1}$.
9. We have $\mathbf{7}$ choices for the first digit and $\mathbf{8}$ choices for the remaining $\mathbf{4}$ digits, giving $\mathbf{7 \cdot 8^4 = 28672}$ such integers.
10. We have $\mathbf{6}$ choices for the first digit and $\mathbf{7}$ choices for the remaining $\mathbf{4}$ digits, giving $\mathbf{6 \cdot 7^4 = 14406}$ such integers.
11. We use inclusion-exclusion. From figure 3, the numbers inside the circles add up to $\mathbf{85854}$. Thus the desired number is $\mathbf{90000 - 85854 = 4146.}$

34 See figure 4 for a tree diagram. We have

$$\begin{aligned}
\mathbf{P\,(cracked)} &= \mathbf{P\,(cracked|R'Us)\,P\,(R'Us) + P\,(cracked|YR)\,P\,(YR) + P\,(cracked|ChE)\,P\,(ChE)} \\
&= \mathbf{\frac{1}{3} \cdot \frac{1}{100} + \frac{1}{2} \cdot \frac{2}{100} + \frac{1}{6} \cdot \frac{5}{100}} \\
&= \mathbf{\frac{13}{600}}
\end{aligned}$$

and so,

$$\begin{aligned}
\mathbf{P\,(ChE|cracked)} &= \mathbf{\frac{P\,(ChE \cap cracked)}{P\,(cracked)}} \\
&= \mathbf{\frac{P\,(cracked|ChE) \cdot P\,(ChE)}{P\,(cracked)}} \\
&= \mathbf{\frac{\frac{5}{100} \cdot \frac{1}{6}}{\frac{13}{600}}} \\
&= \mathbf{\frac{5}{13}}
\end{aligned}$$

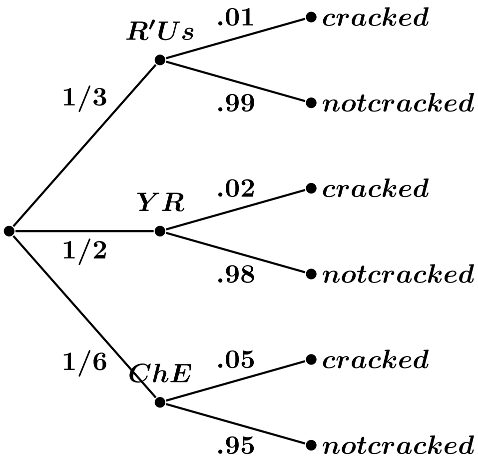

Figure 4: Example 34.

35 Let A, B, C denote the events that the plug is type A, B, C respectively, and D the event that the plug is defective. We have

$$\begin{aligned} \mathbf{P}(D) &= \mathbf{P}(D|A)\cdot\mathbf{P}(A)+\mathbf{P}(D|B)\cdot\mathbf{P}(B)+\mathbf{P}(D|C)\cdot\mathbf{P}(C) \\ &= \frac{6}{100}\cdot\frac{50}{100}+\frac{4}{100}\cdot\frac{30}{100}+\frac{2}{100}\cdot\frac{20}{100} \\ &= \frac{23}{500}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{P}(A|D) &= \frac{\mathbf{P}(A\cap D)}{\mathbf{P}(D)} \\ &= \frac{\mathbf{P}(D|A)\cdot\mathbf{P}(A)}{\mathbf{P}(D)} \\ &= \frac{\frac{6}{100}\cdot\frac{50}{100}}{\frac{23}{500}} \\ &= \frac{15}{23}. \end{aligned}$$

36 We have

$$\mathbf{P}(HHT)=\frac{1}{4}\cdot\frac{4}{5^3}+\frac{1}{4}\cdot\frac{12}{5^3}+\frac{1}{4}\cdot\frac{18}{5^3}+\frac{1}{4}\cdot\frac{16}{5^3}=\frac{1}{10}.$$

Hence

$$\mathbf{P}(A|HHT)=\frac{\frac{1}{4}\cdot\frac{4}{5^3}}{\mathbf{P}(HHT)}=\frac{2}{25},$$

$$\mathbf{P}(B|HHT)=\frac{\frac{1}{4}\cdot\frac{12}{5^3}}{\mathbf{P}(HHT)}=\frac{6}{25},$$

$$\mathbf{P}(C|HHT)=\frac{\frac{1}{4}\cdot\frac{18}{5^3}}{\mathbf{P}(HHT)}=\frac{9}{25},$$

$$\mathbf{P}(D|HHT)=\frac{\frac{1}{4}\cdot\frac{16}{5^3}}{\mathbf{P}(HHT)}=\frac{8}{25},$$

so it is more likely to be coin C.

