1 Functions. Continuity. Limits

1 Problem The curve $y = x - \dfrac{2}{x^2}$ undergoes the following successive transformations:

1. a translation one unit to the right,
2. a reflection about the y-axis,
3. a translation one unit down.

What is the equation of the resulting curve?

2 Problem If the function f given by

$$f(x) = \begin{cases} \dfrac{\tan 2ax}{\sin 3bx} & \text{if } x < 0 \\ c & \text{if } x = 0 \\ \dfrac{|x|}{x} + \dfrac{x^3 - 1}{x - 1} & \text{if } 0 < x < 1 \\ b & \text{if } x \geq 1 \end{cases}$$

is everywhere continuous, find a, b, and c.

3 Problem Let $f(x) = x^2 - 2x$ and $g(x) = 2x + 1$. Find all x for which

$$(f \circ g)(x) = (g \circ f)(x).$$

4 Problem Let f be a function defined for all real numbers and such that $f(2x - 1) = 1 - 4x^2$. Find $(f \circ f)(x)$.

5 Problem Find $\lim_{x \to 0} \dfrac{\sin(\tan ax)}{\tan(\sin a^2 x)}$, where $a \neq 0$ is a constant.

6 Problem Find

$$\lim_{x \to 2-} \frac{|x - 2|}{(2 - x)[\![x]\!]}.$$

7 Problem Determine the domain of definition of the function f given by $f(x) = \sqrt{-\sin x} + \sqrt{16 - x^2}$. Write a brief explanation for your choice.

8 Problem How many solutions does the equation

$$\sin x = \frac{x}{100}$$

have?

9 Problem How many solutions does the equation

$$\sin(\sin(\sin(\sin(\sin(x))))) = \frac{x}{3}$$

have?

10 Problem Determine the set of points of discontinuity of the function $f : \mathbb{R} \to \mathbb{R}$, $f : x \mapsto \llbracket x \rrbracket + \sqrt{x - \llbracket x \rrbracket}$.

11 Problem What are the points of discontinuity of the function $f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \end{array}$?

12 Problem What are the points of discontinuity of the function $f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \end{array}$?

13 Problem What are the points of discontinuity of the function $f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \end{array}$?

14 Problem What are the points of discontinuity of the function $f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} \cos x & \text{if } x \in \mathbb{Q} \\ \sin x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \end{array}$?

15 Problem Let $f : [a\,;b] \to [a\,;b]$ be continuous. Then f has a fixed point, that is, there is $c \in [a\,;b]$ such that $f(c) = c$.

16 Problem Given the general shape of the curve $y = \dfrac{1}{x}$, graph the following curves in succession, noting all asymptotes, poles, and intercepts. Graphing devices are not allowed!

1. $y = \dfrac{1}{x-1}$
2. $y = \dfrac{1}{1-x}$
3. $y = \dfrac{1}{-x-1}$
4. $y = \dfrac{1}{x-1} + 2$
5. $y = \dfrac{1}{\left|x\right| - 1} + 2$
6. $y = \left|\dfrac{1}{x-1} + 2\right|$
7. $y = \left|\dfrac{1}{\left|x\right| - 1} + 2\right|$
8. $y = \left|\dfrac{1}{-\left|x\right| - 1} + 2\right|$

17 Problem Let $p(x), q(x)$ be polynomials with real coefficients such that

$$p(x^2 + x + 1) = p(x)q(x).$$

Prove that p must have even degree.

18 Problem A function f defined over all real numbers is continuous and for all real x satisfies

$$(f(x)) \cdot ((f \circ f)(x)) = 1.$$

Given that $f(1000) = 999$, find $f(500)$.

19 Problem Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x \to -\infty} f(x) = 0 = \lim_{x \to +\infty} f(x)$. Prove that if f is strictly negative somewhere on $\mathbb{R}$ then f attains a finite absolute minimum on $\mathbb{R}$, and that if f is strictly positive somewhere on $\mathbb{R}$ then f attains a finite absolute maximum on $\mathbb{R}$.

20 Problem Let $f : [0\,;1] \to [0\,;1]$ be continuous. Prove that there is no $c \in [0\,;1]$ such that $f^{-1}(\{c\})$ has exactly two elements.

21 Problem Let f, g be continuous functions from $[0\,;1]$ to $[0\,;1]$ such that

$$\forall x \in [0\,;1] \qquad f(g(x)) = g(f(x)).$$

Prove that f and g have a common fixed point in $[0\,;1]$.

22 Problem (Cauchy's Functional Equation) Complete the following steps in order to prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $f(a + b) = f(a) + f(b)$ for all $(a, b) \in \mathbb{R}^2$, then $f(x) = xf(1)$ for all $x \in \mathbb{R}$, that is, f must be linear.

1. Prove that $f(0) = 0$.
2. Prove by induction on n, that if $n \in \mathbb{N}$, $x \in \mathbb{R}$, then $f(nx) = nf(x)$.
3. Prove that if $m \in \mathbb{N}$ then $f(-mx) = -mf(x)$.
4. Prove that if $a \in \mathbb{Z}$, $b \in \mathbb{Z} \setminus \{0\}$, then $f\left(\dfrac{a}{b}\right) = \left(\dfrac{a}{b}\right) f(1)$.
5. Conclude now that for all $x \in \mathbb{R}$, $f(x) = xf(1)$.

23 Problem A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\forall x \in \mathbb{R} \qquad f(x + f(x)) = f(x).$$

Prove that f is constant.

