1 Sets on the Line

1 Problem Consider the set

$$\{x \in \mathbb{R} : x^2 - x - 6 \leq 0\}.$$

1. Draw a sign diagram depicting this set.
2. Using the sign diagram above, express the set $\{x \in \mathbb{R} : x^2 - x - 6 \leq 0\}$ as an interval, or as a union of intervals. Draw a graph of the set on the line.
3. Using the sign diagram above, express the set $\left\{x \in \mathbb{R} : \dfrac{x-3}{x+2} \geq 0\right\}$ as an interval, or as a union of intervals. Draw a graph of the set on the line.

2 Problem Consider the equation

$$|x-3| + |x+2| = 5.$$

1. Write $|x-3| + |x+2|$ without absolute values in $]-\infty; -2[$. You may wish to draw a sign diagram.
2. Write $|x-3| + |x+2|$ without absolute values in $]-2; +3[$.
3. Write $|x-3| + |x+2|$ without absolute values in $]+3; +\infty[$.
4. Find all solutions to the equation $|x-3| + |x+2| = 5$.

3 Problem Write the infinite repeating decimal $0.10101010\ldots = 0.\overline{10}$ as a quotient of two integers.

4 Problem Let $\mathbb{R} \setminus \mathbb{Q}$ be the set of irrational numbers. Is this set closed under multiplication? Is this set closed under addition?

5 Problem Consider the decimal

$$0.101001000010000000001\ldots$$

where between successive 1's there are 2^0, 2^1, 2^2, 2^3, etc. 0's. Argue why this number must be irrational.

6 Problem Express as a single interval: $[-2\,;4\,[\ \setminus\ [1\,;5]$.

7 Problem Use interval notation to list the values of x satisfying the inequality

$$\frac{x^2+x-6}{x^2-x-6} \geq 0.$$

8 Problem How many real solutions does the equation $|x^2 - 4x| = 3$ have?

9 Problem Solve the equation $\left|\dfrac{2x}{x-1}\right| = |x+1|$ for x.

10 Problem Express $|1 - |2 - x||$ without absolute values given that $x > 3$.

11 Problem Write the set

$$\{x \in \mathbb{R} : |x+1| - |x-2| = -3\}$$

as an interval, or as a union of intervals.

12 Problem Write the set

$$\{x \in \mathbb{R} : (x-1)(x+2) \geq 0\} \cap \left\{x \in \mathbb{R} : \frac{x^3 - 2x^2}{x^2 - x - 6} \leq 0\right\}$$

as an interval or as a union of intervals.

13 Problem Write the set

$$\left\{x \in \mathbb{R} : x^2 - x - 6 \leq 0\right\} \cap \left\{x \in \mathbb{R} : \frac{1-x}{x+3} \geq 1\right\}$$

as an interval or as a union of intervals.

2 Sets on the Plane. Distance between two points.

14 Problem What is the distance between the points $(a, 0)$ and $(0, b)$?

15 Problem What is the distance between the points $(a+1, b+1)$ and $(a+2, b+2)$?

16 Problem Find x if the point $(x, 1)$ is at distance 2 from the point $(3, x)$.

17 Problem Draw the plane region

$$\left\{(x, y) \in \mathbb{R}^2 : |x - 1| \leq 1, \quad |y| \geq 2\right\}.$$

18 Problem Draw the plane region

$$\left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2, x \geq 1\right\}.$$

19 Problem Draw the plane region

$$\left\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\right\}.$$

20 Problem Find the point equidistant from $A = (-1, 3)$, $B = (2, 4)$ and $C = (1, 1)$.

3 Circles.

21 Problem A circle has diameter with endpoints at $(-1, 2)$ and $(3, 4)$.

1. Find its centre.
2. Find its radius.
3. Find its canonical equation.
4. Draw the circle.
5. Find the equation of the line tangent to the circle at $(3, 4)$.

22 Problem Find the center and the radius of the circle

$$x^2 + 4x + y^2 - 6y = 2.$$

23 Problem Find the equation of the circle passing through $(0, 1)$, $(1, 0)$ and $(1, 1)$.

24 Problem Find the equation of the circle passing through $(1, 1)$, $(0, 1)$ and $(1, 2)$.

25 Problem A point (x, y) moves such that it is always equidistant from the point $(2, 3)$ and the line $x = -4$. Find the equation of its locus.

26 Problem A point (x, y) moves such that it is equidistant from the points $(2, 5)$ and $(3, -1)$. Determine an equation for its locus.

27 Problem Consider the region

$$\mathscr{R} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\} \setminus \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}.$$

1. Draw it.
2. Find its area.

28 Problem If a, b, c are real constants, find the center and the radius of the circle

$$x^2 + 4ax + y^2 - 2by = c^2.$$

29 Problem Let $0 < a < b$. What is the area of the region in the first quadrant bounded by the x-axis, the line $y = x$ and the circles $x^2 + y^2 = a$ and $x^2 + y^2 = b$?

4 Affine Curves. Distance between a Point and a Line. Absolute Values with Affine Terms

30 Problem The lines with equations $L : ax + by = c$ and $L' : dx + ey = f$ are perpendicular, where a, b, c, d, e, f are non-zero constants. What relationship holds between these constants?

31 Problem Consider the line L passing through $A(-1, 2)$ and $B(1, 3)$.

1. Find the slope of L.
2. Find the equation of the line L.
3. Find the equation of the line L' parallel to L and passing through $(4, 4)$.
4. Find the equation of the line L'' perpendicular to L and passing through $(4, 4)$.
5. Find the equation of the line L_1 which is the reflexion of the line L about the x-axis.
6. Find the equation of the line L_2 which is the reflexion of the line L about the y-axis.

32 Problem Consider the curve $\mathscr{C} : y = |x - 1| - |x| + |x + 1|$.

1. Write an expression *without absolute values* for $\mathscr{C}$ when $x \leq -1$.
2. Write an expression *without absolute values* for $\mathscr{C}$ when $-1 \leq x \leq 0$.
3. Write an expression *without absolute values* for $\mathscr{C}$ when $0 \leq x \leq 1$.
4. Write an expression *without absolute values* for $\mathscr{C}$ when $x \geq 1$.
5. Draw $\mathscr{C}$.

33 Problem Problems 1 through 4 refer to the point $P : (1, -1)$ and the line $L : x + 2y = 1$.

1. Find the equation of the line L' passing through P and perpendicular to L.
2. Find the intersection of the line L and the line L' found in problem 1.
3. Find the distance between the point P and the line L.
4. Find the equation of the line L' passing through P and parallel to L.

34 Problem Questions 1 to 5 refer to the straight line L_u given by the equation

$$L_u : \quad (u-2)y = (2u+4)x + 2u,$$

where u is a real parameter.

1. For which value of u is L_u a horizontal line?
2. For which value of u is L_u a vertical line?
3. For which value of u is L_u parallel to the line $y = -2x + 1$?
4. For which value of u is L_u perpendicular to the line $y = -2x + 1$?
5. Is there a point which is on every line L_u regardless the value of u? If so, find it. If not, prove that there is no such point.

35 Problem Figure 1 refers to problems 1 through 3. You may assume that lines L and L' are perpendicular and that they meet at the point P.

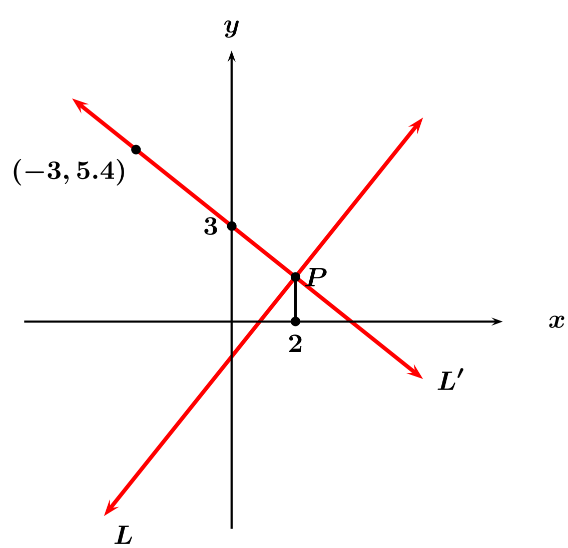


Figure 1: Problems 1 through 3.

