1 Exponential Functions

1 Problem Solve the equation $3^x + \frac{1}{3^x} = 2$.

2 Problem Put

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

and

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Prove that

$$\cosh^2 x - \sinh^2 x = 1.$$

The function $x \mapsto \cosh x$ is known as the *hyperbolic cosine*. The function $x \mapsto \sinh x$ is known as the *hyperbolic sine*.

3 Problem Draw the following curves in succession, without using a graphing device:

1. $y = e^x - 2$
2. $y = e^{|x|} - 2$
3. $y = \left|e^{|x|} - 2\right|$
4. $y = |e^x - 2|$
5. $y = e^{-x} - 2$
6. $y = e^{-|x|} - 2$
7. $y = \left|e^{-|x|} - 2\right|$

2 Logarithmic Functions

I write $\log x$ for what most write $\ln x$.

4 Problem If the number 5^{2000} is written out (in decimal notation), how many digits does it have?

5 Problem Find the exact value of

$$\lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \cdots + \lfloor \log_2 66 \rfloor$$

6 Problem Find the exact value of

$$\frac{1}{\log_2 1996!} + \frac{1}{\log_3 1996!} + \frac{1}{\log_4 1996!} + \cdots + \frac{1}{\log_{1996} 1996!}.$$

7 Problem Compute the following.

1. $\log_{1/3} 243$
2. $\log_{10} .00001$
3. $\log_{.001} 100000$
4. $\log_9 \frac{1}{3}$

5. $\log_{1024} 64$
6. $\log_{5^{2/3}} 625$
7. $\log_{2\sqrt{2}} 32\sqrt[5]{2}$
8. $\log_{2} .0625$
9. $\log_{.0625} 2$
10. $\log_{3} \sqrt[4]{729\sqrt[3]{9^{-1}27^{-4/3}}}$

8 Problem Find real solutions to the following equations for x.

1. $\log_x 3 = 4$
2. $\log_3 x = 4$
3. $\log_4 x = 3$
4. $\log_{x-2} 9 = 2$
5. $\log_{|x|} 16 = 4$
6. $23^x - 2 = 0$
7. $(2^x - 3)(3^x - 2)(6^x - 1) = 0$
8. $4^x - 9 \cdot 2^x + 14 = 0$
9. $49^x - 2 \cdot 7^x + 1 = 0$
10. $36^x - 2 \cdot 6^x = 0$
11. $36^x + 6^x - 6 = 0$
12. $5^x + 12 \cdot 5^{-x} = 7$
13. $\log_2 \log_3 x = 2$
14. $\log_3 \log_5 x = -1$

9 Problem Solve the equation $9^x + 3^x - 6 = 0$.

10 Problem Given that $\log_a p = 2,\ \log_a m = 9,\ \log_a n = -1$ find

1. $\log_a p^7$
2. $\log_{a^7} p$
3. $\log_{a^4} p^2n^3$
4. $\log_{a^6} \dfrac{m^3n}{p^6}$

11 Problem Which number is larger, 3^{1000} or 5^{600}?

12 Problem Find $(\log_3 169)(\log_{13} 243)$ without recourse of a calculator or tables.

13 Problem Find $\dfrac{1}{\log_2 36} + \dfrac{1}{\log_3 36}$ without recourse of a calculator or tables.

14 Problem Given that $\log_a p = b,\ \log_q a = 3b^{-2}$, find $\log_p q$ in terms of b.

15 Problem Given that $\log_2 a = s,\ \log_4 b = s^2,\ \log_{c^2} 8 = \frac{2}{s^3+1}$, write $\log_2 \dfrac{a^2b^5}{c^4}$ as a function of s.

16 Problem Given that $\log_{a^2}(a^2+1) = 16$, find the value of

$$\log_{a^{32}} \left(a + \frac{1}{a}\right).$$

17 Problem Write without logarithms. Assume the proper restrictions on the variables wherever necessary.

1. $(a^\alpha)^{-\beta \log_\alpha s \ N^\gamma}$
2. $-\log_8 \log_4 \log_2 16$
3. $\log_{0.75} \log_2 \sqrt{\sqrt[-2]{0.125}}$
4. $\left(5^{(\log_7 5)^{-1}} + (-\log_{10} 0.1)^{-1/2}\right)^{1/3}$
5. $b^{a^{(\log_b \log_b N)/(\log_b a)}}$
6. $2^{(\log_3 5)} - 5^{(\log_3 2)}$
7. $\left(\dfrac{1}{49}\right)^{1+(\log_7 2)} + 5^{-(\log_{1/5} 7)}$

18 Problem A sheet of paper has approximately 0.1 mm of thickness. Suppose you fold the sheet by halves, thirty times consecutively. (1) What is the thickness of the folded paper?, (2) How many times should you fold the sheet in order to obtain the distance from Earth to the Moon? (the distance from Earth to the Moon is about 384 000 km.)