37 Let Y, F, E denote the events of choosing the 30% heads, the 50% heads, and the 80% heads, respectively. Now,

$$\begin{aligned} \mathbf{P}(HHT) &= \mathbf{P}(HHT|Y)\cdot\mathbf{P}(Y)+\mathbf{P}(HHT|F)\cdot\mathbf{P}(F)+\mathbf{P}(HHT|E)\cdot\mathbf{P}(E) \\ &= \frac{3\times3\times7}{1000}\cdot\frac{1}{3}+\frac{5\times5\times5}{1000}\cdot\frac{1}{3}+\frac{8\times8\times2}{1000}\cdot\frac{1}{3} \\ &= \frac{79}{750}, \end{aligned}$$

whence

$$\begin{aligned} \mathbf{P}(F|HHT) &= \frac{\mathbf{P}(F\cap HHT)}{\mathbf{P}(HHT)} \\ &= \frac{\mathbf{P}(HHT|F)\cdot\mathbf{P}(F)}{\mathbf{P}(HHT)} \\ &= \frac{\frac{5\times5\times5}{1000}\cdot\frac{1}{3}}{\frac{79}{750}} \\ &= \frac{125}{316} \end{aligned}$$

38 Observe that there are 10 ways of getting a sum of six in three dice: the 3 permutations of $(1,1,4)$, the 6 permutation of $(1,2,3)$, and the 1 permutations of $(2,2,2)$. Of these, only $(2,2,2)$ does not require a 1. Let S be the event that the sum of the dice is 6 and let N be the event that no die landed on a 1. We need

$$\mathbf{P}(N|S)=\frac{\mathbf{P}(N\cap S)}{\mathbf{P}(S)}=\frac{\frac{1}{216}}{\frac{10}{216}}=\frac{1}{10}.$$

39 We have

$$\mathbf{P}(A \cap B) = \mathbf{P}(A|B)\,\mathbf{P}(B) = \frac{1}{6}, \implies \mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = 1$$

whence

$$\mathbf{P}(A^c \cap B^c) = \mathbf{P}((A \cup B)^c) = 1 - \mathbf{P}(A \cup B) = 0.$$

40 Let G be the event that the lemon is green and A that it is of grade A, etc. Then we want

$$\mathbf{P}(A|G) = \frac{\mathbf{P}(A \cap G)}{\mathbf{P}(G)} = \frac{.4(.01)}{.4(.01) + .4(.0075) + .2(.005)} = 0.5.$$

41 Let T denote the event that Tom operates the machinery, S the event that Sally operates the machinery and H that two out of three pieces of the output be of high quality. Then

$$\begin{aligned}\mathbf{P}(H) &= \mathbf{P}(H|T)\cdot\mathbf{P}(T) + \mathbf{P}(H|S)\cdot\mathbf{P}(S)\\ &= \tbinom{3}{2}\left(\frac{70}{100}\right)^2\left(\frac{30}{100}\right)\cdot\frac{3}{5} + \tbinom{3}{2}\left(\frac{90}{100}\right)^2\left(\frac{10}{100}\right)\cdot\frac{2}{5}\\ &= \frac{1809}{5000},\end{aligned}$$

whence

$$\begin{aligned}\mathbf{P}(T|H) &= \frac{\mathbf{P}(H|T)\cdot\mathbf{P}(T)}{\mathbf{P}(H)}\\ &= \frac{\tbinom{3}{2}\left(\frac{70}{100}\right)^2\left(\frac{30}{100}\right)\cdot\frac{3}{5}}{\frac{1809}{5000}}\\ &= \frac{49}{67}.\end{aligned}$$

42 Plainly,

❶

$$\mathbf{P}(R_2) = \frac{4}{9}\cdot\frac{6}{11} + \frac{5}{9}\cdot\frac{3}{5} = \frac{19}{33}.$$

❷

$$\mathbf{P}(R_2 \cap R_1) = \frac{4}{9}\cdot\frac{6}{11} = \frac{8}{33}$$

❸

$$\mathbf{P}(R_1|R_2) = \frac{\mathbf{P}(R_2 \cap R_1)}{\mathbf{P}(R_2)} = \frac{8}{19}.$$