1. Find the equation of the line L'.
2. Find the point P.
3. Find the equation of the line L.

36 Problem A vertical line divides the triangle with vertices $(0,0)$, $(1,1)$ and $(9,1)$ in the plane into two regions of equal area. Find the equation of this vertical line.

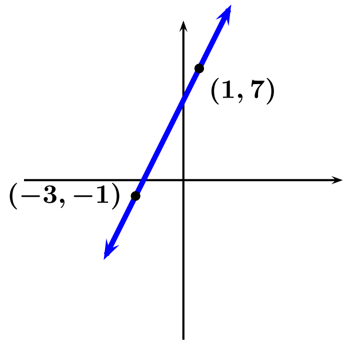
37 Problem Consider the infinite family of lines

$$L_t : (2-t)x + t = (t-3)y,$$

where $t \in \mathbb{R}$ is a real parameter.

1. Find t so that L_t be parallel to the line $L : 2x + y = -1$. Explain your reasoning.
2. Find t so that L_t be perpendicular to the line $L : 2x + y = -1$. Explain your reasoning.
3. Find t so that L_t be equidistant from the points $(2,1)$ and $(0,0)$. Explain your reasoning.

38 Problem The figure below shews a curve of the form $y = ax + b$. Find (a, b).



5 Functions

I write $f : \begin{array}{ccc} \mathbf{Dom}(f) & \to & \mathbf{Target}(f) \\ x & \mapsto & f(x) \end{array}$ for a function with name f, input variable x, assignment rule $x \mapsto f(x)$, domain $\mathbf{Dom}(f)$ and target set $\mathbf{Target}(f)$. The *image* $\mathbf{Im}(f)$ of f is the set

$$\mathbf{Im}(f) = \{y \in \mathbf{Target}(f) : \exists x \in \mathbf{Dom}(f), y = f(x)\}.$$

39 Problem (*Contributed by E. Chertok*) Consider the assignment rules $f(x) = \sqrt{x^3 + 5x^2 + 4x}$ and $g(x) = 2\sqrt{x}$.

1. find the natural domain of f,
2. find the x- and y-intercepts of the curve $y = f(x)$,
3. if $f(4) = a$, express $f(1)$ in terms of a,
4. Solve equation $f(x) = g(x)$.

40 Problem (*Contributed by E. Chertok*) Consider the assignment rules $f(x) = \sqrt{16 + 6x - x^2}$ and $g(x) = 2 - x$.

1. find the natural domain of f,
2. which quantity is larger $f(4) + f(7)$ or $f(5) + f(6)$?,
3. draw the curve $y = f(x - 2)$,
4. Solve equation $f(x) = g(x)$.

41 Problem (*Contributed by E. Chertok*) Consider the assignment rule $f(x) = \sqrt{x^2 - 4x + 3}$.

1. find the natural domain of $g(x) = f(x) - f(x-2)$,
2. find the x- and y-intercepts of the curve $y = f(x)$,
3. find the x- and y-intercepts of the curve $y = g(x)$,
4. If $0 < x < 1$, simplify the expression
$$\frac{f(x)f(x-2)}{(x-3)\sqrt{x^2 - 6x + 5}}.$$

42 Problem Find the natural domain of definition of the assignment rule $x \mapsto \dfrac{x}{(x-1)\sqrt{4-x^2}}$.

43 Problem Suppose $f(x) = ax + b$, $g(x) = bx + a$ (a, b integers). If $f(1) = 8$ and $f(g(50)) - g(f(50)) = 28$, find the product of a and b.

44 Problem If $f(x) = x^2$, express $f(x+1) - f(x-1)$ as a polynomial in x.

45 Problem Let $f(x) = \sqrt{x}$. Demonstrate that
$$\frac{f(x+h) - f(x)}{h} = \frac{1}{\sqrt{x+h} + \sqrt{x}}.$$

46 Problem Let $f(x) = \dfrac{1}{x}$. Demonstrate that
$$\frac{f(x+h) - f(x)}{h} = -\frac{1}{x(x+h)}.$$

47 Problem Determine, with proof, whether the assignment rule $x \mapsto \dfrac{x}{|x|^3}$ is even, odd, or neither in its domain of definition.

48 Problem Let f be an odd function defined at $x = 0$. Find, with proof, the value of $f(0)$.

49 Problem Find, with proof, all functions defined on the set of real numbers which are simultaneously even and odd. Draw their graphs.

50 Problem Let
$$(2x-1)^5(x^2+x-1)^{20} = a_{45}x^{45} + a_{44}x^{44} + \cdots + a_1x + a_0$$
be a polynomial of degree 45. Find

1. a_0
2. $a_0 + a_1 + a_2 + \cdots + a_{44} + a_{45}$
3. $a_0 - a_1 + a_2 - a_3 + \cdots - a_{43} + a_{44} - a_{45}$
4. $a_0 + a_2 + a_4 + \cdots + a_{42} + a_{44}$
5. $a_1 + a_3 + a_5 + \cdots + a_{43} + a_{45}$

51 Problem How many functions are there from the set $\{a, b, c\}$ to the set $\{0, 1\}$?

52 Problem How many functions are there from the set $\{0, 1\}$ to the set $\{a, b, c\}$?

53 Problem What is the natural domain of definition of the assignment rule $f(x) = \dfrac{1}{\sqrt{x^2-4}}$?

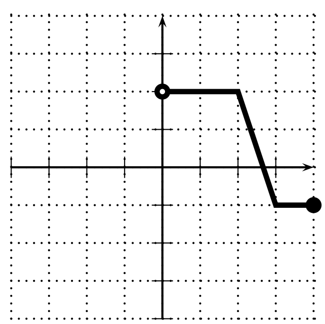
54 Problem What is the natural domain of definition of the assignment rule $f(x) = \dfrac{1}{\sqrt{-|x|}}$?

55 Problem What is the natural domain of definition of the assignment rule $f(x) = \sqrt{x} + \sqrt{-x}$?

56 Problem What is the natural domain of definition of the assignment rule $f(x) = \sqrt{x - 1 - \sqrt{2x-1}}$?

57 Problem The point x is said to be a *fixed point* for the function f if $f(x) = x$. Find all the fixed points of $f(x) = (x-2)^2$.

58 Problem Complete the graph fragment below so that the curve be (i) even, (ii) odd.



59 Problem If $f(2x) = \dfrac{2}{x}$, find $f(1)$.

60 Problem If $f(1-2x) = x^2 - 1$, find $f(2)$.

61 Problem If $f(a)f(b) = f(a+b)\ \forall\, a, b \in \mathbb{R}$ and $f(x) > 0\ \forall\, x \in \mathbb{R}$, find $f(0)$.

62 Problem Find all the functions f that satisfy $f(xy) = yf(x)$.

63 Problem Find all functions f for which

$$f(x) + 2f\left(\frac{1}{x}\right) = x.$$

64 Problem Find all functions $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ such that

$$(f(x))^2 \cdot f\left(\frac{1-x}{1+x}\right) = 64x.$$

65 Problem Let f and g be functions defined over $\mathbb{R}$ such that for all x and y

$$f(x+g(y)) = 2x + y + 5.$$

Find an explicit expression for $g(x+f(y))$.

5.1 Algebra of Functions

66 Problem Consider the assignment rules

$$a(x) = 1 - x^2, \quad b(x) = 1 + 2x, \quad c(x) = 4.$$

Find

1. (1 mark) $(a+b+c)(2)$
2. (1 mark) $(abc)(2)$
3. (1 mark) $(a \circ b \circ c)(2)$
4. (1 mark) $(c \circ b \circ a)(2)$

67 Problem If f is an even function defined over all real numbers and g is an odd function defined over all real numbers, how many of the following functions are even?

$$a(x) = (f \circ g)(x); \qquad b(x) = f(|x|) + g(|x|); \qquad c(x) = f(x^3) + g(x^2); \qquad d(x) = xf(x)g(x).$$

68 Problem f and g are odd functions defined on all real numbers and satisfying $f(-2) = 1$ and $g(-1) = -2$. What is the value of $(f \circ g)(1)$?

69 Problem Prove that in the product

$$(1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100})(1 + x + x^2 + x^3 + \cdots + x^{99} + x^{100})$$

after multiplying and collecting terms, there does not appear a term in x of odd degree.