19 Problem How many digits does 11^{2000} have?

20 Problem Let $A = \log_6 16, B = \log_{12} 27$. Find integers a, b, c such that $(A + a)(B + b) = c$.

21 Problem Given that $\log_{ab} a = 4$, find

$$\log_{ab} \frac{\sqrt[3]{a}}{\sqrt{b}}.$$

22 Problem The number 5^{100} is written in binary (base-2) notation. How many binary digits does it have?

23 Problem Prove that if $x > 0, a > 0, a \neq 1$ then $x^{1/\log_a x} = a.$

24 Problem Find the exact value of $\log_{3\sqrt{3}} 729$.

25 Problem If a and b are consecutive integers such that $a < \log_5 100 < b$ find them.

26 Problem Which number is larger, 3^{100} or 5^{30}.

27 Problem Find all real solutions to the equation $\log_2 \log_3 \log_2 x = 1$.

28 Problem Given that $(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)\cdots(\log_{1023} 1024)$ is an integer, find it.

29 Problem Given that $a > 1$, $t > 0$, $s > 0$ and that

$$\log_a t^3 = p, \qquad \log_{\sqrt{a}} s^2 = q,$$

find $\log_a st$ in terms of p and q.

30 Problem Given that $a > 1$, $s > 1$, $t > 1$, and that

$$\log_a \sqrt{t} = p, \qquad \log_s a^2 = 2p^2,$$

find $\log_s t$ in terms of p.

31 Problem What is the domain of definition of $x \mapsto \log_x(1 - x^2)$?

32 Problem How many digits does $5^{2000}3^{1000}$ have?

33 Problem What is $5^{2000}3^{1000}$ approximately?

34 Problem Let $a > 1$, $x > 1$, $y > 1$. If $\log_a x^3 = N$ and $\log_{a^{1/3}} y^4 = M$, find $\log_{a^2} xy$ in terms of N and M. Also, find $\log_x y$.

35 Problem Draw the following curves in succession and without using a graphing device:

1. $y = \log(-x)$
2. $y = \log|x|$
3. $y = |\log x|$
4. $y = |\log|x||$
5. $y = \log|x + 1|$
6. $y = (\log|x|) + 1$
7. $y = \log(|x| + 1)$
8. $y = \log||x| + 1|$

36 Problem How many real positive solutions does the equation

$$x^{(x^x)} = (x^x)^x$$

have?

37 Problem The non-negative integers smaller than 10^n are split into two subsets A and B. The subset A contains all those integers whose decimal expansion does not contain a 5, and the set B contains all those integers whose decimal expansion contains at least one 5. Given n, which subset, A or B is the larger set? One may use the fact that $\log_{10} 2 := .3010$ and that $\log_{10} 3 := .4771.$

38 Problem Shew that if $a, b, c,$ are real numbers with $a^2 = b^2 + c^2, a + b > 0, a + b \neq 1, a - b > 0, a - b \neq 1,$ then

$$\log_{a-b} c + \log_{a+b} c = 2(\log_{a-b} c)(\log_{a+b} c).$$

39 Problem If $\log_{12} 27 = a$ prove that $\log_6 16 = \dfrac{4(3-a)}{3+a}$.

40 Problem Solve the equation

$$|x - 3|^{(x^2-8x+15)/(x-2)} = 1$$

41 Problem Solve the equation

$$\log_{2x-1} \frac{x^4 + 2}{2x + 1} = 1$$

42 Problem Solve the equation

$$\log_{3x} x = \log_{9x} x$$

43 Problem Solve

$$\log_2 x + \log_4 y + \log_4 z = 2,$$
$$\log_3 x + \log_9 y + \log_9 z = 2,$$
$$\log_4 x + \log_{16} y + \log_{16} z = 2.$$