24 Problem Let $f : [0\,;+\infty\,[\; \to [0\,;+\infty\,[, x \mapsto \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}$. Is f right-continuous at 0?

25 Problem Let I be a closed and bounded interval on the line and let f be continuous on I. Suppose that for each $x \in I$, there exists a $y \in I$ such that

$$|f(y)| \leq \frac{1}{2}|f(x)|.$$

Prove the existence of a $t \in I$ such that $f(t) = 0$.

26 Problem Find all continuous functions that satisfy the functional equation

$$f(x) + f(y) = f\left(\frac{x + y}{1 - xy}\right),$$

for all $-1 < x, y < 1$.

27 Problem (Putnam 1947) A real valued continuous function satisfies for all real x, y the functional equation

$$f(\sqrt{x^2+y^2}) = f(x)f(y).$$

Prove that $f(x) = (f(x))^{x^2}$.

28 Problem Suppose that $f : [0\,;1] \to [0\,;1]$ is continuous. Prove that there is a number c in $[0\,;1]$ such that $f(c) = 1 - c$.

29 Problem (Universal Chord Theorem) Suppose that f is a continuous function of $[0\,;1]$ and that $f(0) = f(1)$. Let n be a strictly positive integer. Prove that there is some number $x \in [0\,;1]$ such that $f(x) = f(x + 1/n)$.

30 Problem Under the same conditions of problems 29 prove that there are no universal chords of length $a, 0 < a < 1, a \neq 1/n$.

31 Problem (Putnam, 1999) Find polynomials $f(x)$,$g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x+2 & \text{if } -1 \leq x \leq 0 \\ -2x+2 & \text{if } x > 0. \end{cases}$$

2 Derivatives

32 Problem Consider an *even* function $f : \mathbb{R} \to \mathbb{R}$ whose derivative f' exists everywhere and which satisfies

$$f(1) = 2; \quad f(-2) = 1; \quad f'(-1) = 1; \quad f'(2) = 2, \quad f''(-1) = 0; \quad f''(2) = 3.$$

Problems 1 through 2 refer to this situation.

1. If $a(x) = xf(x^2)$, find $a''(-1)$.
2. If $b(x) = \dfrac{x}{f(x^2)}$, find $b'(\sqrt{2})$.

33 Problem Figures 1 and 2 show two functions f and g.

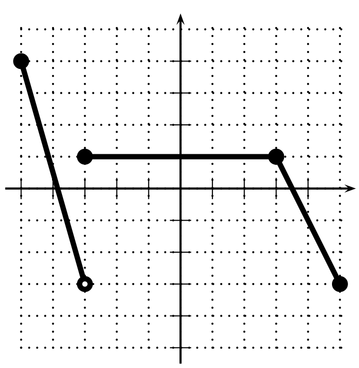


Figure 1: Problem 33. The function f.

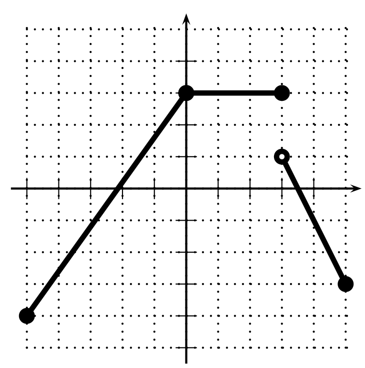


Figure 2: Problem 33. The function g.

Determine $\left(\dfrac{g}{f} + \dfrac{f}{g}\right)'(-1)$.

34 Problem Let p be the polynomial of degree 100

$$a_0x^{100} + a_1x^{99} + a_2x^{98} + \cdots + a_{99}x + a_{100} = (x-2)^{50}(2x-1)^{50}.$$

Find $100a_0 + 99a_1 + 98a_2 + \cdots + 2a_{98} + a_{99}$.

35 Problem Let f be a function defined for all real numbers, which satisfies

$$f(x+h) - f(x) = 2xh + h^2 - h.$$

Problems 1 through 2 refer to this situation.

1. Find $f'(2)$.
2. Find $f''(2)$.

36 Problem Let f be an everywhere differentiable function and let $g = f \circ f$. It is known that

$$f(1) = 2; \quad f(2) = 3; \quad f'(1) = 1; \quad f'(2) = 1, \quad f''(1) = -1, \quad f''(2) = -1.$$

Problems 1 through 2 refer to this situation.

1. Find $g'(1)$.
2. Find $g''(1)$.

37 Problem Find the 100-th derivative of $x \mapsto x^2 \sin x$.

38 Problem Demonstrate that the polynomial $p(x) \in \mathbb{R}[x]$ has a zero at $x = a$ of multiplicity k if and only if

$$p(a) = p'(a) = \cdots = p^{(k-1)}(a) = 0.$$

39 Problem Demonstrate that if for all $x \in \mathbb{R}$ there holds the identity

$$\sum_{k=0}^{n} a_k(x-a)^k = \sum_{k=0}^{n} b_k(x-b)^k,$$

then $a_k = \sum_{j=k}^{n} \binom{n}{j} b_j(a-b)^{j-k}$.

40 Problem Let a, b, c be three functions such that $a' = b$, $b' = c$, and $c' = a$. Prove that the function $a^3 + b^3 + c^3 - 3abc$ is constant.

3 Applications of Differentiation

41 Problem Let a, b be strictly positive constants. Find the equation of the line that is tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the first quadrant and forms with the coordinate axes the triangle with smallest possible area.