70 Problem If $f(x) = \sqrt{x-2}$ and $g(x) = x^2 - 2$, find $(g \circ f)(4)$.

71 Problem Let

$$f : \begin{array}{ccc} \mathbb{R} \setminus \left\{-\frac{3}{2}\right\} & \to & \mathbb{R} \setminus \left\{\frac{c}{2}\right\} \\ x & \mapsto & \dfrac{cx}{2x+3} \end{array}$$

be such that

$$(f \circ f)(x) = x.$$

Find the value of c.

72 Problem Let $f^{[1]} = f$ be given by $f(x) = \dfrac{1}{1-x}$ and let $f^{[n+1]} = f \circ f^{[n]}$ for $n \geq 1$. Find $f^{[69]}(x)$.

73 Problem Problems 1 through 3 refer to a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = 2x + 1$.

1. Determine $f(3)$.
2. Determine $f(x)$.
3. Find $(f \circ f)(x)$.

6 Inverse Functions

74 Problem If $f(x) = 2x^3 + 1$ find $f^{-1}(x)$.

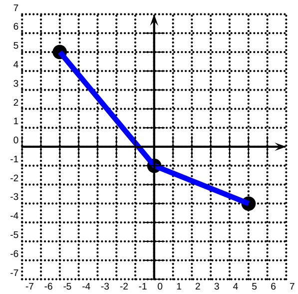
75 Problem Find f^{-1} for the assignment rule $f(x) = \sqrt[3]{\dfrac{x+2}{x-1}}$.

76 Problem f and g are invertible functions such that

$$f(1) = 2, \quad f(2) = 3, \quad f(3) = 1, \qquad g(1) = -1, \quad g(2) = 3, \quad g(4) = -2.$$

Find $(f \circ g)^{-1}(1)$.

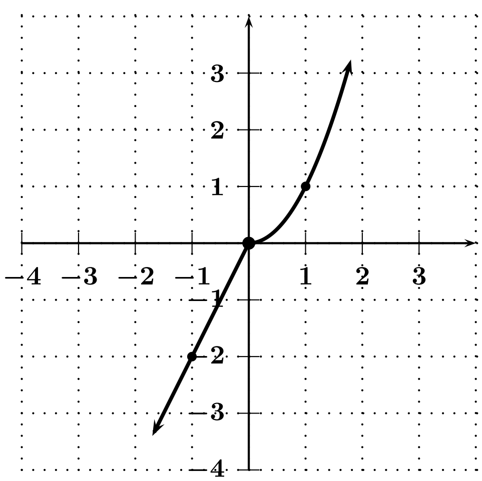
77 Problem Let $f : [-5\,;5] \to [-3\,;5]$ be the function whose graph appears below, which is composed of two lines. Observe that f is invertible.



1. Find formulæ for f and f^{-1} on $[-5\,;0]$.
2. Find formulæ for f and f^{-1} on $[0\,;5]$.
3. Draw the graph of f^{-1}.

78 Problem A function f appears below.

$$y = f(x) = \begin{cases} 2x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$$



1. Is f invertible? Why? Write a brief sentence justifying your answer.
2. If the answer above is affirmative, draw the graph of f^{-1}.
3. If the answer above is affirmative, write a formula for f^{-1}. Your formula will be a piecewise function.

79 Problem Prove, starting from the definition, that $f : \begin{array}{ccc} \mathbb{R} \setminus \{-1\} & \to & \mathbb{R} \setminus \{1\} \\ x & \mapsto & \dfrac{x-1}{x+1} \end{array}$ is a bijection and then find f^{-1}.

80 Problem Consider $f : x \mapsto x^2 - 4x + 5$ defined on all real numbers. Determine the two largest intervals I_1 and I_2 where f is injective, and find inverses for the restrictions $f\Big|_{I_1}$ and $f\Big|_{I_2}$. Select I_1 and I_2 such that $I_1 \cap I_2$ has exactly one point.

81 Problem How many injections are there from $\{a, b\}$ to $\{1, 2, 3\}$? How many surjections?

82 Problem How many injections are there from $\{1, 2, 3\}$ to $\{a, b\}$? How many surjections?

83 Problem Consider the assignment rule $f(x) = x^2 - 2x + 3$.

1. Is it true that f is invertible when $x \geq 0$? Why or why not? If so, find its inverse when $x \geq 0$.
2. Is it true that f is invertible when $x \geq 1$? Why or why not? If so, find its inverse when $x \geq 1$.

84 Problem Find f^{-1} for the assignment rule $f(x) = \dfrac{(x-1)^3}{x^3}$.

7 Transformations of Functions

85 Problem The curve $y = 2x + 1$ undergoes the following successive transformations: a translation one unit left, and a reflexion about the y-axis. What is the equation of the resulting curve?

86 Problem The curve $y = x^2 - 2x$ experiences the following successive transformations: (i) a translation one unit right, (ii) a reflexion about the x-axis, (iii) a translation two units up. Find the equation of the resulting curve.

87 Problem Draw the following curves in succession, without electronic recourse.

1. $y = x^2 - 4x$.
2. $y = -(x^2 - 4x)$.
3. $y = x^2 + 4x$.
4. $y = \left|x^2 - 4x\right|$.
5. $y = x^2 - 4\,|x|$.
6. $y = x^2 + 4\,|x|$.
7. $y = \left|x^2 - 4\,|x|\right|$.
8. $y = \left|x^2 + 4\,|x|\right|$.

88 Problem Draw the following curves in succession, without electronic recourse.

1. $y = \dfrac{1}{x-1}$.
2. $y = \dfrac{1}{-x-1}$.
3. $y = -\dfrac{1}{x-1}$.
4. $y = \dfrac{1}{x-1} + 2$.
5. $y = -\left(\dfrac{1}{x-1} + 2\right)$.
6. $y = \left|\dfrac{1}{x-1} + 2\right|$.
7. $y = \dfrac{1}{|x| - 1} + 2$.
8. $y = \left|\dfrac{1}{|x| - 1} + 2\right|$.

9. $y = \left|\dfrac{1}{-|x|-1} + 2\right|$.

89 Problem Draw the following curves in succession, without electronic recourse.

1. $y = 1 + x$.
2. $y = 1 + |x|$.
3. $y = |1 + x|$.
4. $y = |1 + |x||$.
5. $y = |1 + |1 + |x|||$.

90 Problem Given below is the graph of a curve $y = f(x)$. You are to find the graphs of $y = -f(x)$, $y = |f(x)|$, $y = |f(x)| - 1$, $y = |f(x-1)|$, $y = f(|x|)$.

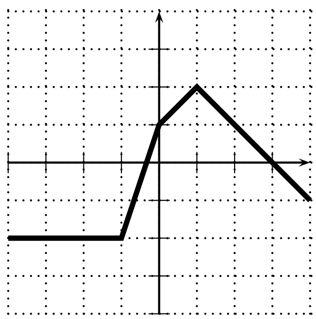


Figure 2: $y = f(x)$

91 Problem The curve $y = \dfrac{x}{x^2 - x - 1}$ undergoes the following successive transformations:

1. a translation one unit right,
2. a reflexion about the x-axis,
3. a translation two units up,
4. a reflexion about the y-axis.

Find the equation of the resulting curve.

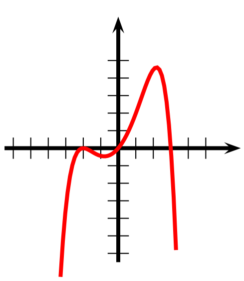


Figure 3: Problem 92. $y = f(x)$

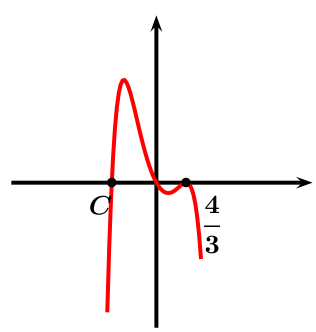


Figure 4: Problem 92. $y = f(ax)$

92 Problem Given in figures 3 and 4 are the graphs of two curves, $y = f(x)$ and $y = f(ax)$ for some real constant $a < 0$. Determine the value of the constant a. Determine the value of C.