44 Problem Solve the equation

$$x^{0.5 \log_{\sqrt{x}}(x^2 - x)} = 3^{\log_9 4}.$$

3 Goniometric Functions

45 Problem Write in the form $a \sin x + b \cos x$, with real constants a, b.

$$A(x) = \sin\left(\frac{\pi}{2} - x\right) + \cos(5\pi - x) + \cos\left(\frac{3\pi}{2} - x\right) + \sin\left(\frac{3\pi}{2} + x\right)$$

46 Problem Draw the curve $y = \arctan \tan x$.

47 Problem Draw the curve $y = \tan \arctan x$.

48 Problem Find $\sin(\arcsin 4)$.

49 Problem Find $\arcsin(\sin 10)$.

50 Problem Let $\tan x + \cot x = a$. Find $\tan^3 x + \cot^3 x$ as a polynomial in a.

51 Problem Draw the following curves in succession, without recourse of a graphing device:

1. $y = \cos(x - 1)$
2. $y = |\cos(x - 1)|$
3. $y = \cos(|x| - 1)$

4. $y = |\cos(|x| - 1)|$
5. $y = (\cos x) - \dfrac{1}{2}$
6. $y = \left|(\cos x) - \dfrac{1}{2}\right|$

52 Problem Given that

$$3\sin x + 4\cos x = 5,$$

find $\sin x$ and $\cos x$.

53 Problem Find k such that the expression

$$(\sin x + \cos x)^2 + k\sin x\cos x = 1$$

becomes an identity.

54 Problem Let $\sin x = \frac{1}{3}$ and $\sin y = \frac{1}{4}$ where x and y are acute angles. Problems 1 through 6 refer to this situation.

1. Find $\cos x$.
2. Find $\cos 2x$.
3. Find $|\cos\frac{x}{2}|$.
4. Find $\cos y$.
5. Find $\sin(x + y)$.
6. Find $\cos(x + y)$.

55 Problem If it is known that $\cos\frac{2\pi}{5} = \dfrac{\sqrt{5}-1}{2}$, find the exact value of $\cos\frac{\pi}{5}$ without using goniometric functions.

56 Problem If it is known that $\cos\frac{2\pi}{5} = \dfrac{\sqrt{5}-1}{2}$, find the exact value of $\cos\frac{4\pi}{5}$ without using goniometric functions.

57 Problem Prove that the equation

$$\cos\left(\left(\frac{3}{2}\right)^x - 1\right) = \frac{1}{2},$$

has only 4 solutions lying in the interval $[0; 2\pi]$.

58 Problem Prove that the equation

$$\cos(\log_3 x - 2) = \frac{1}{2},$$

has only 2 solutions lying in the interval $[0; 2\pi]$.

59 Problem (AHSME 1976) If $\sin x + \cos x = \frac{1}{5}$ and $x \in]0; \pi[$, find $\cos x$ and $\sin x$.

60 Problem (AIME 1983) Find the minimum value of the function

$$x \mapsto \frac{9x^2 \sin^2 x + 4}{x \sin x}$$

over the interval $]0; \pi[$.

61 Problem Find the solutions of the equation

$$\log_{\sqrt{2}\sin x}(1 + \cos x) = 2$$

in the interval $[0; 2\pi]$.

62 Problem Find the set of all the real solutions to

$$2^{\sin^2 x} + 5(2^{\cos^2 x}) = 7$$

63 Problem Find all the real solutions of the equation

$$\cos^{2000} x - \sin^{2000} x = 1.$$

64 Problem Find all the real solutions of the equation

$$\cos^{2001} x - \sin^{2001} x = 1.$$

65 Problem If $\sqrt[4]{\tan x} + \sqrt[4]{\cot x} = 3$, prove that $\sin 2x = \dfrac{2}{47}$.

66 Problem Prove that if $\dfrac{a}{\sin x} = \dfrac{b}{\cos x} = \dfrac{c}{\csc x}$, then $a^2 + b^2 = ac$.

4 Trigonometry

Note: In $\triangle ABC$, the vertices are labelled with angles A, B, and C and the sides measure $AB = c$, $BC = a$, and $CA = b$.