42 Problem The derivative of a function f is $f'(x) = (x-1)^2(x+2)^3(2-\sin x)^2$. Find, with explanation, all the values of x for which has a local minimum.

43 Problem Let g be a continuous function on $[a;b]$ and suppose that g'' exists for all x in the open interval $]a;b[$. Prove that if there are three values of x in $[a;b]$ for which $g(x) = 0$, then there is at least one value of x in $]a;b[$ where $g''(x) = 0$.

44 Problem A polynomial p has a local maximum $(-2,4)$, a local minimum at $(1,1)$, a local maximum at $(5,7)$ and no other critical points. How many real zeroes does p have?

45 Problem If the graph of the curve $y = x^3 + ax^2 + bx - 4$ has an inflexion point at $(1,-6)$, what is the value of b?

46 Problem Shew, by means of Rolle's Theorem, that $5x^4 - 4x + 1 = 0$ has a solution in $[0;1]$.

47 Problem Let $a_0, a_1 \ldots, a_n$ be real numbers satisfying

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.$$

Shew that the polynomial

$$a_0 + a_1x + \cdots + a_nx^n$$

has a root in $]0\,;1[$.

48 Problem Suppose that $f : [0\,;1] \to \mathbb{R}$ is differentiable, $f(0) = 0$ and $f(x) > 0$ for $x \in]0\,;1[$. Is there a number $d \in]0\,;1[$ such that

$$\frac{2f'(c)}{f(c)} = \frac{f'(1-c)}{f(1-c)}?$$

49 Problem Let $n \geq 1$ be an integer and let $f : [0;1] \to \mathbb{R}$ be differentiable and such that $f(0) = 0$ and $f(1) = 1$. Prove that there exist distinct points $0 < a_0 < a_2 < \cdots < a_{n-1} < 1$ such that

$$\sum_{k=0}^{n-1} f'(a_k) = n.$$

50 Problem Let $n \geq 1$ be an integer and let $f : [0;1] \to \mathbb{R}$ be differentiable and such that $f(0) = 0$ and $f(1) = 1$. Prove that there exist distinct points $0 < a_0 < a_2 < \cdots < a_{n-1} < 1$ such that

$$\sum_{k=0}^{n-1} \frac{1}{f'(a_k)} = n.$$

51 Problem (Putnam 1946) Let $p(x)$ is a quadratic polynomial with real coefficients satisfying $\max_{x\in[-1\,;1]} \left|f(x)\right| \leq 1$. Prove that $\max_{x\in[-1\,;1]} \left|f'(x)\right| \leq 4$.

52 Problem If f' exists on an interval containing c, prove that

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c-h)}{2h}.$$

53 Problem If f'' exists on an interval containing c, prove that

$$f''(c) = \lim_{h\to 0} \frac{f(c+h) + f(c-h) - 2c}{h^2}.$$

54 Problem Let f be a polynomial with real coefficients of degree n such that $\forall x \in \mathbb{R} \quad f(x) \geq 0$. Prove that

$$\forall x \in \mathbb{R} \qquad f(x) + f'(x) + f''(x) + \cdots + f^{(n)}(x) \geq 0.$$

55 Problem (Putnam 1991) A there any polynomials $p(x)$ with real coefficients of degree $n \geq 2$ all whose n roots are distinct real numbers and all whose $n - 1$ zeroes of $p'(x)$ are the midpoints between consecutive roots of $p(x)$?

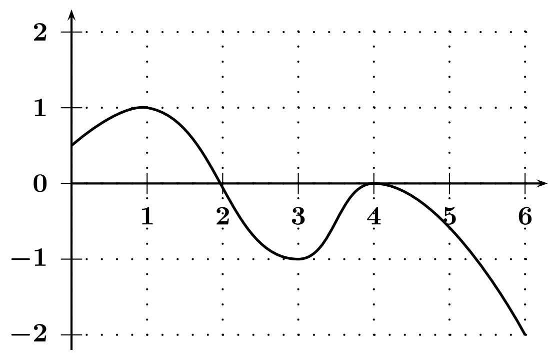
56 Problem Prove that the inflexion points of $x \mapsto \dfrac{x}{\tan x}$ are aligned.

57 Problem Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

58 Problem Problems 1 and 2 refer to the graph below, which is the graph of the *derivative* f' of a certain function f defined on $[0; 6]$.



1. Find the x-coordinate of the global maximum of f.
2. How many inflexion points does the graph of f have?

59 Problem The curve

$$y = x^3 + 3x^2 + ax + b$$

has one inflexion point. The tangent line at this inflexion point is $y = 3x + 4$. Find the constants a and b.

60 Problem (Putnam, 1998) Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

61 Problem (Putnam, 1998) Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

62 Problem (Putnam, 1999) Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

63 Problem (Putnam, 2003) Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

4 Integration

64 Problem The graph of a function f defined on $[-3, 4]$ is given in figure 3 below. It is composed of a semicircle and lines.

