1 Applications of Integration.

Note: I use log for what engineers write ln.

1 Problem Find the work needed to pour sand for an embankment in the form of a cone of height H and base radius R. The specific weight of the sand is γ.

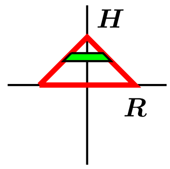

Figure 1: Problems 1.

2 Problem A cylinder of volume V is filled with steam under a pressure p. Find the work needed to compress the steam to the volume $\frac{V}{2}$, if the temperature of the steam is to remain constant. (Hint: From thermodynamics one knows that $\mathrm{d}W = p\,\mathrm{d}v$, where W is the work, and v is the volume. Also, if the temperature is constant, then $pv = C$, a constant.)

3 Problem Find the work necessary to pump the water out of a horizontal cylinder of length L and base radius R. The specific weight of the water is γ.

4 Problem Two very long cylinders of the same radius R intersect at right angles as in figure 2. Find the volume of the overlap.

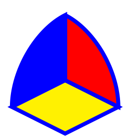

Figure 2: Problem 4.

5 Problem Find the volume when the region

$$\mathscr{R} = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 2, \qquad 0 \le y \le \llbracket x^2 \rrbracket\}$$

is revolved

1. about the x-axis.
2. about the y-axis.
3. about the line $y = -1$.
4. about the line $x = -1$.

2 Techniques of Integration

6 Problem Find $\int e^{e^x+x}\ \mathrm{d}x$.

7 Problem Find $\int \tan x \log(\cos x)\ \mathrm{d}x$.

8 Problem Find $\int \frac{\log\log x}{x\log x}\ \mathrm{d}x$.

9 Problem Find $\int \frac{x^{18}-1}{x^3-1}\ \mathrm{d}x$.

10 Problem Find $\int \frac{1}{x^8+x}\ \mathrm{d}x$.

11 Problem Find $\int \frac{4^x}{2^x+1}\ \mathrm{d}x$.

12 Problem Find $\int \frac{x^2}{(x+1)^{10}}\ \mathrm{d}x$.

13 Problem Find $\int \frac{1}{1+e^x}\ \mathrm{d}x$.

14 Problem Find $\int \frac{1}{1-\sin x}\ \mathrm{d}x$.

15 Problem Find $\int \sqrt{1+\sin 2x}\ \mathrm{d}x$.

16 Problem Find $\int \frac{x}{\sqrt{1-x^4}}\ \mathrm{d}x$.

17 Problem Let $a>0, b>0$, and f a continuous strictly increasing function with $f(0)=0$. Prove that

$$ab \leq \int_0^a f(x)\,\mathrm{d}x + \int_0^b f^{-1}(x)\,\mathrm{d}x.$$

Prove, moreover, that equality occurs if and only if $b=f(a)$.

18 Problem Find $\int \sec^4 x\,\mathrm{d}x$.

19 Problem Find $\int \sec^5 x\,\mathrm{d}x$.

20 Problem Find $\int e^{x^{1/3}}\,dx$.

21 Problem Find $\int \log(x^2+1)\,dx$.

22 Problem Find $\int xe^x \cos x\,dx$.

23 Problem Find $\int x^{2/3}\log x\,dx$.

24 Problem Find $\int \sin(\log x)\,dx$.

25 Problem Find $\int \frac{\log\log x}{x}\,dx$.

26 Problem Prove that $\frac{1}{\cos x} = \frac{\cos x}{2(1+\sin x)} + \frac{\cos x}{2(1-\sin x)}$. Use this to find $\int \sec x\,dx$.

27 Problem Using $\sin 2\theta = 2\sin\theta\cos\theta$ shew that $\int \csc x\,dx = \log|\tan\frac{x}{2}| + C$. Now use $\csc(\frac{\pi}{2}+x) = \sec x$ to find yet another formula for $\int \sec x\,dx$.

28 Problem Find $\int (\arcsin x)^2\,dx$.

29 Problem Find $\int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}}$.

30 Problem $\int x\arctan x\,dx$.

31 Problem Find $\int \sqrt{\tan x}\,dx$.

32 Problem Find $\int \frac{2x+1}{x^2(x-1)}\,dx$.

33 Problem Find $\int \log(x+\sqrt{x})\,dx$.

34 Problem Find $\int \frac{1}{x^4+1}\,dx$.

35 Problem Find $\int \frac{1}{x^3+1}\, dx$.

36 Problem Verify that $\int f(x)\, dx = F(x) + C$.

$f(x)$	$F(x)$
$\frac{1}{\sin x \sin 4x}$	$-\frac{1}{4\sin x} + \frac{1}{8}\log\left\lvert\frac{1-\sin x}{1+\sin x}\right\rvert - \frac{1}{2\sqrt{2}}\log\left\lvert\frac{1-\sqrt{2}\sin x}{1+\sqrt{2}\sin x}\right\rvert$
$\frac{\tan x}{1+\tan x}$	$\frac{x}{2} - \frac{1}{2}\log\lvert\cos x + \sin x\rvert$
$\cos x\sqrt{\cos 2x}$	$\frac{\sin x\sqrt{\cos 2x}}{2} + \frac{1}{2\sqrt{2}}\arcsin(\sqrt{2}\sin x)$
$\frac{1}{\sin x + \sin 2x}$	$\frac{1}{6}\log(1-\cos x) + \frac{1}{2}\log(1+\cos x) - \frac{2}{3}\log\lvert 1+2\cos x\rvert$
$\frac{1}{\cos x\cos 2x}$	$\sqrt{2}\text{arctanh}(\sqrt{2}\sin x) - \text{arctanh}(\sin x)$
$\frac{1}{\sin x\sqrt{\sin x(1+\sin x)}}$	$-2\sqrt{\frac{1-\sin x}{\sin x}} + \sqrt{2}\arctan\sqrt{\frac{1-\sin x}{2\sin x}}$ (put $u = 1/\sin x$)
$\frac{a\sin x}{\cos x\sqrt{\cos^2 x - a^2\sin^2 x}}$	$-\arctan\left(\frac{\sqrt{\cos^2 x - a^2\sin^2 x}}{a}\right)$

37 Problem Verify that $\int f(x)\, dx = F(x) + C$.

$f(x)$	$F(x)$
$\frac{1}{x^3-1}$	$\frac{1}{3}\log\lvert x-1\rvert - \frac{1}{6}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$
$\frac{1}{(x^3-1)^2}$	$-\frac{2}{9}\log\lvert x-1\rvert + \frac{1}{9}\log(x^2+x+1) + \frac{2}{3\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{x}{3(x^3-1)}$
$\frac{1}{x^3(1+x^3)}$	$-\frac{1}{2x^2} + \frac{1}{6}\log\left[\frac{x^2-x+1}{(x+1)^2}\right] - \frac{1}{\sqrt{3}}\arctan\left[\frac{2x-1}{\sqrt{3}}\right]$
$\frac{x^2+x+1}{(x^2-1)^2}$	$-\frac{3}{4(x-1)} - \frac{1}{4(x+1)}$
$\frac{1}{1+x^4}$	$\frac{1}{4\sqrt{2}}\log\left[\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}\right] + \frac{1}{2\sqrt{2}}\left[\arctan(1+x\sqrt{2}) - \arctan(1-x\sqrt{2})\right]$
$\frac{x^2}{1+x^4}$	$\frac{1}{4\sqrt{2}}\log\left[\frac{1-x\sqrt{2}+x^2}{1+x\sqrt{2}+x^2}\right] + \frac{1}{2\sqrt{2}}\left[\arctan(1+x\sqrt{2}) - \arctan(1-x\sqrt{2})\right]$
$\frac{x}{(x^4+1)^2}$	$\frac{\arctan x^2}{4} + \frac{x^2}{4(x^4+1)}$
$\frac{x^2+x+1}{x^3-2x-4}$	$\frac{7}{10}\log\lvert x-2\rvert + \frac{3}{20}\log(x^2+2x+2) - \frac{1}{10}\arctan(x+1)$
$\frac{x^2-4}{x^6-2x^4+x^2}$	$\frac{4}{x} + \frac{3x}{2(x^2-1)} + \frac{11}{4}\log\left\lvert\frac{x-1}{x+1}\right\rvert$
$\frac{1}{x^{20}-1}$	$\frac{1}{10}\sum_{k=1}^{9}\left[\frac{1}{2}\cos k\alpha\log(x^2-2x\cos k\alpha+1) - \sin k\alpha\arctan\left(\frac{x-\cos k\alpha}{\sin k\alpha}\right)\right] + \frac{1}{20}\log\left\lvert\frac{x-1}{x+1}\right\rvert, \quad \alpha = \frac{\pi}{10}$
$\frac{1}{(x-a)^n(x-b)}$	$\frac{1}{(b-a)^n}\log\left\lvert\frac{x-b}{x-a}\right\rvert + \sum_{k=1}^{n-1}\frac{1}{k(b-a)^{n-k}(x-a)^k}, n \geq 2.$