93 Problem Figure 5 shows a functional curve $y = f(x)$. You are to match the letters of figures 6 to 16 with the equations on α through μ below. Some figures may not match with any equation, or viceversa.

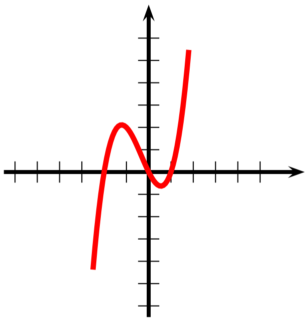


Figure 5: $y = f(x)$

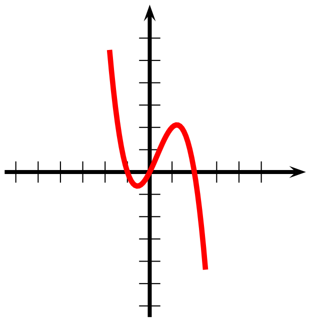


Figure 6: A

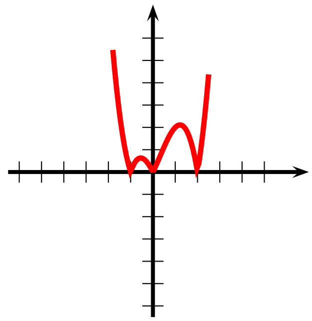


Figure 7: B

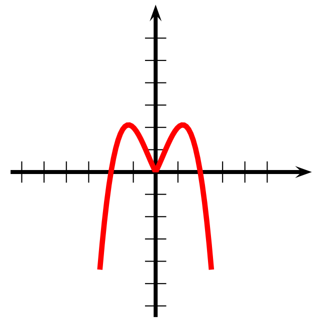


Figure 8: C

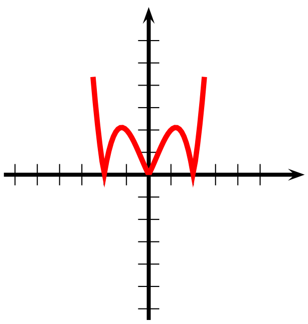


Figure 9: D

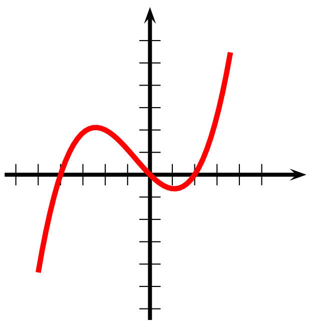


Figure 10: E

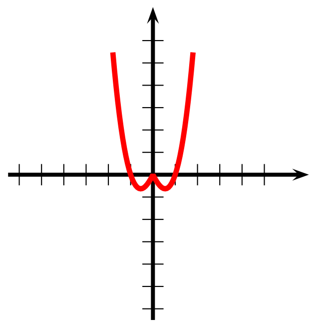


Figure 11: F

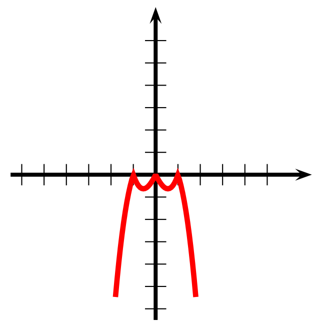


Figure 12: G

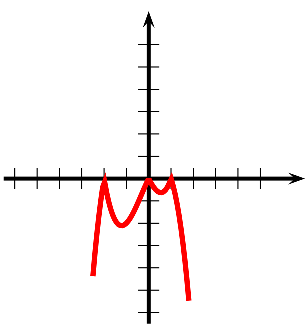


Figure 13: H

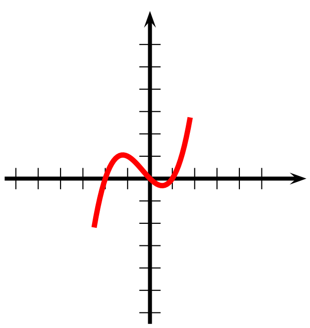


Figure 14: I

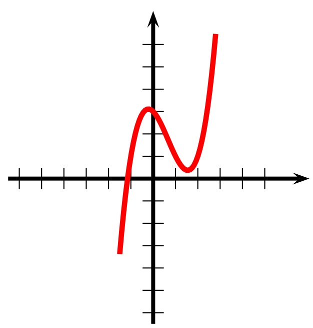


Figure 15: J

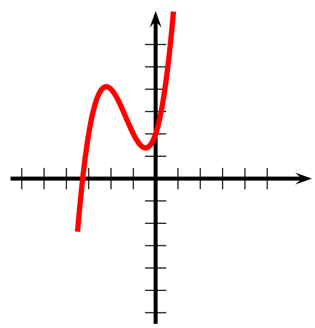


Figure 16: K

α. $y = f(-x) =$ ____________

β. $y = -f(-x) =$ ____________

γ. $y = f(-|x|) =$ ____________

δ. $y = f(x+1) + 2 =$ ____________

ϵ.$y = |f(-|x|)| =$ ____________

ζ. $y = -|f(|x|)| =$ ____________

η. $y = |f(-x)| =$ ____________

θ.$y = |f(-|x|/2)| =$ ____________

ι. $y = f(x/2) =$ ____________

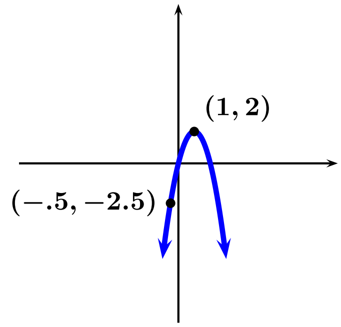
κ. $y = -|f(x)| =$ ____________

λ. $y = \frac{1}{2}f(x) =$ ____________

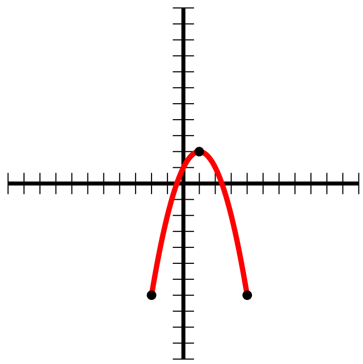
μ. $y = f(x-1) + 1 =$ ____________

8 Quadratic Functions

94 Problem The figure below shews a curve of the form $y = a(x - h)^2 + k$. Find (a, h, k).



95 Problem Find the equation of the quadratic function below. Each tick represents one unit.



96 Problem Find the vertex of the parabola $y = ax^2 - 6abx - ab^2$.

97 Problem What is the discriminant of the quadratic function $x \mapsto \sqrt{2}x^2 - 2x + 1$? What does it tell you about the nature of the roots of this function?

98 Problem Let $a \in \mathbb{R}$ be a real parameter. Prove that the quadratic function $x \mapsto x^2 + (a + 1)x + 1$ intersects the x-axis if and only if $x \in]-\infty\, ; -3] \cup [1\, ; +\infty\, [$.

99 Problem Of all rectangles having perimeter 20 show that the square has the largest area.

100 Problem If the positive y-axis indicates the direction "north" and you translate every point on the graph of $y = (x - 3)(x - 7)$ a distance of 2 units in the northeast direction, then what is the smallest y-value on the new graph?

101 Problem An orchard currently has 25 trees, which produce 600 fruits each. It is known that for each additional tree planted, the production of each tree diminishes by 15 fruits. Find:

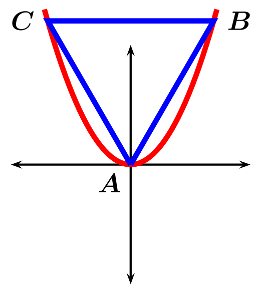
❶ the current fruit production of the orchard,

❷ a formula for the production obtained from each tree upon planting x more trees,

❸ a formula $P(x)$ for the production obtained from the orchard upon planting x more trees.

❹ How many trees should be planted in order to yield maximum production?

102 Problem Points A , B, and C are on the parabola $y = \frac{x^2}{2}$ as shown in the diagram. If $\triangle ABC$ is equilateral, determine the x-coordinate of point B.



103 Problem Find the maximum value of the function $a : \mathbb{R} \to \mathbb{R}, x \mapsto \sqrt{(1-x)(x+2)}$.

104 Problem Find all the real values of the parameter t for which the equation in x

$$t^2x - 3t = 81x - 27$$

has a solution.