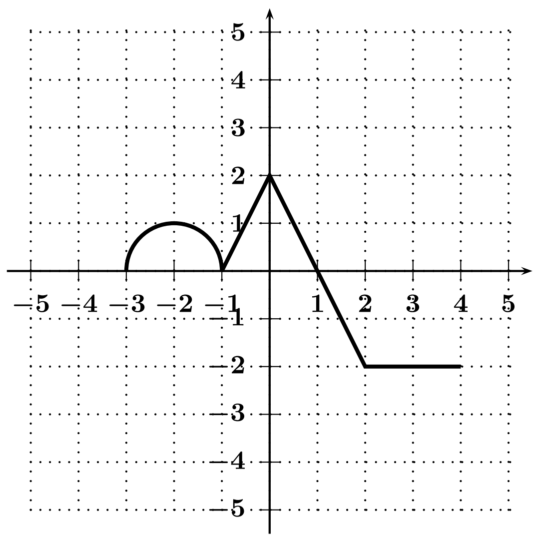


Figure 3: Problems 64 and 2.

1. If $g(x) = \int_{-3}^{x} f(t)dt$, find $g(2)$.
2. Where does g attain its absolute maximum value?

65 Problem Compute $\int_0^3 x\llbracket x \rrbracket dx$.

66 Problem Let f be a differentiable function such that

$$f(x+h) - f(x) = e^{x+h} - h - e^x$$

and $f(0) = 3$. Find $f(x)$.

67 Problem Let f be a continuous function such that $f(x)f(a-x) = 1$ and let $a > 0$. Find $\int_0^a \frac{1}{f(x)+1}dx$.

68 Problem Let n be a fixed integer. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ 2^n & \text{if } 2^n - 2^{n-2} < x \leq 2^{n+1} - 2^{n-1} \end{cases}$$

Prove that $\int_0^{2^n} f(x)\mathrm{d}x = \int_0^{2^n} x\mathrm{d}x = 2^{2^n-1}$.

69 Problem (Putnam 1938) Evaluate the limit

$$\lim_{t\to 0} \frac{\int_0^t (1+\sin 2x)^{1/x}\mathrm{d}x}{t}.$$

70 Problem Find the value of $\int_0^1 \max(x^2, 1-x)\mathrm{d}x$.

71 Problem Let $a > 0$. Let f be a continuous function on $[0\,;a]$ such that $f(x)+f(a-x)$ does not vanish on $[0\,;a]$. Evaluate $\int_0^a \frac{f(x)\mathrm{d}x}{f(x)+f(a-x)}$.

72 Problem Each line passing through the origin $O(0,0)$ and with slope $\lambda > 0$ cuts the curve $y = x^n$, ($x \geq 0$, $n \neq -1$) at the point $A(x_1, y_1)$ which projects to the point $B(x_1, 0)$ on the x-axis. $\triangle OAB$ is divided into two regions by the curve $y = x^n$. Demonstrate that the ratio of the areas of these two regions is independent of λ. For which value of n are the areas of these two regions equal?

73 Problem Let $a > 0$. Let F be a differentiable function such that $\forall x \in [0\,;a]$ $F'(a-x) = F'(x)$. Evaluate $\int_0^a F(x)\mathrm{d}x$.

74 Problem Let $n \geq 0$ be an integer. Let a be the unique differentiable function such that $\forall x \in \mathbb{R}$

$$(a(x))^{2n+1} + a(x) = x.$$

Evaluate $\int_0^x a(t)\mathrm{d}t$.

75 Problem Find $\int_0^{\pi/2} \frac{\sin x\mathrm{d}x}{\sin x + \cos x}$.

76 Problem Find $\int_0^{\pi/2} \frac{1\mathrm{d}x}{1+(\tan x)^{\sqrt{2}}}$.

77 Problem Let p be a polynomial of degree at most 4 such that $p(-1) = p(1) = 0$ and $p(0) = 1$. If $p(x) \leq 1$ for $x \in [-1; 1]$, find the largest value of $\int_{-1}^1 p(x)\mathrm{d}x$.

78 Problem (Putnam, 1999) Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C\int_{-1}^1 |p(x)|\,dx.$$

5 Some Answers

1 Put $a(x) = x - \frac{2}{x^2}$ After a translation one unit right we get $a(x-1) = x - 1 - \frac{2}{(x-1)^2} = b(x)$, say. After a reflexion about the y axis we get $b(-x) = -x-1-\frac{2}{(-x-1)^2} = -x-1-\frac{2}{(x+1)^2} = c(x)$, say. After a translation one unit down we get $c(x)-1=-x-1-\dfrac{2}{(x+1)^2}-1=-x-2-\dfrac{2}{(x+1)^2}$, whence the resulting curve is $y=-x-2-\dfrac{2}{(x+1)^2}$.

2 We have

$$f(0-)=\frac{2a}{3b};\quad f(0)=c;\quad f(0+)=2;\quad f(1-)=4;\quad f(1)=b;\quad f(1+)=b.$$

Because of continuity,

$$\frac{2a}{3b}=c=2;\quad 4=b.$$

Hence $a=12, b=4, c=2$.

3 We have

$$(f\circ g)(x)=(g\circ f)(x) \iff (2x+1)^2-2(2x+1)=2(x^2-2x)+1 \iff x\in\{-1-\sqrt{2},-1+\sqrt{2}\}.$$

4 Since x is a dummy variable, rename: $f(2u-1)=1-4u^2$. Now put $x=2u-1 \implies u=\dfrac{x+1}{2}$. Hence

$$f(x)=f(2u-1)=1-4u^2=1-4\left(\frac{x+1}{2}\right)^2=1-(1+x)^2=-2x-x^2.$$

Thus

$$(f\circ f)(x)=f(f(x))=-2f(x)-(f(x))^2=-2(-2x-x^2)-(-2x-x^2)^2=4x-2x^2-4x^3-x^4.$$

5 Using $\dfrac{\tan y}{y}\to 1$ and $\dfrac{\sin y}{y}\to 1$ as $y\to 0$ several times,

$$\frac{\sin(\tan ax)}{\tan(\sin a^2x)}=\frac{\sin(\tan ax)}{\tan ax}\cdot\frac{\tan ax}{\sin a^2x}\cdot\frac{\sin a^2x}{\tan(\sin a^2x)}=\frac{\sin(\tan ax)}{\tan ax}\cdot\frac{\tan ax}{ax}\cdot\frac{a^2x}{\sin a^2x}\cdot\frac{\sin a^2x}{\tan(\sin a^2x)}\cdot\frac{1}{a}.\to\frac{1}{a},$$

as $x\to 0$.