38 Problem Verify that $\int f(x)\,\mathrm{d}x = F(x) + C$.

$f(x)$	$F(x)$
$\dfrac{x+1}{\sqrt{x^2-3x+2}}$	$\sqrt{x^2-3x+2}+\dfrac{5}{2}\log\left\|2x-3+2\sqrt{x^2-3x+2}\right\|$
$\dfrac{4x-3}{\sqrt{-4x^2+12x-5}}$	$-\sqrt{-4x^2+12x-5}+\dfrac{3}{2}\arcsin(x-3/2)$
$\dfrac{1}{2x-x^2+\sqrt{2x-x^2}}$	$\dfrac{1-\sqrt{2x-x^2}}{x-1}$
$\dfrac{1}{2+\sqrt{1+x}+\sqrt{3-x}}$	$\sqrt{1+x}-\sqrt{3-x}-\arcsin\left(\dfrac{x-1}{2}\right)$ (put $x=1+2\cos\varphi$)
$\dfrac{2+\sqrt{x+3}}{1+\sqrt{x+4}}$	$(\sqrt{x+3}+4)(\sqrt{x+4}-2)-4\log(1+\sqrt{x+4})+\log(\sqrt{x+3}+\sqrt{x+4})$
$x+\sqrt{a^2+x^2}$	$\dfrac{(x+\sqrt{a^2+x^2})^2}{4}+\dfrac{a^2}{2}\log(x+\sqrt{a^2+x^2})$
$(x+\sqrt{a^2+x^2})^n$	$\dfrac{(x+\sqrt{a^2+x^2})^{n+1}}{2(n+1)}+a^2\dfrac{(x+\sqrt{a^2+x^2})^{n-1}}{2(n-1)}$ $(n\neq 1)$
$\dfrac{1}{\sqrt[3]{1+x^3}}$	$\dfrac{1}{6}\log\left[\dfrac{u^2+u+1}{(u-1)^2}\right]-\dfrac{1}{\sqrt{3}}\arctan\dfrac{2u+1}{\sqrt{3}}, u=\sqrt[3]{1+1/x^3}$ (put $v=1/x^3$)

39 Problem Verify that $\int f(x)\,\mathrm{d}x = F(x) + C$.

$f(x)$	$F(x)$
$x^k\log x$	$\dfrac{x^{k+1}}{k+1}\left(\log x-\dfrac{1}{k+1}\right)$
$\log(1+x^2)$	$x\log(1+x^2)-2x+2\arctan x$
$\dfrac{x^2+a}{x^2+1}\arctan x$	$\dfrac{1}{2}\big((2x+(a-1)\arctan x)\arctan x-\log(1+x^2)\big)$
$\left(1-\dfrac{1}{x}\right)e^{1/x}$	$xe^{1/x}$
$\dfrac{x}{\cos^2 x}$	$x\tan x+\log\left\|\cos x\right\|$
$\dfrac{1}{\sqrt{e^x-1}}$	$2\arctan\sqrt{e^x-1}$
$\arctan\sqrt{\dfrac{x+1}{x+3}}$	$(x+2)\arctan\sqrt{\dfrac{x+1}{x+3}}-\log(\sqrt{x+1}+\sqrt{x+3})$
$\arcsin\sqrt{\dfrac{x}{x+1}}$	$x\arcsin\sqrt{\dfrac{x}{x+1}}-\sqrt{x}+\arctan\sqrt{x}$
$e^{\arcsin x}$	$\dfrac{x+\sqrt{1-x^2}}{2}e^{\arcsin x}$
$x(\cos^2 x)e^{-x}$	$\dfrac{e^{-x}}{50}\big((3-5x)\cos 2x+(4+10x)\sin 2x-25(x+1)\big)$
$(x^2+x+1)e^{2x}\cos x$	$(\dfrac{2x^2}{5}+\dfrac{4x}{25}+\dfrac{39}{125})e^{2x}\cos x+(\dfrac{x^2}{5}-\dfrac{3x}{25}+\dfrac{27}{125})e^{2x}\sin x$

40 Problem Let f be twice continuously differentiable in $[0; 2\pi]$ and convex. Prove that $\int_0^{2\pi} f(x)\cos x\,\mathrm{d}x \geq 0$.

41 Problem Find $\int \frac{\mathrm{d}x}{\sqrt{1+\sqrt{1+\sqrt{x}}}}$.

42 Problem Let $f : [a; b] \to \mathbb{R}$ a bounded integrable function for which $\forall x \in [a; b]$, $f(a + b - x) = f(x)$. Demonstrate that

$$\int_a^b x f(x)\,\mathrm{d}x = \frac{a+b}{2} \cdot \int_a^b f(x)\,\mathrm{d}x.$$

Use this to calculate $\int_0^{\pi} \frac{x \sin x\,\mathrm{d}x}{1+\cos^2 x}$ and $\int_0^{\pi} \frac{x\,\mathrm{d}x}{1+\sin x}$.

3 Improper Integrals

43 Problem (Putnam, 1995) For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^{\infty} \left(\sqrt{\sqrt{x+a}-\sqrt{x}} - \sqrt{\sqrt{x}-\sqrt{x-b}}\right) dx$$

converge?

44 Problem (Putnam, 1997) Evaluate

$$\int_0^{\infty} \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right)\,\mathrm{d}x.$$

45 Problem (Putnam, 2000) Show that the improper integral

$$\lim_{B\to+\infty} \int_0^B \sin(x)\sin(x^2)\,dx$$

converges.

46 Problem Prove that improper integral $I = \int_0^{\pi} \log \sin x\,\mathrm{d}x$ converges and that $I = -\pi \log 2$.

4 L'Hôpital's Rule

47 Problem Find $\lim_{x\to 0} \frac{\cos x - \cos 3x}{x^2}$.

48 Problem Find $\lim_{x\to 1} (1-x)\tan \frac{\pi x}{2}$.

49 Problem Find

$$\lim_{x\to\pi/6} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}$$

50 Problem Find

$$\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt[3]{1+x} - \sqrt{1-x}}$$

51 Problem Find
$$\lim_{x\to\pi/3}\frac{\sin(x-\pi/3)}{1-2\cos x}$$

52 Problem Find
$$\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}$$

53 Problem Find
$$\lim_{x\to 1^-}\frac{x\pi/2-\arcsin x}{1-x}$$

54 Problem Find
$$\lim_{x\to 0}(1+2x)^{1/\sin x}$$

55 Problem Find
$$\lim_{x\to\pi/2}\frac{\log(\sin x)}{1-\sin x}$$

56 Problem Find
$$\lim_{x\to 0}(\arcsin x - x)\csc^3 x$$

57 Problem Find
$$\lim_{x\to 0}(1-e^{-x})^{e^x}$$

5 Sequences and Series

58 Problem Test $\sum_{n=1}^{\infty}\frac{1}{n^{1+1/n}}$ for convergence by comparing it to a suitable p-series. Use the direct comparison test.

59 Problem Test $\sum_{n=2}^{\infty}\frac{1}{n^{1+1/\log n}}$ for convergence by comparing it to a suitable p-series. Use the direct comparison test.

60 Problem Test $\sum_{n=2}^{\infty}\frac{1}{n^{1+1/\log\log n}}$ for convergence by comparing it to a suitable p-series. Use the direct comparison test.