105 Problem Find a quadratic function $f : x \mapsto ax^2 + bx + c$ such that $f(1) = 2$, $f(2) = 5$, $f(3) = 4$.

106 Problem A quadratic function has its graph tangent to the x-axis. If its graph passes through $(1, 2)$ and $(3, 2)$, find the formula of this function.

9 Polynomial Functions

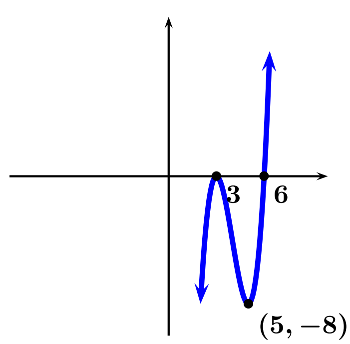
107 Problem A polynomial of degree 5 has real zeros only at $x = -1$, $x = 1$, and $x = 2$. It is known that its graph is tangent to the x-axis at $x = 1$ and $x = 2$. If $p(-2) = 1$, find its equation.

108 Problem Evaluate the expression $E = (2 + a)(2 + b)(2 + c)(2 + d)$; where a; b; c; d are distinct roots of the polynomial $P(x) = 2x^4 + x^3 - 8x^2 - x + 6$.

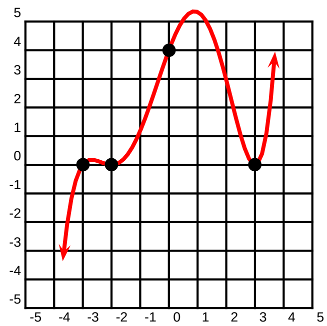
109 Problem Factor the polynomial $x^3 + 6x^2 + 11x + 6$ completely, if it is known that all its roots are integers.

110 Problem A polynomial of degree 3 satisfies $p(1) = 1$, $p(2) = 4$, and $p(3) = 9$ and has leading coefficient 1. Find it.

111 Problem The figure below shews a curve of the form $y = A(x - r)(x - s)^2$. Find (A, r, s).



112 Problem The polynomial in figure 112 has degree 5. You may assume that the points marked below with a dot through which the polynomial passes have integer coordinates.



1. Determine $p(0)$.
2. Determine $p(x)$ for all real numbers x.
3. Determine $(p \circ p)(0)$.

113 Problem The polynomial $p(x)$ has integral coefficients and $p(x) = 7$ for four different values of x. Shew that $p(x)$ never equals 14.

114 Problem Find the value of a so that the polynomial

$$t(x) = x^3 - 3ax^2 + 12$$

be divisible by $x + 4$..

115 Problem Let $f(x) = x^4 + x^3 + x^2 + x + 1$. Find the remainder when $f(x^5)$ is divided by $f(x)$.

116 Problem If $p(x)$ is a cubic polynomial with $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5$, find $p(6)$.

117 Problem The polynomial $p(x) = x^3 + ax + b$, with a and b constants, is divisible by $x + 1$. Moreover, when it is divided by $x - 1$ it leaves remainder 4. Find the constants a and b.

118 Problem A polynomial leaves remainder 1 when divided by $x + 1$ and remainder -4 when divided by $x + 2$. Find its remainder when it is divided by $x^2 + 3x + 2$.

119 Problem The polynomial $p(x)$ satisfies $p(-x) = -p(x)$. When $p(x)$ is divided by $x - 3$ the remainder is 6. Find the remainder when $p(x)$ is divided by $x^2 - 9$.

120 Problem Factorise $x^3 + 3x^2 - 4x + 12$ over $\mathbb{Z}[x]$.

121 Problem Factorise $3x^4 + 13x^3 - 37x^2 - 117x + 90$ over $\mathbb{Z}[x]$.

122 Problem Find a, b such that the polynomial $x^3 + 6x^2 + ax + b$ be divisible by the polynomial $x^2 + x - 12$.

10 Rational and Algebraic Functions

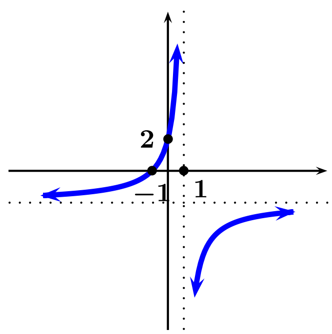
123 Problem Consider the curve $y = -\dfrac{x}{x^2 + 3x + 2}$.

1. Find its zeroes.
2. Find its poles.
3. Find its behaviour as $x \to \pm\infty$. That is, if it is known that it behaves like an expression of the form $\dfrac{a}{x^b}$, where a, b are integers, find a and b.
4. Draw a rough sketch of it.

124 Problem Sketch the graph of the curve $y = \sqrt{\dfrac{1-x}{x+1}}$.

125 Problem Sketch the graph of the curve $y = \sqrt{\dfrac{x-1}{x+1}}$.

126 Problem The figure below shews a curve of the form $y = \dfrac{ax+b}{c-x}$. Find (a, b, c).



127 Problem If the graph of $x \mapsto \dfrac{2x^2+6x}{x^2+3x-4}$ is symmetric about the line $x = a$, find a.

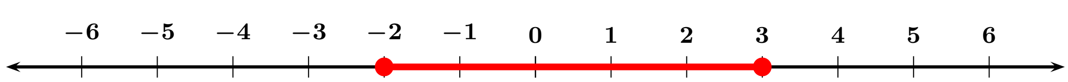
11 Some Answers, Solutions, and Hints

1

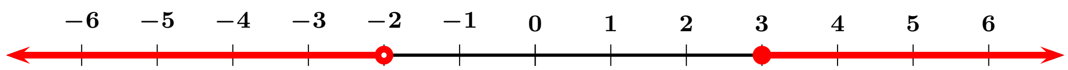
1. Observe that $x^2 - x - 6 = (x-3)(x+2)$. Hence we must observe the behaviour of the product in a neighbourhood of $x = -2$ and $x = 3$.

$x \in$	$]-\infty\,;-2]$	$[-2\,;3]$	$[3\,;+\infty\,[$
$x+2$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
$(x+2)(x-3)$	$+$	$-$	$+$

2. From above this is the interval $[-2\,;3]$. The set is shaded in red below.



3. From above this is the union of intervals $]-\infty\,;-2[\,\cup\,[3\,;+\infty\,[$. The set is shaded in red below. Observe that the point $x=-2$ must be excluded.



2

1. If $x<-2$ then $|x-3|=-x+3$ and $|x+2|=-x-2$. Hence
$$|x-3|+|x+2|=-2x+1, \quad \text{when } x<-2.$$
(Observe that this in fact valid for all $x\le -2$.)
2. If $-2<x<3$ then $|x-3|=-x+3$ and $|x+2|=x+2$. Hence
$$|x-3|+|x+2|=5, \quad \text{when } -2<x<3.$$
(Observe that this in fact valid for all $-2\le x\le 3$.)
3. If $x>3$ then $|x-3|=x-3$ and $|x+2|=x+2$. Hence
$$|x-3|+|x+2|=2x-1, \quad \text{when } x>3.$$
(Observe that this in fact valid for all $x\ge 3$.)
4. We must look for solutions
$$\begin{array}{lll} -2x+1=5 & \Longrightarrow x=-2 & \text{in }]-\infty\,;-2] \\ 5=5 & \Longrightarrow \text{tautology} & \text{in } [-2\,;3] \\ 2x-1=5 & \Longrightarrow x=3 & \text{in } [3\,;+\infty\,[\end{array}$$
From this we gather that the solution set is the interval $[-2\,;3]$.

3 If $x=0.10101010\ldots=0.\overline{10}$ then $100x=10.10101010\ldots$, whence
$$99x=100x-x=10.10101010\ldots-0.10101010\ldots=10 \implies 99x=10 \implies x=\frac{10}{99}.$$

4 The answer is no to both questions. $\sqrt{2}+(-\sqrt{2})$ is the sum of two irrational numbers giving the integer 0 and $\sqrt{2}\cdot(-\sqrt{2})$ is the product of two irrationals giving the integer -2.

5 If this number were rational, then eventually, it would be a repeating decimal, say, with period p. But we can then take a sufficiently large power of 2, larger than p, and so eventually, there would be more than p 0's between 1's, a contradiction.