Aliter: As $x\to 0$, since $\tan x\sim x$ and $\sin x\sim x$, we have $\sin\tan ax\sim\tan ax\sim ax$ and $\tan(\sin a^2x)\sim\sin a^2x\sim a^2x$. Thus

$$\frac{\sin(\tan ax)}{\tan(\sin a^2x)}\sim\frac{ax}{a^2x}\sim\frac{1}{a}.$$

6

$$\frac{|x-2|}{(2-x)[\![x]\!]}=\frac{|x-2|}{(2-x)}\cdot\frac{1}{[\![x]\!]}\to(1)\cdot(1)=1,$$

as $x\to 2-$.

7 We require

$$\{x\in\mathbb{R}:4-x^2\geq 0\}\cap\{x\in\mathbb{R}:-\sin x\geq 0\}=[-4;4]\cap\left(\bigcup_{-\infty}^{+\infty}[-\pi+2\pi n\,;0+2\pi n]\right)=[-\pi\,;0]\cup[\pi\,;4].$$

17 If p had odd degree, then, by the Intermediate Value Theorem it would have a real root. Let α be its largest real root. Then

$$0=p(\alpha)q(\alpha)=p(\alpha^2+\alpha+1)$$

meaning that $\alpha^2+\alpha+1>\alpha$ is a real root larger than the supposedly largest real root α, a contradiction.

18 Observe that $f(1000)f(f(1000))=1 \implies f(999)=\dfrac{1}{999}$. So the range of f include all numbers from $\dfrac{1}{999}$ to 999. By the intermediate value theorem, there is a real number a such that $f(a)=500$. Thus

$$f(a)f(f(a))=1 \implies f(500)=\frac{1}{500}.$$

24 $f(0)=0$, but for $x>0$, $f(x)=\dfrac{1+\sqrt{1+4x}}{2}$, so f is not right-continuous at $x=0$.

28 If either $f(0) = 1$ or $f(1) = 0$, we are done. So assume that $0 \geq f(0) < 1$ and $0 < f(1) \leq 1$. Put $g(x) = f(x) + x - 1$. Then $g(0) = f(0) - 1 < 0$ and $g(1) = f(1) > 0$. By Bolzano's Theorem there is a $c \in]0\,;1[$ such that $g(c) = 0$, that is, $f(c) + c - 1 = 0$, as required.

29 Consider $g(x) = f(x) - f(x + 1/n)$, which is clearly continuous. If g is never 0 in $[0\,;1]$ then by Corollary **??** g must be either strictly positive or strictly negative. But then

$$0 = f(0) - f(1) = \left(f(0) - f\left(\frac{1}{n}\right)\right) + \left(f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right)\right) + \left(f\left(\frac{2}{n}\right) - f\left(\frac{3}{n}\right)\right) + \cdots + \left(f\left(\frac{n-1}{n}\right) - f\left(\frac{n}{n}\right)\right).$$

The sum of each parenthesis on the right is strictly positive or strictly negative and hence never 0, a contradiction.

30 Consider the function $f : [0\,;1] \to [0\,;1]$, $x \mapsto \dfrac{\sin\frac{2\pi x}{a}}{\sin\frac{2\pi}{a}} - x$.

31 Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r, s) = (r + s + |r - s|)/2.$$

Therefore, if $F(x)$ is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x - 3, 0\} - \max\{5x, 0\} + 3x + 2 \\ &= (-3x - 3 + |3x - 3|)/2 \\ &\quad - (5x + |5x|)/2 + 3x + 2 \\ &= |(3x - 3)/2| - |5x/2| - x + \frac{1}{2}, \end{aligned}$$

so we may set $f(x) = (3x - 3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

32 Since f is even, f' is odd and f'' is even. Thus we have

$$f(1) = 2; \quad f(-1) = 2; \quad f(-2) = 1; \quad f(2) = 1$$

$$f'(1) = -1 \quad f'(-1) = 1; \quad f'(2) = 2; \quad f'(-2) = -2;$$

$$f''(1) = 0; \quad f''(-1) = 0; \quad f''(2) = 3; \quad f''(-2) = 3.$$

Now, $a'(x) = f(x^2) + 2x^2 f'(x^2)$ and

$$a''(x) = 2xf'(x^2) + 4xf'(x^2) + 4x^3 f''(x^2),$$

whence

$$a''(-1) = -2f'(1) - 4f'(1) - 4f''(1) = -2(-1) - 4(-1) = 6.$$

Moreover,

$$b'(x) = \frac{f(x^2) - 2x^2 f'(x^2)}{(f(x^2))^2} \implies b'(\sqrt{2}) = \frac{f(2) - 4f'(2)}{(f(2))^2} = \frac{1 - 4(2)}{(1)^2} = -7.$$

34 We have

$$100a_0x^{99} + 99a_1x^{98} + 98a_2x^{97} + \cdots + 2a_{98}x + a_{99} = p'(x) = 50(x - 2)^{49}(2x - 1)^{50} + 100(x - 2)^{50}(2x - 1)^{49}$$

and the desired quantity is

$$100a_0 + 99a_1 + 98a_2 + \cdots + 2a_{98} + a_{99} = p'(1) = -50 + 100 = 50.$$

35 We have $\dfrac{f(x + h) - f(x)}{h} = 2x + h - 1 \to 2x - 1$ as $h \to 0$. Hence $f'(x) = 2x - 1$ and $f'(2) = 3$. Moreover, $f''(x) = 2$, whence $f''(2) = 2$.