61 Problem Test $\sum_{n=1}^{\infty}\frac{3^n}{n^{2n}}$ using both direct comparison and the root test.

62 Problem (Putnam, 1996) Show that for every positive integer n,
$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\cdots(2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

63 Problem Give an example of two series $\sum_{n=1}^{+\infty}a_n$ (with $a_n\neq 0$ for all n) and $\sum_{n=1}^{+\infty}b_n$ such that the first series is conditionally convergent and the second divergent, yet $\lim_{n\to+\infty}\frac{a_n}{b_n}=1$.

64 Problem Prove that the sequence $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n$ converges. Its limit is the Euler-Mascheroni constant

$$\gamma \approx 0.57721566490153286060651209008240243104 2\ldots .$$

It is not known whether γ is rational or irrational.

65 Problem Prove that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\log n+\gamma+\mathrm{O}\left(\frac{1}{n}\right).$$

66 Problem Prove that $\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}=\log 2$.

67 Problem Prove that $\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{2n-1}=\frac{\pi}{4}$.

68 Problem Let $b(n)$ denote the number of ones in the binary expansion of the positive integer n, for example $b(3)=b(11_2)=2$. Prove that $\sum_{n=1}\frac{b(n)}{n(n+1)}=\log 4$.

69 Problem Find the sum $\sum_{k=1}^{+\infty}\frac{1}{k(k+1)(k+2)}$.

70 Problem Prove that $\sum_{k=1}^{+\infty}\frac{1}{(2k-1)(2k)(2k-1)}=\log 2-\frac{1}{2}$.

71 Problem The *Fibonacci Numbers* f_n are defined recursively as follows:

$$f_0=1,\quad f_1=1,\quad f_{n+2}=f_n+f_{n+1},\quad n\geq 0.$$

Prove that $\sum_{n=1}^{+\infty}\frac{f_n}{3^n}=\frac{3}{5}$.

72 Problem Prove that the series $\sum_{n=1}^{+\infty}\frac{1}{n^{1.8+\sin n}}$ diverges.

73 Problem Find the exact numerical value of the sum $\sum_{n=1}^{+\infty}n2^{1-n}$.

74 Problem Find the exact numerical value of the sum $\sum_{n=1}^{+\infty}n^2 2^{1-n}$.

75 Problem Find the exact numerical value of the sum $\sum_{n=0}^{+\infty}\arctan\frac{1}{n^2+n+1}$.

76 Problem Find the sum of the series $\sum_{n=2}^{+\infty}\frac{1}{4n^2-1}$.

77 Problem Find the exact numerical value of the infinite sum

$$\sum_{n=1}^{+\infty}\frac{\sqrt{(n-1)!}}{(1+\sqrt{1})(1+\sqrt{2})(1+\sqrt{3})\cdots(1+\sqrt{n})}.$$

78 Problem Determine whether the series $\sum_{n=1}^{+\infty}\frac{\cos n}{n}$ converges.

79 Problem Determine whether the series $\sum_{n=1}^{+\infty} \frac{\cos(\log n)}{n}$ converges.

80 Problem Determine whether the series $\sum_{n=1}^{+\infty} \frac{\cos(\log \log n)}{n}$ converges.

81 Problem Prove that

$$\int_0^1 \left(\frac{1}{x} - \left\llbracket \frac{1}{x} \right\rrbracket\right) \, dx = 1 - \gamma,$$

where γ is the Euler-Mascheroni constant from problem 64.

82 Problem Determine whether $\sum_{n \geq 2} a_n$ converges, when a_n is given as below.

1. $\left(1 + \frac{1}{n}\right)^n - e$.
2. $\cosh^\alpha n - \sinh^\alpha n$.
3. $2\log(n^3 + 1) - 3\log(n^2 + 1)$.
4. $\sqrt[n]{n+1} - \sqrt[n]{n}$.
5. $\arccos\left(\frac{n^3+1}{n^3+2}\right)$.
6. $\frac{a^n}{1 + a^{2n}}$.
7. $\frac{(-1)^n}{\sqrt{n^2+n}}$.
8. $\frac{(-1)^n}{\log n}$.
9. $\frac{1 + (-1)^n\sqrt{n}}{n}$.
10. $\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{n^n}$.
11. $\frac{1! + 2! + \cdots + n!}{(n+2)!}$.
12. $\frac{1! - 2! + \cdots \pm n!}{(n+1)!}$.
13. $\frac{(-1)^n}{\log n + \sin(2n\pi/3)}$.
14. $\sqrt{1 + \frac{(-1)^n}{\sqrt{n}}} - 1$.
15. $\frac{(-1)^n}{\sqrt{n} + (-1)^n}$.
16. $\frac{(-1)^{\llbracket \sqrt{n} \rrbracket}}{n}$.
17. $\frac{(\log n)^n}{n^{\log n}}$.
18. $\frac{1}{(\log n)^{\log n}}$.

83 Problem Find the radius of convergence of $\sum_{n=2}^{+\infty} a_n x^n$, for the following a_n.

1. if $a_n \to l \neq 0$ as $n \to +\infty$.
2. There is a strictly positive integer p such that $a_n = a_{n+p}$ for all $n \geq 1$
3. $a_n = \sum_{d|n} d^2$.
4. $a_n = \frac{n^n}{n!}$.
5. $a_{2n} = a^n$, $a_{2n+1} = b^n$, $0 < a < b$.
6. $a_{n^2} = n!$, $a_k = 0$ if $\sqrt{k} \notin \mathbb{N}$.
7. $a_n = (\log n)^{-\log n}$.
8. $a_n = e^{\sqrt{n}}$.
9. $a_n = \frac{1.4.7\ldots(3n-2)}{n!}$.
10. $a_n = \frac{1}{\sqrt{n}^{\sqrt{n}}}$.
11. $a_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)^{\log n}$.
12. $a_{n+2} = 2a_{n+1} + a_n$, $a_0 = a_1 = 1$.
13. $a_n = \binom{kn}{n}$.
14. $a_n = e^{(n+1)^2} - e^{(n-1)^2}$.
15. $a_n = \int_{t=0}^{1} (1+t^2)^n \, dt$.

16. $a_n = \sqrt[n]{n} - \sqrt[n+1]{n+1}$.

17. $a_n = \dfrac{\cos n\theta}{\sqrt{n} + (-1)^n}$.

6 Some Answers

1 Refer to figure 1. If a slab of the cone is at height y from the x-axis, the weight of the slab is

$$\pi \left(\frac{R(H-y)}{H}\right)^2 \gamma \, dy$$

and the work required to pour the sand is

$$\pi \int_0^H y \left(\frac{R(H-y)}{H}\right)^2 \gamma \, dy = \frac{\pi\gamma R^2 H^2}{12}.$$

2 This is plainly

$$C \int_V^{V/2} \frac{dv}{v} = C \log v \Big|_V^{V/2} = -C \log 2 = -pV \log 2.$$

3 Here we assume that a hole is at the top of the cylinder. Locate coordinate axes on a side of the cylinder so the centre of circular face lies on the origin. A "rectangular" slab of the cylinder has weight

$$2\sqrt{R^2 - y^2} L\gamma \, dy.$$

The distance that a slab will be pumped is $R - y$. Thus the desired work is

$$2\gamma \int_{-R}^{R} \sqrt{R^2 - y^2}(R - y) L \, dy = 2\gamma R L \int_{-R}^{R} \sqrt{R^2 - y^2} \, dy - 2\gamma L \int_{-R}^{R} y\sqrt{R^2 - y^2}(R - y) \, dy = \pi\gamma R^3 L,$$

as the first integral is the area of a semicircle of radius R and the second is the integral of an odd function over a domain symmetric around zero.

4 The horizontal slices cut are squares of area $x^2 = R^2 - y^2$. Thus the volume sought is $8\int_0^R (R^2 - y^2) \, dy = \frac{16}{3} R^3$.