6 $[-2\,;1\,[$

7 The solution set is: $\{x\in\mathbb{R}: x\in\,]-\infty\,;-3]\cup\,]-2\,;2]\cup\,]3\,;+\infty[\}$.

8 Four. Either $x^2-4x=-3$ or $x^2-4x=3$. Thus $x\in\{1,3,2-\sqrt{7},2+\sqrt{7}\}$.

9 There four solutions: $\pm1\pm\sqrt{2}$.

10 If $x>2$ then $|2-x|=x-2$. Thus for $x>2$,
$$|1-|2-x||=|1-(x-2)|=|3-x|\,.$$
For $x>3$, $|3-x|=x-3$.

11 $]-\infty\,;-1]$.

14 $\sqrt{a^2+b^2}$

15 $\sqrt{2}$

16 We have

$$\sqrt{(x-3)^2+(1-x)^2}=2 \implies (x-3)^2+(1-x)^2=4 \implies 2x^2-8x+10=4 \implies x^2-4x+3=0 \implies x\in\{1,3\}.$$

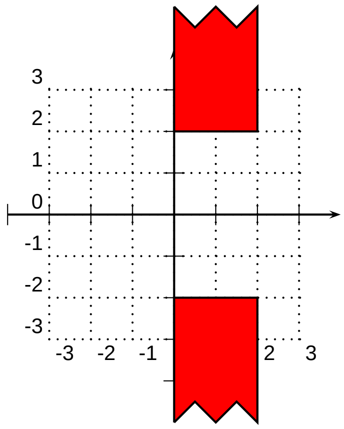
17 Observe that

$$|x-1|\leq 1 \iff -1\leq x-1\leq 1 \iff 0\leq x\leq 2,$$

and

$$|y|\geq 2 \iff y\in\,]-\infty\,;-2]\cup[2\,;+\infty\,[\,.$$

The set appears shaded below.



20 Let (x, y) be the point sought. Then

$$d\langle(x,y),(-1,3)\rangle = d\langle(x,y),(2,4)\rangle \implies (x+1)^2+(y-3)^2=(x-2)^2+(y-4)^2,$$

$$d\langle(x,y),(-1,3)\rangle = d\langle(x,y),(1,1)\rangle \implies (x+1)^2+(y-3)^2=(x-1)^2+(y-1)^2.$$

This gives the two systems of linear equations

$$2x+1-6y+9=-4x+4-8y+16,$$

$$2x+1-6y+9=-2x+1-2y+1,$$

or

$$6x+2y=10,$$
$$4x-4y=-8.$$

This system solves to $(x,y)=(\frac{3}{4},\frac{11}{4})$.

21

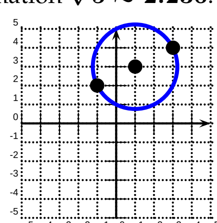
1. Its centre is evidently the midpoint of these two points: $\left(\dfrac{-1+3}{2},\dfrac{2+4}{2}\right)=(1,3)$.
2. If R is the radius of the circle, the equation of the circle has the form

$$(x-1)^2+(y-3)^2=R^2 \implies (-1-1)^2+(2-3)^2=R^2 \implies 5=R^2 \implies R=\sqrt{5}.$$

3. Upon assembling the results above, the canonical equation is clearly

$$(x-1)^2+(y-3)^2=5.$$

4. In order to graph this you may need the approximation $\sqrt{5}\approx 2.236$.



24 The centre (h, k) of the circle is the point equidistant to the given points, which can be found by solving

$$(h-1)^2 + (k-1)^2 = h^2 + (k-1)^2,$$

$$(h-1)^2 + (k-1)^2 = (h-1)^2 + (k-2)^2.$$

The first equation gives $h = \frac{1}{2}$, and the second equation gives $k = \frac{3}{2}$. The centre of the circle is thus $(h, k) = (\frac{1}{2}, \frac{3}{2})$. The radius of the circle is the distance from its centre to any point on the circle, say, to $(0, 1)$:

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2} - 1\right)^2} = \frac{\sqrt{2}}{2}.$$

The equation sought is finally

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{1}{2}.$$

25 The distance of (x, y) to $(2, 3)$ is $\sqrt{(x-2)^2 + (y-3)^2}$. The distance of (x, y) to the line $x = -4$ is $|x - (-4)| = |x + 4|$. We need

$$\sqrt{(x-2)^2 + (y-3)^2} = |x+4| \iff (x-2)^2 + (y-3)^2 = (x+4)^2 \iff x = \frac{y^2}{12} - \frac{y}{2} - \frac{1}{2},$$

so the locus is a parabola with axis of symmetry parallel to the x-axis and opening to the right.

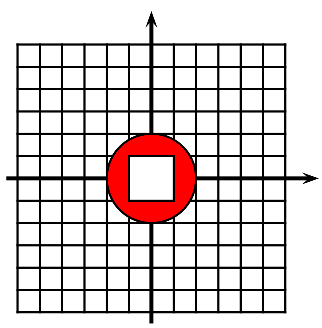
26 The distance of (x, y) to $(2, 5)$ is $\sqrt{(x-2)^2 + (y-5)^2}$. The distance of (x, y) to the point $(3, -1)$ is $\sqrt{(x-3)^2 + (y+1)^2}$. We need

$$\sqrt{(x-2)^2 + (y-3)^2} = \sqrt{(x-3)^2 + (y+1)^2} \iff (x-2)^2 + (y-3)^2 = (x-3)^2 + (y+1)^2 \iff y = \frac{x}{4} + \frac{3}{8},$$

so the locus is a line.

27

1. The figure appears in red below.



2. $4\pi - 4$

28

$$x^2 + 4ax + y^2 - 2by = c^2 \implies x^2 + 4ax + 4a^2 + y^2 - 2by + b^2 = c^2 + 4a^2 + b^2 \implies (x+2a)^2 + (y-b)^2 = c^2 + 4a^2 + b^2.$$

The centre is at $(-2a, b)$ and the radius is $\sqrt{c^2 + 4a^2 + b^2}$.

29 $\dfrac{\pi(b-a)}{8}$.

30 Line L has slope $-\dfrac{a}{b}$ and line L' has slope $-\dfrac{d}{e}$. Now $L \perp L'$ entails that $-\dfrac{a}{b} = \dfrac{e}{d}$, hence $ad + eb = 0$.

31

1. It is plainly $m = \dfrac{3-2}{1-(-1)} = \dfrac{1}{2}$.

2. L has the form $y = \frac{x}{2} + k$. Hence $2 = \frac{-1}{2} + k \implies k = \frac{5}{2}$ so the line sought is $y = \frac{x}{2} + \frac{5}{2}$.
3. L' has the form $y = \frac{x}{2} + k$. Hence $4 = \frac{4}{2} + k \implies k = 2$ so the line sought is $y = \frac{x}{2} + 2$.
4. L'' has the form $y = -2x + k$. Hence $4 = -8 + k \implies k = 12$ so the line sought is $y = -2x + 12$.
5. L is $y = \frac{x}{2} + \frac{5}{2}$. Put $f(x) = \frac{x}{2} + \frac{5}{2}$. A reflexion about the x-axis has equation has equation

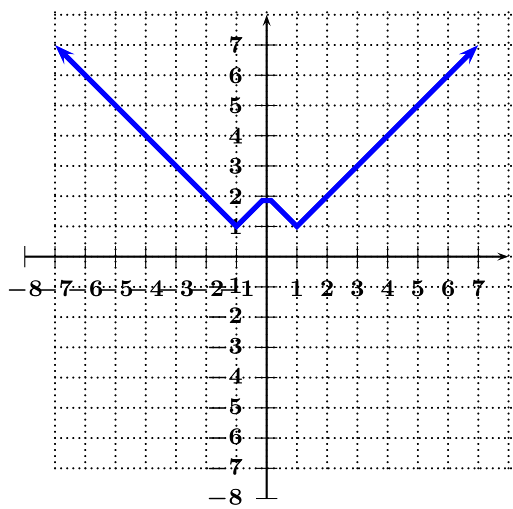
$$y = -f(x) = -\frac{x}{2} - \frac{5}{2}.$$

6. L is $y = \frac{x}{2} + \frac{5}{2}$. Put $f(x) = \frac{x}{2} + \frac{5}{2}$. A reflexion about the y-axis has equation

$$y = f(-x) = -\frac{x}{2} + \frac{5}{2}.$$

32

1. This is $y = (-x+1) - (-x) + (-x-1) = -x$.
2. This is $y = (-x+1) - (-x) + (x+1) = x+2$.
3. This is $y = (-x+1) - (+x) + (x+1) = -x+2$.
4. This is $y = (x-1) - (+x) + (x+1) = x$.
5. $\mathscr{C}$ appears on the grid below .