67 Problem $\triangle ABC$ is right-angled at A, and $AB = 2$ and $\cot \angle B = 4$.

1. Find AC.
2. Find $\sin \angle C$.
3. Find the radius of circumscribed circle to $\triangle ABC$.

68 Problem Find the area of a triangle whose sides measure $2, 3, 4$. Find the radius of its circumcircle.

69 Problem If in a $\triangle ABC$, $a = 5$, $b = 4$, and $\cos(A - B) = \dfrac{31}{32}$, prove that $\cos C = \dfrac{1}{8}$ and that $c = 6$.

70 Problem A triangle with vertices A, B, C on a circle of radius R, has the side opposite to vertex A of length 12, and the angle at $A = \frac{\pi}{4}$. Find diameter of the circle.

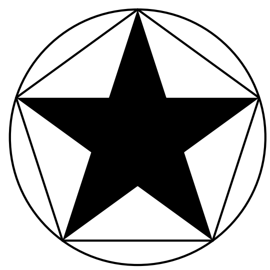
71 Problem $\triangle ABC$ has sides of length a, b, c, and circumradius $R = 4$. Given that the triangle has area 5, find the product abc.

72 Problem $\triangle ABC$ has sides measuring 2, 4, and 5 units.

1. Find its area.
2. Find the radius of its circumscribed circle.
3. Find the radius of its inscribed circle.

73 Problem In $\triangle ABC$, prove that $\dfrac{a-b}{a+b} = \dfrac{\sin A - \sin B}{\sin A + \sin B}$.

74 Problem Consider a regular pentagon inscribed in a circle of radius 1, and the five pointed star obtained by alternately joining the vertices of pentagon, as shewn below.



1. Give an exact expression for the area of the pentagon.
2. Give an exact expression for the perimeter of the pentagon.
3. Give an exact expression for the area of the shaded five-pointed star.

75 Problem A triangle has sides measuring $2, 3, 4$. Find the cosine of the angle opposite the side measuring 3.

76 Problem Find the area and the perimeter of a regular octagon inscribed in a circle of radius 2.

77 Problem The expression

$$\log(\tan 1^\circ) + \log(\tan 2^\circ) + \log(\tan 3^\circ) + \cdots + \log(\tan 89^\circ)$$

is an integer. Find it.

78 Problem Find the exact value of

$$\sin^2 1^\circ + \sin^2 2^\circ + \cdots + \sin^2 90^\circ$$

79 Problem Let $x + y + z = 90^\circ$. Determine

$$(\tan x)(\tan y) + (\tan y)(\tan z) + (\tan z)(\tan x).$$

80 Problem Prove that

$$\tan 51^\circ - \cot 51^\circ = 2\tan 12^\circ.$$

81 Problem If $a + b + c = 36^\circ$, prove that

$$\tan 5a + \tan 5b + \tan 5c - (\tan 5a)(\tan 5b)(\tan 5c) = 0.$$

5 Some Answers, Solutions, and Hints

1 $3^x + \frac{1}{3^x} = 2 \implies 9^x - 2\cdot 3^x + 1 = 0 \implies (3^x - 1)^2 = 0 \implies x = 0.$

4 $\llbracket \log_{10} 5^{2000} \rrbracket + 1 = \llbracket 2000 \cdot \log_{10} 5 \rrbracket + 1 = 1398.$

5 276

6 1

7 (1) -5, (2) -5, (3) $-\frac{5}{3}$, (4) $-\frac{1}{2}$, (5) $\frac{3}{5}$, (6) 6, (7) $\frac{52}{15}$, (8) -4 , (9) $-\frac{1}{4}$, (10) 1

8 (1) $\sqrt[4]{3}$, (2) 81, (3) 64 (4) 5, (5) ± 2, (6) $\log_{23} 2$, (7) $\log_2 3,\ \log_3 2,\ 0$, (8) $\log_2 7, 1$, (9) 0, (10) $\log_6 2$, (11) $\log_6 2$, (12) $\log_5 4, \log_5 3$, (13) 81, (14) $\sqrt[3]{5}$

9 $0 = 9^x + 3^x - 6 = (3^x + 3)(3^x - 2) \implies x = \log_3 2.$

10 (1) 14, (2) $\frac{2}{7}$, (3) $\frac{1}{4}$, (4) $\frac{7}{3}$

11 3^{1000}

12 10

13 $\frac{1}{2}$

14 $\frac{b}{3}$

15 $-3s^3 + 10s^2 + 2s - 3$

16 $\frac{31}{32}$

17 (1) $N^{-\alpha\beta\gamma/s}$, (2) 0, (3) 1 (4) 2, (5) N, (6) 0, (7) $\frac{1373}{196}$

18 (1) About 107.37 km (2) 42 times.