43 Let A be the event that Peter's letter is received by Paul and B be the event that Paul's letter is received by Peter. Then we want $\mathbf{P}(A|B^c)$. Then

$$\mathbf{P}(A|B^c) = \frac{A \cap B^c}{\mathbf{P}(B^c)} = \frac{\mathbf{P}(B^c|A)\cdot\mathbf{P}(A)}{\mathbf{P}(B^c|A)\cdot\mathbf{P}(A) + \mathbf{P}(B^c|A^c)\cdot\mathbf{P}(A^c)} = \frac{\frac{1}{n}\cdot\frac{n-1}{n}}{\frac{1}{n}\cdot\frac{n-1}{n} + 1\cdot\frac{1}{n}} = \frac{n-1}{2n-1}.$$

44 We condition on whether the interchanged card is the one selected on the second half. Let A be the event that the selected on the second half card was an ace, and let I be the event that the card selected was the interchanged one. Then

$$\mathbf{P}(A) = \mathbf{P}(A|I)\,\mathbf{P}(I) + \mathbf{P}(A|I^c)\,\mathbf{P}(I^c) = 1\cdot\frac{1}{27} + \frac{3}{51}\cdot\frac{26}{27} = \frac{43}{459}.$$

45 We want $\mathbf{P}(A|D)$. This can happen in two moves (from D to C to A) with probability $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$, or it can happen in 4 moves (from D to C to D to C to A) with probability $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{16}$, or in six moves, …,etc. We must sum thus the infinite geometric series

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.$$

The required probability is therefore $\frac{1}{3}$.

46 Let $\mathbf{G}$ be the random variable denoting the gain of the player. Then $\mathbf{G}$ has image $\{0, 1, 3, 5\}$ and

$$\mathbf{P}(\mathrm{G} = 0) = \frac{1}{2}, \qquad \mathbf{P}(\mathrm{G} = 1) = \mathbf{P}(\mathrm{G} = 3) = \mathbf{P}(\mathrm{G} = 5) = \frac{1}{6}.$$

Thus

$$E\mathrm{G} = 0\mathbf{P}(\mathrm{G} = 0) + 1\mathbf{P}(\mathrm{G} = 1) + 3\mathbf{P}(\mathrm{G} = 3) + 5\mathbf{P}(\mathrm{G} = 5) = \frac{1 + 3 + 5}{6} = \frac{3}{2},$$

meaning that the fee should be $\$1.50$.

47 Let $\mathbf{G}$ be the random variable denoting Osa's net gain. Then $\mathbf{G}$ has image $\{-1, 1, 12\}$ and

$$\mathbf{P}(\mathrm{G} = -1) = \frac{38}{52}, \quad \mathbf{P}(\mathrm{G} = 1) = \frac{13}{52}, \quad \mathbf{P}(\mathrm{G} = 12) = \frac{1}{52}.$$

Thus

$$E\mathrm{G} = -1\mathbf{P}(\mathrm{G} = -1) + 1\mathbf{P}(\mathrm{G} = 1) + 12\mathbf{P}(\mathrm{G} = 13) = \frac{-38 + 13 + 12}{52} = -\frac{13}{52} = -0.25,$$

and so the net gain is $-\$0.25$.

48 Let $\mathbf{X}_i = 0$ if the i-th pair does not come back, and $\mathbf{X}_i = 1$ if it does. We want

$$E\mathrm{X}_1 + \cdots + E\mathrm{X}_6 = 6E\mathrm{X}_1 = 6\mathbf{P}(\mathrm{X}_1 = 1),$$

since the $\mathbf{X}_i$ have the same distribution. Now

$$\mathbf{P}(\mathrm{X}_1 = 1) = \frac{\binom{2}{2} \cdot \binom{10}{5}}{\binom{12}{7}} = \frac{7}{22},$$

and the required expectation is $\frac{21}{11}$.

49 Let $\mathbf{X}_i = 0$ if the i-th pair does not come back, and $\mathbf{X}_i = 1$ if it does. We want

$$E\mathrm{X}_1 + \cdots + E\mathrm{X}_5 = 5E\mathrm{X}_1 = 5\mathbf{P}(\mathrm{X}_1 = 1),$$

since the $\mathbf{X}_i$ have the same distribution. Now

$$\mathbf{P}(\mathrm{X}_1 = 1) = \frac{\binom{2}{2} \cdot \binom{8}{5}}{\binom{10}{7}} = \frac{7}{15},$$

and the required average is $\frac{7}{3}$.