37

1. The slope of L_t is obtained by solving for y and looking at the coefficient of x:

$$(t-3)y = (2-t)x + t \implies y = \frac{2-t}{t-3}x + \frac{t}{t-3},$$

whence L_t has slope $\frac{2-t}{t-3}$. Since L has slope -2, we need

$$\frac{2-t}{t-3} = -2 \implies 2 - t = -2(t-3) \implies t = 4.$$

2. From the work done above we need

$$\frac{2-t}{t-3} = \frac{1}{2} \implies 2(2-t) = t - 3 \implies t = \frac{7}{3}.$$

3. L_t will be equidistant from these two points if (i) L_t passes through their midpoint, (ii) L_t is perpendicular to the line joining these two points. The midpoint of the points is $\left(1, \frac{1}{2}\right)$ and so the t needed is found by solving

$$(2-t)(1)+t=(t-3)\left(\frac{1}{2}\right) \implies t=7.$$

The line joining these two points has slope $\dfrac{1}{2}$, and so the line perpendicular has slope -2. Thus L_t will be perpendicular to this line when

$$\frac{2-t}{t-3}=-2 \implies 2-t=-2(t-3) \implies t=4.$$

Since $7\neq 4$, there is no such value of t.

38 $(a,b)=(2,5)$

42 We need

$$x-1\neq 0 \text{ and } 4-x^2>0 \implies x\neq 1 \text{ and } x\in\,]-2\,;2[\implies x\in\,]-2\,;+1[\,\cup\,]+1\,;2[\,.$$

47 Since $x\mapsto |x|$ is even, $x\mapsto |x|^3$, is even. Since $x\mapsto x$ is odd and $x\mapsto |x|^3$ is even, their quotient $x\mapsto \dfrac{x}{|x|^3}$ gives an odd rule. Alternatively one may prove this directly from the definition:

$$\frac{-x}{|-x|^3}=-\frac{x}{|x|^3},$$

whence the rule is odd.

49 We need $f(x)=f(-x)$ and $-f(x)=f(-x)$, whence

$$f(x)=-f(x) \implies 2f(x)=0 \implies f(x)=0,$$

thus only the constant function $x\mapsto 0$ is both even and odd. Its graph is the x-axis.

50 We have

1. $a_0=p(0)=(-1)^5(-1)^{20}=-1$
2. $a_0+a_1+a_2+\cdots+a_{44}+a_{45}=p(1)=(1)^5(1)^{20}=1$
3. $a_0-a_1+a_2-a_3+\cdots-a_{43}+a_{44}-a_{45}=p(-1)=(-3)^5(-1)^{20}=-243$
4. $a_0+a_2+a_4+\cdots+a_{42}+a_{44}=\dfrac{p(1)+p(-1)}{2}=-121$
5. $a_1+a_3+a_5+\cdots+a_{43}+a_{45}=\dfrac{p(1)-p(-1)}{2}=122$

51 8

52 9

53 $]-\infty\,;-2[\,\cup\,]2\,;+\infty[$

54 $\varnothing$

55 $\{0\}$

59 4

60 $-\dfrac{3}{4}$

61 Setting $a=b=0$ we get $f(0)f(0)=f(0+0)$, that is,

$$(f(0))^2=f(0) \implies f(0)(f(0)-1)=0.$$

Since $f(0)$ must be >0 we gather that $f(0)=1$.

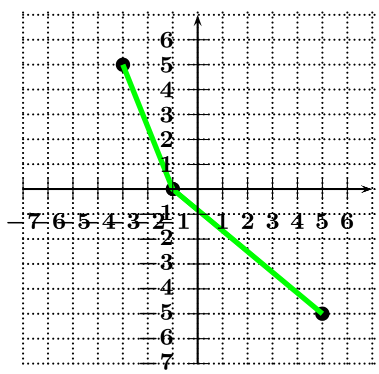
62 Letting $x = 1$, $f(y) = yf(1)$. Hence $f(x) = xf(1)$, that is, f must be linear.

66

1. $(a + b + c)(2) = a(2) + b(2) + c(2) = (1 - 2^2) + (1 + 2(2)) + 4 = -3 + 5 + 4 = 6$.
2. $(abc)(2) = a(2)b(2)c(2) = (-3)(5)(4) = -60$
3. $(a \circ b \circ c)(2) = a(b(c(2))) = a(b(4)) = a(9) = -80$
4. $(c \circ b \circ a)(2) = 4$, since c is a constant function.

77

1. The first piece of f is a line with endpoints at $(-5, 5), (0, -1)$, which has slope $-\frac{6}{5}$. Hence the equation for f is $f(x) = -\frac{6}{5}x - 1$ (observe that the y-intercept is $(0, -1)$. Putting $y = -\frac{6}{5}x - 1$ and solving for x we get $x = -\frac{5}{6}y - \frac{5}{6}$. Hence $f^{-1}(x) = -\frac{5}{6}x - \frac{5}{6}$. For $x \in [-5; 0]$, $-1 \leq f(x) \leq 5$, so the formula for f^{-1} is only good for $-1 \leq x \leq 5$.
2. The second piece is a line with endpoints at $(0, -1), (5, -3)$, which has slope $-\frac{2}{5}$. Hence the equation for f is $f(x) = -\frac{2}{5}x - 1$ (observe that the y-intercept is $(0, -1)$. Putting $y = -\frac{2}{5}x - 1$ and solving for x we get $x = -\frac{5}{2}y - \frac{5}{2}$. Hence $f^{-1}(x) = -\frac{5}{2}x - \frac{5}{2}$. For $x \in [0; 5]$, $-3 \leq f(x) \leq -1$, so the formula for f^{-1} is only good for $-3 \leq x \leq -1$.
3. Recall that if (a, b) is on f, then (b, a) is on f^{-1}.



83

1. Observe that $y = x^2 - 2x + 3 \implies y = (x - 1)^2 + 2$, which has vertex at $(1, 2)$. Thus f is injective for $x \leq 1$ and for $x \geq 1$. Therefore f is not injective for $x \leq 0$, and hence not invertible.
2. Observe that $y = x^2 - 2x + 3 \implies y = (x - 1)^2 + 2$, which has vertex at $(1, 2)$. Thus f is injective for $x \leq 1$ and for $x \geq 1$. Hence f is invertible for $x \geq 1$. Solving for x, and taking the positive square root (since $x \geq 1$),

$$y = (x - 1)^2 + 2 \implies y - 2 = (x - 1)^2 \implies \sqrt{y - 2} = x - 1 \implies x = 1 + \sqrt{y - 2}.$$

   Thus $f^{-1}(x) = 1 + \sqrt{x - 2}$.

84 We have

$$y = \frac{(x - 1)^3}{x^3} \implies \sqrt[3]{y} = \frac{x - 1}{x} \implies x\sqrt[3]{y} = x - 1 \implies (1 - \sqrt[3]{y})x = 1 \implies x = \frac{1}{1 - \sqrt[3]{y}}.$$

Therefore, $f^{-1}(x) = \dfrac{1}{1 - \sqrt[3]{x}}$.