36 We have

$$g(x) = f(f(x)) \implies g'(x) = f'(x)f'(f(x)) \implies g'(1) = f'(1)f'(f(1)) = 1 \cdot f'(2) = 1.$$

Also,

$$g(x) = f(f(x)) \implies g'(x) = f'(x)f'(f(x)) \implies g''(x) = f''(x)f'(f(x)) + (f'(x))^2 f''(f(x)),$$

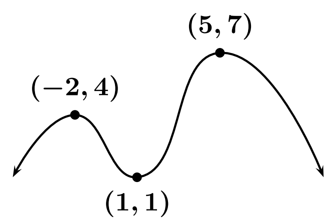
which gives

$$g''(1) = f''(1)f'(f(1)) + (f'(1))^2 f''(f(1)) = -1f'(2) + (1)^2 f''(2) = -1(1) + 1(-1) = -2.$$

37 We use Leibniz's Rule and the observation that the third derivative of $x \mapsto x^2$ is 0. Also $(\sin x)^{(4n)} = \sin x$, $(\sin x)^{(4n+2)} = -\sin x$, $(\sin x)^{(4n+1)} = \cos x$, and $(\sin x)^{(4n+3)} = -\cos x$, Then

$$\frac{\mathrm{d}^{100}}{\mathrm{d}x^{100}}x^2 \sin x = \binom{100}{0}x^2(\sin x)^{(100)}+\binom{100}{1}(x^2)'(\sin x)^{(99)}+\binom{100}{2}(x^2)''(\sin x)^{(98)} = x^2\sin x - 200x\cos x - 9900\sin x.$$

44 The conditions demand that $p(x) \to -\infty$ as $x \to \pm\infty$. The stipulations also demand that the polynomial does not change inflexion at the roots. Hence the number of real roots is two.



45 If $f(x) = x^3 + ax^2 + bx - 4$, since $(1, -6)$ is on the curve,

$$-6 = 1 + a + b - 4 \implies a + b = -3.$$

The second derivative vanishes at $(1, -6)$ and so $f''(1) = 6(1) + 2a = 0 \implies a = -3$. Hence $b = 0$.

46 Put $f(x) = x^5 - 2x^2 + x$. Then $f(0) = f(1) = 0$ and by Rolle's Theorem there is $c \in]0; 1[$ such that $f'(c) = 5c^4 - 4c + 1 = 0$.

47 Set

$$f(x) = a_0x + \frac{a_1x^2}{2} + \frac{a_2x^3}{3} + \cdots + \frac{a_nx^{n+1}}{n+1},$$

and use Rolle's Theorem.

48 Set $g(x) = f(x)^2 f(1 - x)$. Since $g(0) = g(1) = 0$, g satisfies the hypotheses of Rolle's Theorem. There is a $c \in]0 ; 1[$ such that

$$g'(c) = 0 \implies 2f'(c)f(c)f(1 - c) - f(c)^2 f'(1 - c) = 0.$$

Since by assumption $f(c)f(1 - c) \neq 0$ we must have, upon dividing by every term by $f(c)^2 f(1 - c)$, the assertion.

49 For $0 \le k \le n - 1$, consider the interval $\left[\frac{k}{n} ; \frac{k+1}{n}\right]$. By the Mean Theorem, there are $a_k \in \left]\frac{k}{n} ; \frac{k+1}{n}\right[$ such that

$$f'(a_k) = \frac{f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)}{\frac{1}{n}} = n\left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right).$$

Summing from $k = 0$ to $k = n - 1$ and noting that the dextral side telescopes,

$$\sum_{k=0}^{n-1} f'(a_k) = n\sum_{k=0}^{n-1}\left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right) = n(f(1) - f(0)) = n.$$

50 Let $k_i \in [0 ; 1]$ be the smallest number such that $f(k_i) = \frac{i}{n}$, $1 \le i \le n - 1$. Put $k_0 = 0, k_n = 1$. The existence of the k_i is guaranteed by the Intermediate Value Theorem. Moreover, since the k_i are chosen to be the first time f is $\frac{i}{n}$, once again, by the Intermediate Value Theorem we must have

$$0 < k_1 < k_2 < \cdots < k_{n-1} < 1.$$

Hence, by the Mean Value Theorem, there exists $a_i \in]k_i\,; k_{i+1}[$, $0 \leq i \leq n-1$, such that

$$f'(a_i) = \frac{f(k_{i+1}) - f(k_i)}{k_{i+1} - k_i} = \frac{1}{n(k_{i+1} - k_i)} \implies \frac{1}{f'(a_i)} = n(k_{i+1} - k_i).$$

Summing,

$$\sum_{k=0}^{n-1} \frac{1}{f'(a_k)} = n \sum_{k=0}^{n-1} (k_{i+1} - k_i) = n(k_n - k_0) = n.$$

57 It is enough to shew that $|f(x)|$ is bounded for $x \geq 0$, as $f(-x)$ satisfies the same equation as $f(x)$. But then

$$\begin{aligned}\frac{d}{dx}\left((f(x))^2 + (f'(x))^2\right) &= 2f'(x)(f(x) + f''(x)) \\ &= -2xg(x)(f'(x))^2 \leq 0,\end{aligned}$$

so that $(f(x))^2 \leq (f(0))^2 + (f'(0))^2$ for $x \geq 0$.