50 Let X be the random variable counting the number of drawings. Now, $X = 2$ means that matching socks are obtained when 2 socks are drawn. Hence

$$\mathbf{P}(X = 2) = \frac{\binom{2}{1}\binom{4}{2}}{\binom{8}{2}} = \frac{3}{7},$$

and thus $\mathbf{P}(X = 3) = \frac{4}{7}$. Therefore

$$EX = 2\mathbf{P}(X = 2) + 3\mathbf{P}(X = 3) = 2 \cdot \frac{3}{7} + 3 \cdot \frac{4}{7} = \frac{18}{7}.$$

51 $\mathbf{X}$ is a binomial random variable with $E\mathrm{X} = np = \frac{72}{6} = 12$ and $\mathrm{var}\mathrm{X} = np(1 - p) = 72\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = 10$. But $E\mathrm{X}^2 = \mathrm{var}(\mathrm{X}) + (E\mathrm{X})^2 = 10 + 12^2 = 154$.

52 For $1 \leq i \leq 15$ put put

$$X_i = \begin{cases} 1 & \text{if the } i\text{−th pair remains in the urn.} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mathbf{P}(X_i = 1) = \frac{\binom{28}{10}\binom{2}{2}}{\binom{30}{10}} = \frac{\dfrac{28!}{18!10!}}{\dfrac{30!}{20!18!}} = \frac{38}{87},$$

and the desired expectation is $\frac{15 \cdot 38}{87} = \frac{190}{29}$.

53 (1) Consider the 48 cards which are not kings and for $1 \leq i \leq 48$ put

$$X_i = \begin{cases} 1 & \text{if the } i\text{−th non − king appears before a king.} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$X = 1 + \sum_{i=1}^{48} X_i$$

is the number of cards turned up in order to obtain a king. Let us prove that $\mathbf{P}\left(X_i = 1\right) = \frac{1}{5}$. For, if I denotes the i-th non-king and $KKKK$ the four kings, we seek permutations of $IKKKK$, with the remaining other cards interspersed between them. There are $\frac{5!}{4!} = 5$ permutations of these cards and only one of them is successful. Thus $\mathbf{P}\left(X_i = 1\right) = \frac{1}{5}$, as claimed. This gives

$$EX = 1 + \frac{48}{5} = \frac{53}{5}.$$

(2) Using a similar reasoning, the expectation is

$$1 + \frac{39}{14} = \frac{53}{14}.$$

54 We start with barrel A. Abusing notation, we will let A the random variable indicating the number of years of wine in barrel A, etc. After the transfer has been made, the mean age of the new wine is 0 years and the mean age of the old wine is a year older than what it was. Hence

$$A = \frac{1}{3}A_{\text{new}} + \frac{2}{3}A_{\text{old}} \implies EA = \frac{1}{3}EA_{\text{new}} + \frac{2}{3}EA_{\text{old}} \implies EA = \frac{1}{3} \cdot 0 + \frac{2}{3}(1 + EA) \implies EA = 2.$$

Thus $EA_{\text{old}} = 3$. Now,

$$B = \frac{1}{3}B_{\text{new}} + \frac{2}{3}B_{\text{old}} = \frac{1}{3}A_{\text{old}} + \frac{2}{3}B_{\text{old}} \implies EB = \frac{1}{3} \cdot 3 + \frac{2}{3}EB_{\text{old}} \implies EB = \frac{3}{3} + \frac{2}{3}(1 + EB) \implies EB = 5.$$

Hence, $EB_{\text{old}} = 6$. Similarly,

$$C = \frac{1}{3}C_{\text{new}} + \frac{2}{3}C_{\text{old}} = \frac{1}{3}B_{\text{old}} + \frac{2}{3}C_{\text{old}} \implies EC = \frac{1}{3} \cdot 6 + \frac{2}{3}EC_{\text{old}} \implies EC = \frac{6}{3} + \frac{2}{3}(1 + EC) \implies EC = 8.$$

55 The fastest way to do this is perhaps the following. Let $X_i = 1$ if the i-th boy is selected, $X_i = 0$ otherwise. Then $\mathbf{P}\left(X_i = 1\right) = \frac{\binom{24}{7}}{\binom{25}{8}} = \frac{8}{25}$ and $EX = \frac{10 \cdot 8}{25} = \frac{16}{5}$. Similarly, let $Y_i = 1$ if the i-th girl is selected, $Y_i = 0$ otherwise. Then $\mathbf{P}\left(Y_i = 1\right) = \frac{\binom{24}{7}}{\binom{25}{8}} = \frac{8}{25}$ and $EY = \frac{15 \cdot 8}{25} = \frac{24}{5}$. Thus $E(X - Y) = EX - EY = -\frac{8}{5}$.