90 The required graphs appear below.

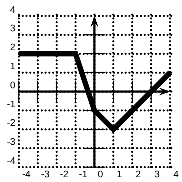


Figure 17: $y = -f(x)$

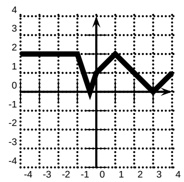


Figure 18: $y = |f(x)|$

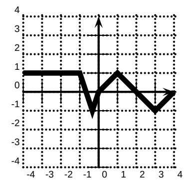


Figure 19: $y = |f(x)| - 1$

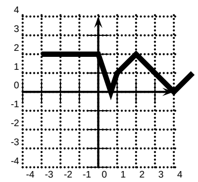


Figure 20: $y = |f(x - 1)|$

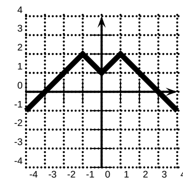


Figure 21: $y = f(|x|)$

91 Put $a(x) := \dfrac{x}{x^2 - x - 1}$. After a translation one unit right we have $a(x-1) = \dfrac{x-1}{(x-1)^2 - (x-1) - 1} = \dfrac{x-1}{x^2 - 3x + 1} = b(x)$, say. After a reflexion about the x-axis we have $-b(x) = -\dfrac{x-1}{x^2 - 3x + 1} = c(x)$, say. After a translation two units up, we have $2 + c(x) = 2 - \dfrac{x-1}{x^2 - 3x + 1} = d(x)$, say. After a reflexion about the y-axis we have $d(-x) = 2 - \dfrac{-x-1}{x^2 + 3x + 1} = 2 + \dfrac{x+1}{x^2 + 3x + 1}$. Conclusion: The resulting curve is $y = 2 + \dfrac{x+1}{x^2 + 3x + 1}$.

92 Notice that the graph of $y = f(ax)$ is a horizontal shrinking of the graph of $y = f(x)$. Put $g(x) = f(ax)$. Since $g(4/3) = 0$ we must have $4a/3 = -2 \implies a = -3/2$, so the point $(-2, 0)$ on the original graph was mapped to the point $(4/3, 0)$ on the new graph. Hence the point $(3, 0)$ in the old graph gets mapped to $(-2, 0)$ and so $C = -2$.

94 $(a, h, k) = (-2, 1, 2)$

95 The equation is of the form $y = a(x-1)^2 + 2$. Now, $-7 = a(4-1)^2 + 2 \implies a = -1$. Thus the parabola is $y = -(x-1)^2 + 2$. One can also use the points $(-2, -7)$ or $(0, 1)$ and get the same result.

96 We have

$$\begin{aligned} y = ax^2 - 6abx - ab^2 &\iff \frac{y}{a} = x^2 - 6bx - b^2 \\ &\iff \frac{y}{a} + 9b^2 = x^2 - 6bx + 9b^2 - b^2 \\ &\iff \frac{y}{a} = (x - 3b)^2 - 10b^2 \\ &\iff y = a(x - 3b)^2 - 10ab^2, \end{aligned}$$

whence the vertex is $(3b, -10ab^2)$.

97 The discriminant is

$$(-2)^2 - 4(\sqrt{2})(1) = 4 - 4\sqrt{2} = 4(1 - \sqrt{2}) < 0,$$

hence there are no real roots.

100 $-4 + \sqrt{2}$

102 Take $x > 0$ and put $A = (0, 0)$, $B = \left(x, \dfrac{x^2}{2}\right)$ and $C = \left(-x, \dfrac{x^2}{2}\right)$. Then

$$AB = BC \implies \sqrt{(x-0)^2 + \left(\frac{x^2}{2} - 0\right)^2} = \sqrt{(x - (-x))^2 + \left(\frac{x^2}{2} - \frac{x^2}{2}\right)^2} \implies x^2\left(1 + \frac{x^2}{4}\right) = 4x^2 \implies x = 2\sqrt{3}.$$

The points are $A(0, 0)$, $B(2\sqrt{3}, 6)$ and $C(-2\sqrt{3}, 6)$.

103 $\sqrt{(1-x)(x+2)} = \sqrt{\dfrac{9}{4} - \left(\dfrac{1}{2} + x\right)^2} \leq \dfrac{3}{2}$.

104 Transposing,

$$(t^2 - 81)x = 3(t - 9). \tag{1}$$

If $t \notin \{-9, 9\}$ then (1) has a unique solution $x = \dfrac{t-9}{t^2 - 81} = \dfrac{1}{t+9}$. If $t = 9$, then (1) becomes $-27 = -27$, which is a tautology for all $x \in \mathbb{R}$ (thus every real number is a solution). If $t = -9$ then (1) becomes the contradiction $0 = -54$.

105 We have

$$2 = f(1) = a + b + c; \quad 5 = 4a + 2b + c; \quad 4 = 9a + 3b + c.$$

Solving this system, $a = -2, b = 9, c = -5$, whence $f(x) = -2x^2 + 9x - 5$.

106 The conditions stipulate that the vertex of the function is on the x-axis. Since the points $(1, 2)$ and $(3, 2)$ lie at the same height, the vertex is halfway between them, so the vertex is at $x = \dfrac{1+3}{2} = 2$ and so the vertex is $(2, 0)$. The curve has equation $y = a(x-2)^2$. Now,

$$2 = a(1-2)^2 \implies a = 2.$$

Hence the desired curve is $y = 2(x-2)^2$.

108 0, since -2 is one of the four roots, and so must be one of the letters a, b, c, d.

110 The polynomial $g(x) = p(x) - x^2$ has degree 3 and leading coefficient 1. It vanishes at $1, 2, 3$ and hence it must have the form $p(x) - x^2 = g(x) = (x-1)(x-2)(x-3)$. Thus

$$p(x) = (x-1)(x-2)(x-3) + x^2.$$

111 $(A, r, s) = (2, 6, 3)$

113 The polynomial $g(x) = p(x) - 7$ vanishes at the 4 different integer values a, b, c, d. In virtue of the Factor Theorem,

$$g(x) = (x-a)(x-b)(x-c)(x-d)q(x),$$

where $q(x)$ is a polynomial with integral coefficients. Suppose that $p(t) = 14$ for some integer t. Then $g(t) = p(t) - 7 = 14 - 7 = 7$. It follows that

$$7 = g(t) = (t-a)(t-b)(t-c)(t-d)q(t),$$

that is, we have factorised 7 as the product of at least 4 different factors, which is impossible since 7 can be factorised as $7(-1)1$, the product of at most 3 distinct integral factors. From this contradiction we deduce that such an integer t does not exist.

114 By Ruffini's Factor Theorem, we must have

$$0 = t(-4) = (-4)^3 - 3a(-4)^2 + 40$$

$$\iff 0 = -24 - 48a$$

$$\iff a = -\frac{1}{2}.$$

115 Observe that $f(x)(x-1) = x^5 - 1$ and

$$f(x^5) = x^{20} + x^{15} + x^{10} + x^5 + 1 = (x^{20} - 1) + (x^{15} - 1) + (x^{10} - 1) + (x^5 - 1) + 5.$$

Each of the summands in parentheses is divisible by $x^5 - 1$ and, a fortiori, by $f(x)$. The remainder sought is thus 5.

116 Put $g(x) = p(x) - x$, which is a polynomial of degree 3 and vanishes at $1, 2$, and 3. Thus $g(x) = A(x-1)(x-2)(x-3)$. Since $g(4) = p(4) - 4 = 1$ we get $1 = g(4) = A(4-1)(4-2)(4-3)$ so $A = \dfrac{1}{6}$. Thus

$$p(x) = x + g(x) = x + \frac{(x-1)(x-2)(x-3)}{6}$$

and $p(6) = 16$.

117 By Ruffini's Theorem, $p(-1) = 0$ and $p(1) = 4$. Hence

$$0 = p(-1) = -1 - a + b \implies 1 = -a + b,$$

$$4 = p(+1) = 1 + a + b \implies 3 = a + b.$$

Solving this system of equations, $a = 1, b = 2$.

118 By Ruffini's Theorem we know that $p(-1) = 1$ and $p(-2) = -4$. Observe that $x^2 + 3x + 2 = (x + 1)(x + 2)$. When p is divided by a polynomial of degree two, by the division algorithm, its remainder is a polynomial of degree 1 or less, say $ax + b$. Thus

$$p(x) = (x + 1)(x + 2)q(x) + ax + b \implies 1 = p(-1) = -a + b; \quad -4 = p(-2) = -2a + b.$$

Solving this system of equations $a = 5$ and $b = 6$. Hence the remainder sought is $5x + 6$.

120 $(x - 2)(x + 2)(x - 3)$

121 $(x - 3)(x + 3)(x + 5)(3x - 2)$

122 $a = -7, b = -60$

126 $(a, b, c) = (2, 2, 1)$