19 2083

20 $a = 4, b = 3, c = 24$

21 $\frac{17}{6}$

36 Assuming $x > 0$ we have $x^x \log_e x = x \log_e x^x$ or $x^x \log_e x = x^2 \log_e x$. Thus $(\log_e x)(x^x - x^2) = 0$. Thus either $\log_e x = 0$, in which case $x = 1$, or $x^x = x^2$, in which case $x = 2$. The equation has therefore only two positive solutions.

37 The set B contains $10^n - 9^n$ elements and the set A contains 9^n elements. Now if $10^n - 9^n > 9^n$ then $10^n > 2 \cdot 9^n$ and taking logarithms base 10 we deduce

$$n > \log_{10} 2 + 2n \log_{10} 3.$$

Thus

$$n > \frac{\log_{10} 2}{1 - 2\log_{10} 3} := 6.57\ldots$$

Therefore, if $n \leq 6$, A has more elements than B and if $n > 6$, B has more elements than A.

38 As $c^2 = a^2 - b^2 = (a-b)(a+b)$, upon taking logarithms base $a + b$ we have

$$2\log_{a+b} c = \log_{a+b} (a-b)(a+b) = 1 + \log_{a+b} (a-b) \tag{1}$$

Similarly, taking logarithms base $a - b$ on the identity $c^2 = (a-b)(a+b)$ we obtain

$$2\log_{a-b} c = \log_{a-b} (a-b)(a+b) = 1 + \log_{a-b} (a+b) \tag{2}$$

Multiplying these last two identities,

$$\begin{aligned} 4(\log_{a-b} c)(\log_{a+b} c) &= (1+\log_{a+b}(a-b))(1+\log_{a-b}(a+b)) \\ &= 1+\log_{a-b}(a+b)+\log_{a+b}(a-b) \\ &\quad +(\log_{a-b}(a+b))(\log_{a+b}(a-b)) \\ &= 2+\log_{a-b}(a+b)+\log_{a+b}(a-b) \\ &= 2+\log_{a-b}\tfrac{c}{a-b}+\log_{a+b}\tfrac{c}{a+b} \\ &= \log_{a-b} c+\log_{a+b} c, \end{aligned}$$

as we wanted to shew.

39 First notice that $a = \log_{12} 27 = 3\log_{12} 3 = \dfrac{3}{\log_3 12} = \dfrac{3}{1+2\log_3 2}$, whence $\log_3 2 = \dfrac{3-a}{2a}$ or $\log_2 3 = \dfrac{2a}{3-a}$. Also

$$\begin{aligned} \log_6 16 &= 4\log_6 2 \\ &= \tfrac{4}{\log_2 6} \\ &= \tfrac{4}{1+\log_2 3} \\ &= \tfrac{4}{1+\frac{2a}{3-a}} \\ &= \tfrac{4(3-a)}{3+a}, \end{aligned}$$

as required.

48 $\arcsin 4$ is not a real number. The given expression is not a real number.

49 $3\pi - 10$.

50 We have $a^2 = (\tan x + \cot x)^2 = \tan^2 x + 2 + \cot^2 x \implies \tan^2 x + \cot^2 x = a^2 - 2$. Hence

$$\tan^3 x + \cot^3 x = (\tan x + \cot x)(\tan^2 x - (\tan x)(\cot x) + \cot^2 x) = a(a^2 - 2 - 1)a^3 - 3a.$$

52 We have

$$3\sin x + 4\cos x = 5 \iff \sin x = \frac{5-4\cos x}{3}.$$

Putting this in the identity $\cos^2 x + \sin^2 x = 1$ we obtain

$$\cos^2 x + \left(\frac{5-4\cos x}{3}\right)^2 = 1$$

$$\cos^2 x + \frac{25 - 40\cos x + 16\cos^2 x}{9} = 1$$

$$9\cos^2 x + 25 - 40\cos x + 16\cos^2 x = 9$$

$$25\cos^2 x - 40\cos x + 16 = 0$$

$$(5\cos x - 4)^2 = 0$$

$$\cos x = \frac{4}{5}$$

Substituting this value we obtain

$$\sin x = \frac{5-4\cos x}{3} = \frac{5-\frac{16}{5}}{3} = \frac{3}{5}.$$

53 We have

$$\begin{aligned} 1 &= (\sin x + \cos x)^2 + k\sin x\cos x \\ &= \sin^2 x + 2\sin x\cos x + \cos^2 x + k\sin x\cos x \\ &= 1 + (k+2)\sin x\cos x \end{aligned}$$

We thus have $(k+2)\sin x\cos x = 0$. This will hold for all real numbers x if $k = -2$.