56 There are 8 words in the sentence, and they have 2, 3, 4, or 9 letters. Then

$$EX = \frac{2 \cdot 1 + 3 \cdot 5 + 4 \cdot 1 + 9 \cdot 1}{8} = \frac{15}{4},$$

$$EX^2 = \frac{2^2 \cdot 1 + 3^2 \cdot 5 + 4^2 \cdot 1 + 9^2 \cdot 1}{8} = \frac{73}{4},$$

whence

$$\operatorname{var}(X) = \frac{73}{4} - \left(\frac{15}{4}\right)^2 = \frac{67}{16}.$$

57 The expected number is plainly

$$3(1/8) + 5(1/4) + 2(1/2) = \frac{21}{8}.$$

58 The player may have:

- three wins, with probability $\binom{3}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ and his fortune increases eightfold.
- two wins, and one loss, with probability $\binom{3}{2}\left(\frac{1}{2}\right)^3 = \frac{3}{8}$ and his fortune doubles.
- one win, and two losses, with probability $\binom{3}{1}\left(\frac{1}{2}\right)^3 = \frac{3}{8}$, and his fortune halves.

- three losses, with probability $\binom{3}{0}\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ and his fortune reduces by a factor of 8.

His expected fortune is thus

$$8\left(8 \cdot \frac{1}{8} + 2 \cdot \frac{3}{8} + \frac{1}{2} \cdot \frac{3}{8} + \frac{1}{8} \cdot \frac{1}{8}\right) = \frac{125}{8}.$$

59 Let $T_k, k = 0, 1, 2, 3, 4$ be the number of walks accomplished when there are k pairs of shoes at the front door. By conditioning, for $k = 1, 2, 3$,

$$E(T_k) = \frac{1}{4}(1 + E(T_{k-1})) + \frac{1}{2}(1 + E(T_k)) + \frac{1}{4}(1 + E(T_{k+1})) \implies E(T_k) = 2 + \frac{1}{2}E(T_{k-1}) + \frac{1}{2}E(T_{k+1}).$$

Also,

$$E(T_4) = \frac{1}{2}(0) + \frac{1}{4}(1 + E(T_4)) + \frac{1}{4}(1 + E(T_3)) \implies E(T_4) = \frac{1}{3}\left(2 + E(T_3)\right).$$

and

$$E(T_0) = \frac{1}{2}(0) + \frac{1}{4}(1 + E(T_1)) + \frac{1}{4}(1 + E(T_0)) \implies E(T_0) = \frac{1}{3}\left(2 + E(T_1)\right).$$

This gives the system of five linear equations in five unknowns, for which we write the augmented matrix (with variables $E(T_k), k = 0, \ldots, 4$ in order)

$$\left[\begin{array}{ccccc|c} -1 & 2 & -1 & 0 & 0 & 4 \\ 0 & -1 & 2 & -1 & 0 & 4 \\ 0 & 0 & -1 & 2 & -1 & 4 \\ 3 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 & 3 & 2 \end{array}\right]$$

Solving this system, we get $E(T_0) = 4$, $E(T_1) = 10$, $E(T_2) = 12$, $E(T_3) = 10$, and $E(T_4) = 4$. The quantity we want is $E(T_2) = 12$.

60 Performing row operations on the augmented matrix of the system

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ A & -B & C \end{array}\right] \xrightarrow{R_2 - AR_1 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 2A - B & C - 3A \end{array}\right].$$

For infinitely many solutions, we need $2A = B; 3A = C$, hence B is even and C is a multiple of 3, giving $(A, B, C) = (1, 2, 3) \quad \textbf{or} \quad (2, 4, 6).$ The probability of infinitely many solutions is thus $\dfrac{2}{216} = \dfrac{1}{108}$.