58 f' is zero at $x = 2$ and $x = 4$. At $x = 2$ f' changes from positive to negative, hence f increases and then decreases, which means $x = 2$ is a local minimum. This is also the global minimum. Also, f'' will be zero where f' has local extrema, which happens thrice.

60 If at least one of $f(a)$, $f'(a)$, $f''(a)$, or $f'''(a)$ vanishes at some point a, then we are done. Hence we may assume each of $f(x)$, $f'(x)$, $f''(x)$, and $f'''(x)$ is either strictly positive or strictly negative on the real line. By replacing $f(x)$ by $-f(x)$ if necessary, we may assume $f''(x) > 0$; by replacing $f(x)$ by $f(-x)$ if necessary, we may assume $f'''(x) > 0$. (Notice that these substitutions do not change the sign of $f(x)f'(x)f''(x)f'''(x)$.) Now $f''(x) > 0$ implies that $f'(x)$ is increasing, and $f'''(x) > 0$ implies that $f'(x)$ is convex, so that $f'(x+a) > f'(x) + af''(x)$ for all x and a. By letting a increase in the latter inequality, we see that $f'(x+a)$ must be positive for sufficiently large a; it follows that $f'(x) > 0$ for all x. Similarly, $f'(x) > 0$ and $f''(x) > 0$ imply that $f(x) > 0$ for all x. Therefore $f(x)f'(x)f''(x)f'''(x) > 0$ for all x, and we are done.

61 Notice that

$$\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)} = \\ (x+1/x)^3 - (x^3 + 1/x^3) = 3(x + 1/x)$$

(difference of squares). The latter is easily seen (e.g., by AM-GM) to have minimum value 6 (achieved at $x = 1$).

62 Suppose that P does not have n distinct roots; then it has a root of multiplicity at least 2, which we may assume is $x = 0$ without loss of generality. Let x^k be the greatest power of x dividing $P(x)$, so that $P(x) = x^k R(x)$ with $R(0) \neq 0$; a simple computation yields

$$P''(x) = (k^2 - k)x^{k-2}R(x) + 2kx^{k-1}R'(x) + x^k R''(x).$$

Since $R(0) \neq 0$ and $k \geq 2$, we conclude that the greatest power of x dividing $P''(x)$ is x^{k-2}. But $P(x) = Q(x)P''(x)$, and so x^2 divides $Q(x)$. We deduce (since Q is quadratic) that $Q(x)$ is a constant C times x^2; in fact, $C = 1/(n(n-1))$ by inspection of the leading-degree terms of $P(x)$ and $P''(x)$.

Now if $P(x) = \sum_{j=0}^{n} a_j x^j$, then the relation $P(x) = Cx^2 P''(x)$ implies that $a_j = Cj(j-1)a_j$ for all j; hence $a_j = 0$ for $j \leq n-1$, and we conclude that $P(x) = a_n x^n$, which has all identical roots.

63 Write

$$\begin{aligned} f(x) &= \sin x + \cos x + \tan x + \cot x + \sec x + \csc x \\ &= \sin x + \cos x + \frac{1}{\sin x \cos x} + \frac{\sin x + \cos x}{\sin x \cos x}. \end{aligned}$$

We can write $\sin x + \cos x = \sqrt{2}\cos(\pi/4 - x)$; this suggests making the substitution $y = \pi/4 - x$. In this new coordinate,

$$\sin x \cos x = \frac{1}{2}\sin 2x = \frac{1}{2}\cos 2y,$$

and writing $c = \sqrt{2}\cos y$, we have

$$\begin{aligned} f(y) &= (1+c)\left(1 + \frac{2}{c^2 - 1}\right) - 1 \\ &= c + \frac{2}{c-1}. \end{aligned}$$

We must analyze this function of c in the range $[-\sqrt{2}, \sqrt{2}]$. Its value at $c = -\sqrt{2}$ is $2 - 3\sqrt{2} < -2.24$, and at $c = \sqrt{2}$ is $2 + 3\sqrt{2} > 6.24$. Its derivative is $1 - 2/(c-1)^2$, which vanishes when $(c-1)^2 = 2$, i.e., where $c = 1 \pm \sqrt{2}$. Only the value $c = 1 - \sqrt{2}$ is in bounds, at which the value of f is $1 - 2\sqrt{2} > -1.83$. As for the pole at $c = 1$, we observe that f decreases as c approaches from below (so takes negative values for all $c < 1$) and increases as c approaches from above (so takes positive values for all $c > 1$); from the data collected so far, we see that f has no sign crossings, so the minimum of $|f|$ is achieved at a critical point of f. We conclude that the minimum of $|f|$ is $2\sqrt{2} - 1$.