60 Hint: Use the Arithmetic-Geometric-Mean Inequality $\frac{a+b}{2} \geq \sqrt{ab}$, for non-negative real numbers a, b.

61 If the logarithmic expression is to make sense, then $\sqrt{2}\sin x > 0$, $\sqrt{2}\sin x \neq 1$ and $1 + \cos x > 0$. For this we must have

$$x \in \left]0; \frac{\pi}{4}\right[\cup \left]\frac{\pi}{4}; \frac{3\pi}{4}\right[\cup \left]\frac{3\pi}{4}; \pi\right[.$$

Now, if x belongs to this set

$$\log_{\sqrt{2}\sin x}(1+\cos x) = 2 \iff 2\sin^2 x = 1 + \cos x.$$

Using $\sin^2 x = 1 - \cos^2 x$, the last equality occurs if and only if

$$(2\cos x - 1)(\cos x + 1) = 0.$$

If $\cos x + 1 = 0$, then $x = \pi$, a value that must be discarded (why?). If $\cos x = \frac{1}{2}$, then $x = \frac{\pi}{3}$, which is the only solution in $[0; 2\pi]$.

62 Observe that

$$\begin{aligned} 2^{\sin^2 x} + 5(2^{\cos^2 x}) - 7 &= 2^{\sin^2 x} + 5(2^{1-\sin^2 x}) - 7 \\ &= 2^{\sin^2 x} + 5(2^1 \cdot 2^{-\sin^2 x}) - 7 \\ &= 2^{\sin^2 x} + \left(\frac{10}{2^{\sin^2 x}}\right) - 7 \\ &= u + \frac{10}{u} - 7. \end{aligned}$$

with $u = 2^{\sin^2 x}$. From this, $0 = u^2 - 7u + 10 = (u-5)(u-2)$. Thus either $u = 2$,, meaning $2^{\sin^2 x} = 2$ which is to say $\sin x = \pm 1$ or $x = (-1)^n(\frac{\pm\pi}{2}) + n\pi$. When $2^{\sin^2 x} = 5$ one sees that $\sin^2 x = \log_2 5$. Since the sinistral side of the last equality is at most 1 and its dextral side is greater than 1, there are no real roots in this instance. The solution set is thus

$$\left\{(-1)^n(\frac{\pm\pi}{2}) + n\pi,\ n \in \mathbb{Z}\right\}.$$

63 Transposing

$$\cos^{2000} x = \sin^{2000} x + 1.$$

The dextral side is ≥ 1 and the sinistral side is ≤ 1. Thus equality is only possible if both sides are equal to 1, which entails that $\cos x = 1$ or $\cos x = -1$, whence $x = \pi n, n \in \mathbb{Z}$.

64 Since $|\cos x| \leq 1$ and $|\sin x| \leq 1$, we have

$$\begin{aligned} 1 &= \cos^{2001} x - \sin^{2001} x \\ &= \cos^{2001}(-x) + \sin^{2001}(-x) \\ &\leq |\cos^{2001}(-x)| + |\sin^{2001}(-x)| \\ &= |\cos^{1999}(-x)|\cos^2(-x) + |\sin^{1999}(-x)|\sin^2(-x) \\ &\leq \cos^2(-x) + \sin^2(-x) \\ &= 1. \end{aligned}$$

The inequalities are tight, and so equality holds throughout. The first inequality above is true if and only if $\cos(-x) \geq 0$ and $\sin(-x) \geq 0$. The second inequality is true if and only if $|\cos(-x)| = 1$ or $|\sin(-x)| = 1$. Hence we must have either $\cos(-x) = 1$ or $\sin(-x) = 1$.This means $x = 2n\pi$ or $x = -\frac{\pi}{2} + 2n\pi$ where $n \in \mathbb{Z}$.