6 Put $u = e^x$, etc.

$$\int e^{e^x + x} \, dx = \int e^x e^{e^x} \, dx = \int e^{e^x} \, de^x = e^{e^x} + C$$

7 Put $u = \log(\cos x)$, etc.

$$\int \tan x \log(\cos x) \, dx = \int (\log(\cos x)) \, d(-\log(\cos(x))) = -\frac{(\log(\cos x))^2}{2} + C$$

8 Put $u = \log \log x$, etc.

$$\int \frac{\log \log x}{x \log x} \, dx = \int \log \log x \, d\,(\log \log x) = \frac{(\log \log x)^2}{2} + C$$

9 Carry out the long division.

$$\int \frac{x^{18} - 1}{x^3 - 1} \, dx = \int (x^{15} + x^{12} + x^9 + x^6 + x^3 + 1) \, dx = \frac{x^{16}}{16} + \frac{x^{13}}{13} + \frac{x^{10}}{10} + \frac{x^7}{7} + \frac{x^4}{4} + x + C$$

10 After an algebraic trick, put $u = 1 + x^{-7}$, etc.

$$\int \frac{1}{x^8 + x} \, dx = \int \frac{x^{-8}}{1 + x^{-7}} \, dx = -\frac{1}{7} \int \frac{d(1 + x^{-7})}{1 + x^{-7}} = -\frac{1}{7} \log|1 + x^{-7}| + C$$

11 Put $u = 2^x + 1$

$$\int \frac{2^x 2^x}{2^x + 1} \, dx = \frac{1}{\log 2} \int \frac{2^x}{2^x + 1} \, d(2^x + 1) = \frac{1}{\log 2} \int \frac{u - 1}{u} \, du = \frac{1}{\log 2} (u - \log|u|) + C = \frac{1}{\log 2} (2^x + 1 - \log|2^x + 1|) + C$$

12 Put $u = x + 1$. Then $x^2 = (u-1)^2 = u^2 - 2u + 1$, and hence

$$\begin{aligned} \int \frac{x^2}{(x+1)^{10}}\,dx &= \int \frac{u^2-2u+1}{u^{10}}\,du \\ &= \int u^{-8} - 2u^{-9} + u^{-10}\,du \\ &= -\frac{u^{-7}}{7} + \frac{u^{-8}}{4} - \frac{u^{-9}}{9} + C \\ &= -\frac{(x+1)^{-7}}{7} + \frac{(x+1)^{-8}}{4} - \frac{(x+1)^{-9}}{9} + C \end{aligned}$$

13 Algebraic trick, and then $u = e^{-x} + 1$, etc.

$$\int \frac{1}{1+e^x}\,dx = \int \frac{e^{-x}}{e^{-x}+1}\,dx = -\int \frac{1}{e^{-x}+1}\,d(e^{-x}+1) = -\log|e^{-x}+1| + C$$

14

$$\int \frac{1}{1-\sin x}\,dx = \int \frac{1+\sin x}{1-\sin^2 x}\,dx = \int \frac{1+\sin x}{\cos^2 x}\,dx = \int \sec^2 x + \sec x \tan x\,dx = \tan x + \sec x + C$$

15

$$\begin{aligned} \int \sqrt{1+\sin 2x}\,dx &= \int \sqrt{\sin^2 x + 2\sin x\cos x + \cos^2 x}\,dx \\ &= \int \sqrt{(\sin x + \cos x)^2}\,dx \\ &= \int |\sin x + \cos x|\,dx \\ &= -\cos x + \sin x + C \quad \text{or} + \cos x - \sin x + C \end{aligned}$$

16 Put $u = x^2$, etc.

$$\int \frac{x}{\sqrt{1-(x^2)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin x^2 + C$$

18 We have

$$\begin{aligned} \int \sec^4 x\,dx &= \int \sec^2 x(\tan^2 x + 1)\,dx \\ &= \int \sec^2 x \tan^2 x\,dx + \int \sec^2 x\,dx \\ &= \int (\tan x)^2\,d(\tan x) + \int \sec^2 x\,dx \\ &= \frac{\tan^3 x}{3} + \tan x + C. \end{aligned}$$

19 We have

$$\begin{aligned} \int \sec^5 x\,dx &= \int \sec^3 x \sec^2 x\,dx \\ &= \int \sec^3 x\,d(\tan x) \\ &= \sec^3 x \tan x - \int \tan x\,d(\sec^3 x) \\ &= \sec^3 x \tan x - 3\int \tan^2 x \sec^2 x \sec x\,dx \\ &= \sec^3 x \tan x - 3\int (\sec^2 x - 1)\sec^3 x\,dx \\ &= \sec^3 x \tan x - 3\int \sec^5 x\,dx + 3\int \sec^3 x\,dx \end{aligned}$$

The above implies that

$$\begin{aligned} \int \sec^5 x\,dx &= \frac{\tan x \sec^3 x}{4} + \frac{3}{4}\int \sec^3 x\,dx \\ &= \frac{\tan x \sec^3 x}{4} + \frac{3\tan x \sec x}{8} + \frac{3}{8}\log|\sec x + \tan x| + C, \end{aligned}$$

upon recalling from class that

$$\int \sec^3 x\,dx = \frac{\tan x \sec x}{2} + \frac{1}{2}\log|\sec x + \tan x| + C$$

20 First put $t = x^{1/3}$, then $t^3 = x \implies 3t^2\,\mathrm{d}t = \mathrm{d}x$. Thus

$$\begin{aligned}\int e^{x^{1/3}}\,\mathrm{d}x &= \int 3t^2e^t\,\mathrm{d}t\\ &= 3t^2e^t - 6te^t - 6e^t + C\\ &= 3x^{2/3}e^{x^{1/3}} - 6x^{1/3}e^{x^{1/3}} - 6e^{x^{1/3}} + C,\end{aligned}$$

where the penultimate step results from tabular integration by parts.

21 We have

$$\begin{aligned}\int \log(x^2+1)\,\mathrm{d}x &= x\log(x^2+1) - \int x\,\mathrm{d}(\log(x^2+1))\\ &= x\log(x^2+1) - 2\int \frac{x^2}{x^2+1}\,\mathrm{d}x\\ &= x\log(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1}\,\mathrm{d}x\\ &= x\log(x^2+1) - 2\int \left(1 - \frac{1}{x^2+1}\right)\,\mathrm{d}x\\ &= x\log(x^2+1) - 2(x - \arctan x) + C\end{aligned}$$

22 This method parallels the one in class of "solving for the integral." Put

$$I = \int xe^x\cos x := (Ax+B)e^x\cos x + (Cx+D)e^x\sin x + K.$$

Differentiating both sides,

$$xe^x\cos x = Ae^x\cos x + (Ax+B)e^x\cos x - (Ax+B)e^x\sin x + Ce^x\sin x + (Cx+D)e^x\sin x + (Cx+D)e^x\cos x.$$

Equating coefficients,

$$\begin{array}{lcl} xe^x\cos x & : & 1 = A + C\\ xe^x\sin x & : & 0 = -A + C\\ e^x\cos x & : & 0 = A + B + D\\ e^x\sin x & : & 0 = -B + C + D\end{array}$$

From the first two equations $C = \frac{1}{2}, A = \frac{1}{2}$. Then the third and fourth equations become $-\frac{1}{2} = B + D; -\frac{1}{2} = -B + D$, whence $D = -\frac{1}{2}$, and $B = 0$. We conclude that

$$\int xe^x\cos x = \frac{x}{2}e^x\cos x + \left(\frac{x-1}{2}\right)e^x\sin x + K.$$

23 We will do this one two ways: first, by making the substitution

$$t = \log x \implies e^t = x \implies e^t\,\mathrm{d}t = \mathrm{d}x.$$

Observe also that $x^{2/3} = e^{2t/3}$. Then

$$\begin{aligned}\int x^{2/3}\log x\,\mathrm{d}x &= \int te^{2t/3}e^t\,\mathrm{d}t\\ &= \frac{3t}{5}e^{5t/3} - \frac{9}{25}e^{5t/3} + C\\ &= \frac{3(\log x)}{5}x^{5/3} - \frac{9}{25}x^{5/3} + C.\end{aligned}$$

Aliter: By directly integrating by parts,

$$\begin{aligned}\int x^{2/3}\log x\,\mathrm{d}x &= \int \log x\,\mathrm{d}\left(\frac{3x^{5/3}}{5}\right)\\ &= \frac{3x^{5/3}}{5}\log x - \frac{3}{5}\int x^{5/3}\,\mathrm{d}(\log x)\\ &= \frac{3(\log x)}{5}x^{5/3} - \frac{3}{5}\int x^{2/3}\,\mathrm{d}x\\ &= \frac{3(\log x)}{5}x^{5/3} - \frac{9}{25}x^{5/3} + C,\end{aligned}$$

as before.