65

$$\begin{aligned}\int_0^3 x\lfloor x\rfloor\ \mathrm{d}x &= \int_0^1 x\lfloor x\rfloor\ \mathrm{d}x + \int_1^2 x\lfloor x\rfloor\ \mathrm{d}x + \int_2^3 x\lfloor x\rfloor\ \mathrm{d}x \\ &= 0\int_0^1 x\ \mathrm{d}x + 1\int_1^2 x\ \mathrm{d}x + 2\int_2^3 x\ \mathrm{d}x \\ &= \frac{x^2}{2}\Big|_1^2 + x^2\Big|_2^3 \\ &= (2 - \tfrac{1}{2}) + (9 - 4) \\ &= \frac{13}{2}.\end{aligned}$$

66 We have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h} - h - e^x}{h} = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h} - \lim_{h\to 0}\frac{h}{h} = e^x - 1,$$

whence $f(x) = e^x - x + C$. Since $3 = f(0) = e^0 - 0 + C \implies C = 2$, we deduce that $f(x) = e^x - x + 2$.

67 Put $I = \int_0^a \frac{1}{f(x)+1}\mathrm{d}x$. We have

$$I = \int_0^a \frac{1}{f(u)+1}\mathrm{d}u = \int_0^a \frac{f(u)f(a-u)}{f(u) + f(u)f(a-u)}\mathrm{d}u = \int_0^a \frac{f(a-u)}{1 + f(a-u)}\mathrm{d}u = -\int_a^0 \frac{f(v)}{1+f(v)}\mathrm{d}v = \int_0^a \frac{f(u)}{1+f(u)}\mathrm{d}u,$$

whence

$$2I = \int_0^a \frac{f(u)}{1+f(u)}\mathrm{d}u + \int_0^a \frac{f(a-u)}{1+f(a-u)}\mathrm{d}u = \int_0^a \frac{2 + f(u) + f(a-u)}{2 + f(u) + f(a-u)}\mathrm{d}u = a,$$

and so $I = \dfrac{a}{2}$.

70 Ask yourself, when is $x^2 = 1 - x$? There are two roots, one is clearly negative and is discarded. Now, for $0 \leq x \leq \dfrac{\sqrt{5}-1}{2}$, $1 - x \geq x^2$. For $\dfrac{\sqrt{5}-1}{2} \leq x \leq, 1 - x \leq x^2$. Hence

$$\int_0^1 \max(x^2, 1-x)\mathrm{d}x = \int_0^{(\sqrt{5}-1)/2}(1-x)\mathrm{d}x + \int_{(\sqrt{5}-1)/2}^1 x^2\mathrm{d}x = \frac{5\sqrt{5}}{12} - \frac{1}{4}.$$

An algebraic trick that simplifies the preceding evaluation is the following. Put $a = \dfrac{\sqrt{5}-1}{2}$. *Then* $a^2 = 1 - a$. *and* $a^3 = a - a^2 = a - (1-a) = 2a - 1$. *Thus*

$$\int_0^a (1-x)\mathrm{d}x + \int_a^1 x^2\mathrm{d}x = \left(a - \frac{a^2}{2}\right) + \left(\frac{1}{3} - \frac{a^3}{3}\right) = \left(a - \frac{1-a}{2}\right) + \left(\frac{1}{3} - \frac{2a-1}{3}\right) = \frac{5}{6}a + \frac{1}{6},$$

and now substitute $a = \dfrac{\sqrt{5}-1}{2}$.

72 Since $x_1^n = y_1 = \lambda x_1$, the area of the triangle is $\dfrac{y_1 x_1}{2} = \dfrac{\lambda x_1^2}{2}$ and the area under the curve is

$$\int_0^{x_1} x^n\mathrm{d}x = \frac{x_1^{n+1}}{n+1} = \frac{\lambda x_1^2}{n+1},$$

and therefore the other portion of the triangle has area

$$\frac{\lambda x_1^2}{2} - \frac{\lambda x_1^2}{n+1} = \frac{\lambda x_1^2(n-1)}{2n+2}.$$

The ratio of both portions is therefore

$$\frac{\dfrac{\lambda x_1^2}{n+1}}{\dfrac{\lambda x_1^2(n-1)}{2n+2}} = \frac{2}{n-1},$$

independent of λ. If $\dfrac{2}{n-1} = 1$ then $n = 3$ and so the portions will have equal area for $n = 3$.

77 $\frac{8}{5}$

78 Let $r_1, \ldots, r_{1999}$ be the roots of P. Draw a disc of radius ϵ around each r_i, where $\epsilon < 1/3998$; this disc covers a subinterval of $[-1/2, 1/2]$ of length at most 2ϵ, and so of the 2000 (or fewer) uncovered intervals in $[-1/2, 1/2]$, one, which we call I, has length at least $\delta = (1 - 3998\epsilon)/2000 > 0$. We will exhibit an explicit lower bound for the integral of $|P(x)|/P(0)$ over this interval, which will yield such a bound for the entire integral.

Note that

$$\frac{|P(x)|}{|P(0)|} = \prod_{i=1}^{1999} \frac{|x - r_i|}{|r_i|}.$$

Also note that by construction, $|x - r_i| \geq \epsilon$ for each $x \in I$. If $|r_i| \leq 1$, then we have $\frac{|x-r_i|}{|r_i|} \geq \epsilon$. If $|r_i| > 1$, then

$$\frac{|x - r_i|}{|r_i|} = |1 - x/r_i| \geq 1 - |x/r_i| \geq= 1/2 > \epsilon.$$

We conclude that $\int_I |P(x)/P(0)|\, dx \geq \delta\epsilon$, independent of P.