If the system will have no solutions, then $2A = B$ and $3A \neq C$. For $(A, B) = (1, 2)$ we have 5 choices of C; for $(A, B) = (2, 4)$ we have 5 choices of C; and for $(A, B) = (3, 6)$ we have 6 choices of C. Hence there are $5 + 5 + 6 = 16$ successes, and the probability sought is $\dfrac{16}{216} = \dfrac{2}{27}$.

For the system to have exactly one solution we need $2A \neq B$. If $A = 1, 2$ or 3, then B cannot $B = 2, 4$ or 6, giving $5 + 5 + 5 = 15$ choices of B in these cases. If $A = 4, 5$ or 6, then B can be any of the 6 choices, giving $6 + 6 + 6 = 18$ in these cases. These $15 + 18 = 33$ choices of B can be combined with any 6 choices of C, giving $33 \cdot 6 = 198$ choices. The probability in this case is thus $\dfrac{198}{216} = \dfrac{11}{12}$.

For the system to have $x = 3, y = 0$ as its unique solution, we need $2A \neq B$ and $3A = C$. If $A = 1$ then $C = 3$ and we have 5 choices for B. If $A = 2$ then $C = 6$ and again, we have 5 choices for B. Hence there are 10 successes and the probability sought is $\dfrac{10}{216} = \dfrac{5}{108}$.

61 For a single toss, the expected value is

$$\frac{1 + 2 + 3}{3} = 2,$$

and the variance is

$$\frac{1^2 + 2^2 + 3^2}{3} - 4 = \frac{2}{3}.$$

For 600 tosses, the expected value is $2(600) = 1200$, and the variance is $\dfrac{2}{3}(600) = 400$, hence the standard deviation is $\sqrt{400} = 20$. If S is the sum,

$$\mathbf{P}\left(1200 \leq S \leq 1230\right) = \mathbf{P}\left(\frac{1200 - 1200}{20} \leq \frac{S - 1200}{20} \leq \frac{1230 - 1200}{20}\right) = \mathbf{P}\left(0 \leq Z \leq 1.5\right),$$

where Z is a standard normal random variable. The desired probability is thus

$$\mathbf{P}(0 \leq Z \leq 1.5) = \Phi(1.5) - \Phi(0) = 0.9332 - 0.5 = 0.4332.$$

62 The number X of heads behaves like a binomial random variable with $n = 150$ and $p = \frac{2}{5}$. Hence $\mathbf{E}X = np = 150(\frac{2}{5}) = 60$ and $\mathbf{var}X = np(1-p) = 150(\frac{2}{5})(\frac{3}{5}) = 36$. Thus the desired probability is

$$\begin{aligned}
\mathbf{P}(70 \leq X \leq 90) &\approx \mathbf{P}(69.5 \leq X \leq 90.5) \\
&= \mathbf{P}\left(\frac{69.5-60}{6} \leq \frac{X-60}{6} \leq \frac{90.5-60}{6}\right) \\
&\approx \mathbf{P}(1.58 \leq Z \leq 5.08) \\
&= \mathbf{P}(Z \leq 5.08) - \mathbf{P}(Z \leq 1.58) \\
&= 1 - 0.9429 \\
&= 0.0571.
\end{aligned}$$

63 The transition matrix is

$$\begin{bmatrix} 0.70 & 0.30 \\ 0.40 & 0.60 \end{bmatrix}.$$

Solving

$$\begin{bmatrix} a & 1-a \end{bmatrix} \begin{bmatrix} 0.70 & 0.30 \\ 0.40 & 0.60 \end{bmatrix} = \begin{bmatrix} a & 1-a \end{bmatrix}$$

we get $a = \dfrac{4}{7}$. The probability sought is thus $\dfrac{4}{7}$.

64 The transition matrix is

$$\begin{bmatrix} 0.20 & 0.80 \\ 0.10 & 0.90 \end{bmatrix}.$$

Solving

$$\begin{bmatrix} a & 1-a \end{bmatrix} \begin{bmatrix} 0.20 & 0.80 \\ 0.10 & 0.90 \end{bmatrix} = \begin{bmatrix} a & 1-a \end{bmatrix}$$

we get $a = \dfrac{1}{9}$. The probability sought is thus $\dfrac{1}{9}$.

65 The transition matrix is

$$\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

Solving

$$\begin{bmatrix} a & b & 1-a-b \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} a & b & 1-a-b \end{bmatrix}$$

we get $a = \dfrac{5}{18}, b = \dfrac{1}{3}$. The probability sought is thus $\dfrac{5}{18}$.