24 This integral can be done multiple ways. For example, you may integrate by parts directly and then "solve" for the integral. Another way, which parallels a method shewn in class is the following. Start by putting

$$t = \log x \implies e^t = x \implies e^t\,dt = dx.$$

Then

$$\int \sin(\log x)\,dx = \int e^t \sin t\,dt,$$

an integral that we found in class. We will find it again, using a method similar of problem 22. Put

$$I = \int e^t \cos t\,dt := Ae^t \cos t + Be^t \sin t + K.$$

Differentiating both sides

$$e^t \cos t = Ae^t \cos t - Ae^t \sin t + Be^t \sin t + Be^t \cos t.$$

Equating coefficients,

$$\begin{array}{lcl} e^t \cos t & : & 1 = A + B \\ e^t \sin t & : & 0 = -A + B \end{array}$$

and so $A = B = \frac{1}{2}$. We have thus

$$\begin{aligned} \int \sin(\log x)\,dx &= \int e^t \sin t\,dt \\ &= \tfrac{1}{2}e^t \cos t + \tfrac{1}{2}e^t \sin t + K \\ &= \tfrac{1}{2}x \cos \log x + \tfrac{1}{2}x \sin \log x + K. \end{aligned}$$

25 Put $t = \log\log x \implies e^{e^t} = x \implies e^t e^{e^t}\,dt = dx$. Hence

$$\begin{aligned} \int \frac{\log\log x}{x}\,dx &= \int \frac{te^t e^{e^t}}{e^{e^t}}\,dt \\ &= te^t - e^t + C \\ &= (\log x)(\log\log x) - (\log x) + C, \end{aligned}$$

where the penultimate equality follows from a tabular integration by parts.

26 Simple algebra will yield the identity. We have

$$\begin{aligned} \int \sec x\,dx &= \int \frac{\cos x}{2(1+\sin x)}\,dx + \int \frac{\cos x}{2(1-\sin x)}\,dx \\ &= \tfrac{1}{2}\log|1+\sin x| - \tfrac{1}{2}\log|1-\sin x| + C \\ &= \frac{1}{2}\log\left|\frac{1+\sin x}{1-\sin x}\right| + C \end{aligned}.$$

27 We have

$$\begin{aligned} \int \csc x\,dx &= \int \frac{1}{\sin x}\,dx \\ &= \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx \\ &= \int \frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos^2\frac{x}{2}}\,dx \\ &= \int \frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2}}\,dx \\ &\overset{u=\tan\frac{x}{2}}{=} \int \frac{du}{u} \\ &= \log|\tan\tfrac{x}{2}| + C. \end{aligned}$$

Thus

$$\int \sec x\,dx = \int \csc(\frac{\pi}{2}+x)\,dx = \int \csc(\frac{\pi}{2}+x)\,d(\frac{\pi}{2}+x) = \log\left|\tan(\frac{\pi}{4}+\frac{x}{2})\right| + C.$$

28 Putting $t = \arcsin x$ we have

$$\sin t = x \implies \cos t\,dt = dx,$$

whence

$$\begin{aligned} \int (\arcsin x)^2\,dx &= \int t^2 \cos t\,dt \\ &= t^2 \sin t + 2t\cos t - 2\sin t + C \\ &= (\arcsin x)^2 x + 2(\arcsin x)\cos(\arcsin x) - 2x + C \\ &= (\arcsin x)^2 x + 2(\arcsin x)\sqrt{1-x^2} - 2x + C \end{aligned}$$

29 We have

$$\begin{aligned}\int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}} &= \int \frac{(\sqrt{x+1}-\sqrt{x-1})\,dx}{2} \\ &= \frac{1}{3}(x+1)^{3/2} - \frac{1}{3}(x-1)^{3/2} + C\end{aligned}.$$

30 We have

$$\begin{aligned}\int x \arctan x\, dx &= \int \arctan x\, d\left(\frac{x^2}{2}\right) \\ &= \frac{x^2}{2}\arctan x - \int \frac{x^2}{2}\, d(\arctan x) \\ &= \frac{x^2}{2}\arctan x - \int \frac{1}{2}\frac{x^2}{1+x^2}\, dx \\ &= \frac{x^2}{2}\arctan x - \int \frac{1}{2}\frac{x^2+1-1}{1+x^2}\, dx \\ &= \frac{x^2}{2}\arctan x - \frac{x}{2} + \tfrac{1}{2}\arctan x + C\end{aligned}.$$

31 Put $u = \sqrt{\tan x}$ and so $u^2 = \tan x$, $2u\, du = \sec^2 x\, dx = (\tan^2 x + 1)\, dx = (u^4+1)\, dx$. Hence the integral becomes

$$\int \sqrt{\tan x}\, dx = 2\int \frac{u^2}{u^4+1}\, du.$$

To decompose the above fraction into partial fractions observe (Sophie Germain's trick) that $u^4+1 = u^4+2u^2+1-2u^2 = (u^2+u\sqrt{2}+1)(u^2-u\sqrt{2}+1)$ and hence

$$\begin{aligned}\int \sqrt{\tan x}\, dx &= 2\int \frac{u^2}{u^4+1}\, du \\ &= -\frac{\sqrt{2}}{2}\int \frac{u}{u^2+u\sqrt{2}+1}\, du + \frac{\sqrt{2}}{2}\int \frac{u}{u^2-u\sqrt{2}+1}\, du \\ &= -\frac{\sqrt{2}}{4}\log(u^2+u\sqrt{2}+1) + \frac{\sqrt{2}}{4}\log(u^2-u\sqrt{2}+1) + \frac{\sqrt{2}}{2}\arctan(\sqrt{2}u+1) - \frac{\sqrt{2}}{2}\arctan(-\sqrt{2}u+1) + C \\ &= -\frac{\sqrt{2}}{4}\log(\tan x + \sqrt{2\tan x}+1) + \frac{\sqrt{2}}{4}\log(\tan x - \sqrt{2\tan x}+1) \\ &\quad + \frac{\sqrt{2}}{2}\arctan(\sqrt{2\tan x}+1) - \frac{\sqrt{2}}{2}\arctan(-\sqrt{2\tan x}+1) + C\end{aligned}$$

32 Put

$$\frac{2x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \implies 2x+1 = Ax(x-1) + B(x-1) + Cx^2.$$

Letting $x = 1$ we get $3 = C$. Letting $x = 0$ we get $1 = -B \implies B = -1$. To get A observe that equating the coefficients of x^2 on both sides we get $0 = A + C$, whence $A = -3$. Thus

$$\begin{aligned}\int \frac{2x+1}{x^2(x-1)}\, dx &= -3\int \frac{1}{x}\, dx - \int \frac{1}{x^2}\, dx + 3\int \frac{1}{x-1}\, dx \\ &= -3\log|x| + \frac{1}{x} + 3\log|x-1| + C \\ &= 3\log\left|\tfrac{x-1}{x}\right| + \frac{1}{x} + C.\end{aligned}$$

33 Integrating by parts,

$$\begin{aligned}\int \log(x+\sqrt{x})\, dx &= x\log(x+\sqrt{x}) - \int x\, d\log(x+\sqrt{x}) \\ &= x\log(x+\sqrt{x}) - \int \frac{x(1+\frac{1}{2\sqrt{x}})}{x+\sqrt{x}}\, dx \\ &= x\log(x+\sqrt{x}) - \int \left(1 - \frac{1}{2}\cdot\frac{\sqrt{x}}{x+\sqrt{x}}\right)\, dx \\ &= x\log(x+\sqrt{x}) - x + \frac{1}{2}\int \frac{\sqrt{x}}{x+\sqrt{x}}\, dx \\ &\overset{u=\sqrt{x}}{=} x\log(x+\sqrt{x}) - x + \int \frac{u^2}{u^2+u}\, du \\ &\overset{u=\sqrt{x}}{=} x\log(x+\sqrt{x}) - x + \int 1 - \frac{1}{u+1}\, du \\ &= x\log(x+\sqrt{x}) - x + u - \log(u+1) + C \\ &= x\log(x+\sqrt{x}) - x + \sqrt{x} - \log(\sqrt{x}+1) + C\end{aligned}$$

34 We use Sophie Germain's trick to factor

$$x^4+1=x^4+2x^2+1-2x^2=(x^2+1)^2-2x^2=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1),$$

and seek the partial fraction decomposition

$$\frac{1}{x^4+1}=\frac{Ax+B}{x^2-\sqrt{2}x+1}+\frac{Cx+D}{x^2+\sqrt{2}x+1}\implies 1=(Ax+B)(x^2+\sqrt{2}x+1)+(Cx+D)(x^2-\sqrt{2}x+1).$$

Equating coefficients

$$\begin{array}{rcl} x^3 & : & 0=A+C \\ x^2 & : & 0=B+D+\sqrt{2}(A-C) \\ x & : & 0=A+C+\sqrt{2}(B-D) \\ x^0 & : & 1=B+D \end{array}$$

From the first and third equation it follows that $A=-C$ and that $B=D$. From the fourth equation $B=D=\frac{1}{2}$ and from the second equation $A=-\dfrac{1}{2\sqrt{2}}=-C$. Hence we must integrate

$$\begin{aligned}\int\frac{1}{x^4+1}\,\mathrm{d}x &= \int\frac{\sqrt{2}x+2}{4(x^2+\sqrt{2}x+1)}\,\mathrm{d}x-\int\frac{\sqrt{2}x-2}{4(x^2-\sqrt{2}x+1)}\,\mathrm{d}x\\ &= \frac{\sqrt{2}}{8}\int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}\,\mathrm{d}x+\frac{1}{4}\int\frac{1}{x^2+\sqrt{2}x+1}\,\mathrm{d}x-\frac{\sqrt{2}}{8}\int\frac{2x+\sqrt{2}}{x^2-\sqrt{2}x+1}\,\mathrm{d}x+\frac{1}{4}\int\frac{1}{x^2-\sqrt{2}x+1}\,\mathrm{d}x\\ &= \frac{\sqrt{2}}{8}\log(x^2+x\sqrt{2}+1)-\frac{\sqrt{2}}{8}\log(x^2-x\sqrt{2}+1)+\frac{1}{2}\int\frac{\mathrm{d}x}{(x\sqrt{2}+1)^2+1}+\frac{1}{2}\int\frac{\mathrm{d}x}{(-x\sqrt{2}+1)^2+1}\\ &= \frac{\sqrt{2}}{8}\log(x^2+x\sqrt{2}+1)-\frac{\sqrt{2}}{8}\log(x^2-x\sqrt{2}+1)+\frac{\sqrt{2}}{4}\arctan(x\sqrt{2}+1)-\frac{\sqrt{2}}{4}\arctan(-x\sqrt{2}+1)+C\end{aligned}$$

35 We begin by observing that

$$\frac{1}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}\implies 1=A(x^2-x+1)+(Bx+C)(x+1).$$

Letting $x=-1$ we obtain $1=3A\implies A=\frac{1}{3}$. Letting $x=0$ we obtain $1=A+C\implies C=1-A=\dfrac{2}{3}$. Finally, we must have $A+B=0$, since the coefficient of x^2 must be zero. thus $B=-\frac{1}{3}$. We must then integrate

$$\begin{aligned}\int\frac{\mathrm{d}x}{3(x+1)}-\int\frac{x-2}{3(x^2-x+1)}\,\mathrm{d}x &= \tfrac{1}{3}\log|x+1|-\int\frac{x-\frac{1}{2}}{3(x-\frac{1}{2})^2+\frac{3}{4}}+\frac{1}{2}\int\frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\\ &= \tfrac{1}{3}\log|x+1|-\tfrac{1}{6}\log|(x-\tfrac{1}{2})^2+\tfrac{3}{4}|+\frac{2}{3}\int\frac{1}{\frac{4}{3}(x-\frac{1}{2})^2+1}\\ &= \tfrac{1}{3}\log|x+1|-\tfrac{1}{6}\log|(x-\tfrac{1}{2})^2+\tfrac{3}{4}|+\frac{2}{3}\cdot\frac{\sqrt{3}}{2}\arctan\tfrac{2}{\sqrt{3}}\left(x-\tfrac{1}{2}\right)\\ &= \tfrac{1}{3}\log|x+1|-\tfrac{1}{6}\log|x^2-x+1|+\frac{\sqrt{3}}{3}\arctan\tfrac{2}{\sqrt{3}}(x-\tfrac{1}{2})\end{aligned}$$

40 Integrate by parts twice to obtain $\displaystyle\int_0^{2\pi}f(x)\cos x\,\mathrm{d}x=\int_0^{2\pi}f''(x)(1-\cos x)\,\mathrm{d}x$. Since f is convex, $f''\geq 0$ and the assertion follows.

41 Put $u=\sqrt{1+\sqrt{1+\sqrt{x}}}$, then $x=(u^2-2)^2u^4$ and $\mathrm{d}x=(4u^3(u^2-2)^2+4u^5(u^2-2))\,\mathrm{d}u$. Hence

$$\begin{aligned}\int\frac{\mathrm{d}x}{\sqrt{1+\sqrt{1+\sqrt{x}}}} &= \int\frac{(4u^3(u^2-2)^2+4u^5(u^2-2))\,\mathrm{d}u}{u}\\ &= 4\int u^2(u^2-2)^2\,\mathrm{d}u+4\int u^4(u^2-2)\,\mathrm{d}u\\ &= 4\int(u^6-4u^4+4u^2)\,\mathrm{d}u+4\int(u^6-2u^4)\,\mathrm{d}u\\ &= 8\int u^6\,\mathrm{d}u-24\int u^4\,\mathrm{d}u+16\int u^2\,\mathrm{d}u\\ &= \frac{8}{7}u^7-\frac{24}{5}u^5+\frac{16}{3}u^3+C\\ &= \frac{8}{7}(\sqrt{1+\sqrt{1+\sqrt{x}}})^7-\frac{24}{5}(\sqrt{1+\sqrt{1+\sqrt{x}}})^5+\frac{16}{3}(\sqrt{1+\sqrt{1+\sqrt{x}}})^3+C.\end{aligned}$$

Note: Maple X doesn't quite know how to do this problem!

43 The integral converges iff $a = b$. Use the fact that $(1+x)^{1/2} = 1 + x/2 + O(x^2)$ for $|x| < 1$.
Now,

$$\begin{aligned}\sqrt{x+a} - \sqrt{x} &= x^{1/2}(\sqrt{1+a/x} - 1)\\ &= x^{1/2}(1 + a/2x + O(x^{-2})),\end{aligned}$$

hence

$$\sqrt{\sqrt{x+a} - \sqrt{x}} = x^{1/4}(1 + a/4x + O(x^{-2})$$

and similarly

$$\sqrt{\sqrt{x} - \sqrt{x-b}} = x^{1/4}(1 + b/4x + O(x^{-2}).$$

Hence the integral we're looking at is

$$\int_b^\infty x^{1/4}((a-b)/4x + O(x^{-2})\, dx.$$

The term $x^{1/4}O(x^{-2})$ is bounded by a constant times $x^{-7/4}$, whose integral converges. Thus we only have to decide whether $x^{-3/4}(a-b)/4$ converges. But $x^{-3/4}$ has divergent integral, so we get convergence if and only if $a = b$ (in which case the integral telescopes anyway).

44 Note that the series on the left is simply $x\exp(-x^2/2)$. By integration by parts,

$$\int_0^\infty x^{2n+1}e^{-x^2/2}dx = 2n\int_0^\infty x^{2n-1}e^{-x^2/2}dx$$

and so by induction,

$$\int_0^\infty x^{2n+1}e^{-x^2/2}dx = 2 \times 4 \times \cdots \times 2n.$$

Thus the desired integral is simply

$$\sum_{n=0}^\infty \frac{1}{2^n n!} = \sqrt{e}.$$

45 We use integration by parts:

$$\begin{aligned}\int_0^B \sin x \sin x^2\, dx &= \int_0^B \frac{\sin x}{2x}\sin x^2(2x\, dx)\\ &= -\frac{\sin x}{2x}\cos x^2\Big|_0^B\\ &+ \int_0^B \left(\frac{\cos x}{2x} - \frac{\sin x}{2x^2}\right)\cos x^2\, dx.\end{aligned}$$

Now $\frac{\sin x}{2x}\cos x^2$ tends to 0 as $B \to +\infty$, and the integral of $\frac{\sin x}{2x^2}\cos x^2$ converges absolutely by comparison with $1/x^2$. Thus it suffices to note that

$$\begin{aligned}\int_0^B \frac{\cos x}{2x}\cos x^2\, dx &= \frac{\cos x}{4x^2}\cos x^2(2x\, dx)\\ &= \frac{\cos x}{4x^2}\sin x^2\Big|_0^B\\ &- \int_0^B \frac{2x\cos x - \sin x}{4x^3}\sin x^2\, dx,\end{aligned}$$

and that the final integral converges absolutely by comparison to $1/x^3$.

An alternate approach is to first rewrite $\sin x \sin x^2$ as $\frac{1}{2}(\cos(x^2 - x) - \cos(x^2 + x))$. Then

$$\begin{aligned}\int_0^B \cos(x^2+x)\, dx &= -\frac{2x+1}{\sin(x^2+x)}\Big|_0^B\\ &- \int_0^B \frac{2\sin(x^2+x)}{(2x+1)^2}\, dx\end{aligned}$$

converges absolutely, and $\int_0^B \cos(x^2 - x)$ can be treated similarly.

46 From $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ we get

$$\begin{aligned} I &= \int_0^\pi \log 2\,dx + \int_0^\pi \log\sin\frac{x}{2}\,dx + \int_0^\pi \log\cos\frac{x}{2}\,dx \\ &= \pi\log 2 + 2\int_0^{\pi/2}\log\sin y\,dy + 2\int_0^{\pi/2}\log\cos y\,dy. \end{aligned}$$

Setting $y = \frac{\pi}{2} - u$ and using $\sin(\pi - u) = \sin u = \cos\left(\frac{\pi}{2} - u\right)$ we see that

$$\int_0^{\pi/2}\log\sin y\,dy = \int_0^{\pi/2}\log\cos y\,dy \implies 2\int_0^{\pi/2}\log\sin y\,dy = \int_0^{\pi/2}(\log\sin u + \log\sin(\pi - x))\,du = \int_0^\pi \log\sin u\,du = I,$$

from where

$$I = \pi\log 2 + 2I \implies I = -\pi\log 2.$$

47 We have

$$\begin{aligned} \lim_{x\to 0}\frac{\cos x - \cos 3x}{x^2} &\overset{0/0}{=} \lim_{x\to 0}\frac{-\sin x + 3\sin 3x}{2x} \\ &\overset{0/0}{=} \lim_{x\to 0}\frac{-\cos x + 9\cos 3x}{2} \\ &= 4. \end{aligned}$$

48 We have

$$\begin{aligned} \lim_{x\to 1}(1-x)\tan\frac{\pi x}{2} &\overset{0\infty}{=} \lim_{x\to 1}(1-x)\frac{(1-x)\sin\frac{\pi x}{2}}{\cos\frac{\pi x}{2}} \\ &\overset{0/0}{=} \lim_{x\to 1}\frac{-\sin\frac{\pi x}{2} + (1-x)\frac{\pi}{2}\cos\frac{\pi x}{2}}{-\frac{\pi}{2}\sin\frac{\pi x}{2}} \\ &= \frac{2}{\pi}. \end{aligned}$$

49 We have

$$\begin{aligned} \lim_{x\to\pi/6}\frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1} &\overset{0/0}{=} \lim_{x\to\pi/6}\frac{4\sin x\cos x + \cos x}{4\sin x\cos x - 3\cos x} \\ &= -3. \end{aligned}$$

50 We have

$$\begin{aligned} \lim_{x\to 0}\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt[3]{1+x} - \sqrt{1-x}} &\overset{0/0}{=} \lim_{x\to 0}\frac{\dfrac{1}{2\sqrt{1+x}} + \dfrac{1}{2\sqrt{1-x}}}{\dfrac{1}{3(1+x)^{2/3}} + \dfrac{1}{2\sqrt{1-x}}} \\ &= \frac{6}{5} \end{aligned}$$

51 We have

$$\begin{aligned} \lim_{x\to\pi/3}\frac{\sin(x-\pi/3)}{1-2\cos x} &\overset{0/0}{=} \lim_{x\to\pi/3}\frac{\cos(x-\pi/3)}{2\sin x} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

52 We have

$$\begin{aligned} \lim_{x\to 0}\frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3} &\overset{0/0}{=} \lim_{x\to 0}\frac{\tan x - \sin x}{x^3(\sqrt{1+\tan x} + \sqrt{1+\sin x})} \\ &= \lim_{x\to 0}\frac{\tan x - \sin x}{2x^3} \\ &\overset{0/0}{=} \lim_{x\to 0}\frac{\sec^2 x - \cos x}{6x^2} \\ &\overset{0/0}{=} \lim_{x\to 0}\frac{2\sec^2 x\tan x + \sin x}{12x} \\ &= \lim_{x\to 0}\left(\sec^3 x\frac{\sin x}{6x} + \frac{\sin x}{12x}\right) \\ &= \frac{1}{6} + \frac{1}{12} \\ &= \frac{1}{4}. \end{aligned}$$

53 We have

$$\begin{aligned} \lim_{x\to 1^-}\frac{x\pi/2 - \arcsin x}{1-x} &\overset{0/0}{=} \lim_{x\to 1^-}\frac{\frac{\pi}{2} - \frac{1}{1-x^2}}{-1} \\ &= +\infty \end{aligned}$$

54 We have

$$\begin{aligned}\lim_{x\to 0}\ (1+2x)^{1/\sin x} &\overset{1^\infty}{=} \exp\left(\lim_{x\to 0}\frac{\log(1+2x)}{\sin x}\right)\\ &\overset{0/0}{=} \exp\left(\lim_{x\to 0}\frac{\frac{2}{1+2x}}{\cos x}\right)\\ &= e^2.\end{aligned}$$

55 We have

$$\begin{aligned}\lim_{x\to\pi/2}\ \frac{\log(\sin x)}{1-\sin x} &\overset{0/0}{=} \lim_{x\to\pi/2}\ \frac{\frac{\cos x}{\sin x}}{-\cos x}\\ &= \lim_{x\to\pi/2}\ \frac{-1}{\sin x}\\ &= -1.\end{aligned}$$

56 We have

$$\begin{aligned}\lim_{x\to 0}\ (\arcsin x - x)\csc^3 x &\overset{0\infty}{=} \lim_{x\to 0}\ \frac{\arcsin x - x}{\sin^3 x}\\ &\overset{0/0}{=} \lim_{x\to 0}\ \frac{\frac{1}{\sqrt{1-x^2}}-1}{3\sin^2 x\cos x}\\ &\overset{0/0}{=} \lim_{x\to 0}\ \frac{x(1-x^2)^{-3/2}}{6\sin x\cos^2 x - 3\sin^3 x}\\ &= \lim_{x\to 0}\ \frac{(1-x^2)^{-3/2}}{6\cos^2 x - 3\sin^2 x}\cdot\frac{x}{\sin x}\\ &= \frac{1}{6}\cdot 1\\ &= \frac{1}{6}.\end{aligned}$$

57 This last one was a joke.... Can you see why the answer is 0?

58 By induction $n < 2^n \implies n^{1/n} < 2$ and so $n^{1+1/n} < 2n \implies \frac{1}{2n} < \frac{1}{n^{1+/1n}}$. So the series diverges by direct comparison to $\sum_{n=1}^{\infty}\frac{1}{2n}$.

59 $n = e^{\log n} \implies n^{\frac{1}{\log n}} = e$ and so $n^{1+1/\log n} = en, \qquad n > 1.$. So the series diverges by direct comparison to $\sum_{n=2}^{\infty}\frac{1}{en}$.

60 This one is really tricky! $e^x > \frac{x^2}{2}$ for $x > 0$ as can be seen by considering the monotonicity of $f(x) = e^x - \frac{x^2}{2}$ or considering the Maclaurin expansion of e^x. Now,

$$n^{1/\log\log n} = e^{\log n^{1/\log\log n}} = e^{\frac{\log n}{\log\log n}} > \frac{(\log n)^2}{2(\log\log n)^2}.$$

This gives

$$\frac{2(\log\log n)^2}{n(\log n)^2} > \frac{1}{n^{1+\frac{1}{\log\log n}}}.$$

Now,

$$\sum_{n=2}^{+\infty}\frac{2(\log\log n)^2}{n(\log n)^2}$$

can be shewn to converge by comparing to a series in the De Morgan logarithmic scale.

61 By the root test

$$a_n^{1/n} = \left(\frac{3^n}{n^{2n}}\right)^{1/n} = \frac{3}{n} \to 0 < 1,$$

and the series converges. By direct comparison, for $n \geq 3$ we have

$$\frac{3^n}{n^{2n}} = \frac{3^n}{n^n}\cdot\frac{1}{n^n} \leq 1\frac{1}{n^n} \leq \frac{1}{n^3},$$

and the series converges by direct comparison to $\sum_{n=1}^{\infty}\frac{1}{n^3}$.

62 By estimating the area under the graph of $\log x$ using upper and lower rectangles of width 2, we get

$$\int_1^{2n-1} \log x\, dx \leq 2(\log(3) + \cdots + \log(2n-1)) \leq \int_3^{2n+1} \log x\, dx.$$

Since $\int \log x\, dx = x\log x - x + C$, we have, upon exponentiating and taking square roots,

$$\begin{aligned}\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} &< (2n-1)^{\frac{2n-1}{2}} e^{-n+1} \\ &\leq 1\cdot 3\cdots(2n-1) \\ &\leq (2n+1)^{\frac{2n+1}{2}} \frac{e^{-n+1}}{3^{3/2}} \\ &< \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}},\end{aligned}$$

using the fact that $1 < e < 3$.

63 Take $a_n = \dfrac{(-1)^n}{n}$ and $b_n = a_n + (1-(-1)^{n+1})\dfrac{1}{n\log(1+n)}$, say.

66 We have, using Abel's Theorem

$$\begin{aligned}\log 2 &= \int_0^1 \frac{dx}{1+x} \\ &= \int_0^1 \left(1 - x + x^2 - x^3 + x^4 - \cdots\right) dx \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots,\end{aligned}$$

as wanted.

67 We have, using Abel's Theorem

$$\begin{aligned}\frac{\pi}{4} &= \int_0^1 \frac{dx}{1+x^2} \\ &= \int_0^1 \left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right) dx \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots,\end{aligned}$$

as wanted. Note: this series was known to Leibniz, for which he exclaimed that *Deus numero impare gaudet*, "God delights in odd numbers," quoting Virgil.

69 The sum telescopes: $\dfrac{2}{k(k+1)(k+2)} = \dfrac{1}{k(k+1)} - \dfrac{1}{(k+1)(k+2)}$.

72 For $k \in \mathbb{Z}$, the interval $I_k = \left[(2k+\frac{4}{3})\pi\, ;(2k+\frac{5}{3})\pi\right]$ has length $\dfrac{\pi}{3} > 1$ and $x \in I_k \implies \sin x \leq -\dfrac{\sqrt{3}}{2}$. The gap between I_k and I_{k+1} is $< \dfrac{5\pi}{3} < 6$. Hence, among any seven consecutive integers, at least one must fall into I_k and for this value of n we must have $1.8 + \sin n < 1 - \dfrac{\sqrt{3}}{2} < 1$. This means that

$$\sum_{n=1}^{+\infty} \frac{1}{n^{1.8+\sin n}} = \sum_{m=0}^{+\infty} \sum_{n=7m+1}^{n=7m+7} \frac{1}{n^{1.8+\sin n}} \geq \sum_{m=0}^{+\infty} \frac{1}{7m+7},$$

and since the rightmost series diverges, the original series diverges by the direct comparison test.

73 For $|x| < 1$,
$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$
Differentiating,
$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2} \implies \sum_{n=1}^{+\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$
Letting $x = \frac{1}{2}$,
$$\sum_{n=1}^{+\infty} \frac{n}{2^{n-1}} = 4.$$

74 For $|x| < 1$,
$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$
Differentiating,
$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}.$$
Multiplying by x,
$$x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2}.$$
Differentiating again,
$$1 + 4x + 9x^2 + \cdots = \frac{1+x}{(1-x)^3} \implies \sum_{n=1}^{+\infty} \frac{n^2{}^{n-1}}{x} = \frac{1+x}{(1-x)^3}$$
Letting $x = \frac{1}{2}$,
$$\sum_{n=1}^{+\infty} \frac{n^2}{2^{n-1}} = 12.$$

75 Since $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$, observe that $\arctan \dfrac{1}{n^2+n+1} = \arctan(n+1) - \arctan n$. Hence the series telescopes to $\lim_{n\to+\infty} \arctan(n+1) - \arctan 1 = \dfrac{\pi}{4}$.

76 Observe that
$$\frac{1}{4n^2-1} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}.$$
Hence
$$\sum_{n=2}^{+\infty} \frac{1}{4n^2-1} = \left(\frac{1}{2(1)} - \frac{1}{2(3)}\right) + \left(\frac{1}{2(3)} - \frac{1}{2(5)}\right) + \left(\frac{1}{2(5)} - \frac{1}{2(7)}\right) + \cdots = \frac{1}{2(1)} = \frac{1}{2}.$$

82
1. $a_n \sim -\dfrac{e}{2n} \implies$ diverges.
2. $a_n \sim \dfrac{\alpha}{2^{\alpha-1}} e^{n(\alpha-2)} \implies$ converges iff $\alpha < 2$.
3. $a_n \sim -\dfrac{3}{n^2} \implies$ converges.
4. $a_n \sim \dfrac{1}{n^2} \implies$ converges.
5. $a_n \sim \sqrt{\dfrac{2}{n^3}} \implies$ converges.
6. converges iff $|a| \neq 1$.
7. Alternating series $\implies$ converges.
8. Alternating series $\implies$ converges.
9. Harmonic series plus alternating series $\implies$ diverges.
10. Converges.
11. $a_n \leq \dfrac{(n-1)(n-1)! + n!}{(n+2)!} \leq \dfrac{2}{(n+1)(n+2)} \implies$ converges.
12. $a_n = \dfrac{(-1)^{n-1}}{n+1} + O\left(\dfrac{1}{n^2}\right) \implies$ converges.
13. Decompose into three alternating series $\implies$ converges.
14. $a_n = \dfrac{(-1)^n}{2\sqrt{n}} - \dfrac{1}{8n} + O(n^{-3/2}) \implies$ diverges.
15. Rearranging terms $\implies$ diverges.
16. Converges.
17. $a_n \not\longrightarrow 0 \implies$ diverges.
18. $a_n = \dfrac{1}{n^{\log\log n}} \implies$ converges.

83

$\boldsymbol{R = 1}$.

2. $\boldsymbol{R = 1}$.
3. $\boldsymbol{R = 1}$.
4. $\boldsymbol{R = \frac{1}{e}}$
5. $\boldsymbol{R = \frac{1}{\sqrt{b}}}$.
6. $\boldsymbol{R = 1}$.
7. $\boldsymbol{R = 1}$.
8. $\boldsymbol{R = 1}$.
9. $\boldsymbol{R = \frac{1}{3}}$.
10. $\boldsymbol{R = 1}$.
11. $\boldsymbol{R = 1}$.
12. $\boldsymbol{R = \sqrt{2} - 1}$.
13. $\boldsymbol{R = \frac{(k-1)^{k-1}}{k^k}}$.
14. $\boldsymbol{R = 0}$.
15. $\boldsymbol{R = \frac{1}{2}}$, $\boldsymbol{2t \leq 1 + t^2 \leq 2}$.
16. $\boldsymbol{R = 1}$, $\boldsymbol{a_n \sim \frac{\log n}{n^2}}$.
17. $\boldsymbol{R = 1}$.