1 Puzzles

1 Problem A frog is in a 10 ft well. At the end of each day, it has leaped 5 ft up and 4 down. After how many days, if at all, will the frog escape the well?

2 Problem Dale should have divided a number by 4, but instead he subtracted 4. He got the answer 48. What should his answer have been?

3 Problem When a number is multiplied by 3 and then increased by 16, the result obtained is 37. What is the original number?

4 Problem You and I play the following game. I tell you to write down three 2-digit integers between 10 and 89. Then I write down three 2-digit integers of my choice. The answer comes to 297, no matter which three integers you choose (my choice always depends on yours). For example, suppose you choose $12, 23, 48$. Then I choose $87, 76, 51$. You add

$$12 + 23 + 48 + 87 + 76 + 51 = 297.$$

Again, suppose you chose $33, 56, 89$. I then choose 66, 43, 10. Observe that

$$33 + 56 + 89 + 66 + 43 + 10 = 297.$$

Explain how I choose my numbers so that the answer always comes up to be 297 (!!!).

5 Problem What is the sum

$$1 + 2 + 3 + \cdots + 99 + 100$$

of all the positive integers from 1 to 100?

6 Problem Can we find five even integers whose sum is 25?

7 Problem Iblis entered an elevator in a tall building. She went up 4 floors, down 6 floors, up 8 floors and down 10 floors. She then found herself on the 23rd floor. In what floor did she enter the elevator?

8 Problem Bilbo and Frodo have just consumed a plateful of cherries. Each repeats the rhyme 'Tinker, tailor, soldier, sailor, rich man, poor man, beggar man, thief' over and over again as he runs through his own heap of cherry stones. Bilbo finishes on 'sailor', whereas Frodo finishes on 'poor man'. What would they have finished on if they had run through both heaps together?

9 Problem Four comrades are racing side by side down a dusty staircase. Frodo goes down two steps at a time, Gimli three, Legolas four, and Aragorn five. If the only steps with all four's footprints are at the top and the bottom, how many steps have just one footprint?

2 Arithmetic

10 Problem What is the value of $((1 \times 2 \div (3 \times 4) - 5) \times 6 - 7) \div (8 \times 9)$?

11 Problem Write as a single fraction. If the result is an "improper fraction," leave it in that form.

$$\frac{1}{3} + \frac{2}{5} - \frac{3}{7}$$

12 Problem Write as a single fraction. If the result is an "improper fraction," leave it in that form.

$$\frac{1}{3} \cdot \frac{2}{5} \div \frac{3}{7}$$

13 Problem Convert into a fraction and express in lowest terms: 0.204

14 Problem Convert into a decimal: $\frac{9}{11}$.

15 Problem Calculate: $\frac{(10)^3 + (-5)^3 + (1)^3 - 3(10)(-5)(1)}{(10) + (-5) + (1)}$.

16 Problem Calculate: $1020.4016 \div 637.751$.

3 Symbolic Expression

17 Problem Describe the following sequence of operations by means of algebraic symbols.

1. Think of a number.
2. Multiply your number by 3.
3. To the above result, add 45.
4. Divide the above result by 3.
5. Subtract the number you originally thought of.
6. The answer is always 15 (!!!).

18 Problem Identify the law of formation and conjecture a general formula:

$$1 = 1,$$
$$1 + 2 = \frac{(2)(3)}{2},$$
$$1 + 2 + 3 = \frac{(3)(4)}{2},$$
$$1 + 2 + 3 + 4 = \frac{(4)(5)}{2},$$
$$1 + 2 + 3 + 4 + 5 = \frac{(5)(6)}{2}.$$

19 Problem Identify the law of formation and conjecture a general formula:

$$1^2 = \frac{(1)(2)(3)}{6},$$
$$1^2 + 2^2 = \frac{(2)(3)(5)}{6},$$
$$1^2 + 2^2 + 3^2 = \frac{(3)(4)(7)}{6},$$

$$1^2 + 2^2 + 3^2 + 4^2 = \frac{(3)(4)(9)}{6},$$
$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \frac{(5)(6)(11)}{6}.$$

20 Problem Complete the following table with the appropriate algebraic translation. Use the letter k to refer to the unknown number.

A number increased by 18	
One third of a number	
Three quarters of a number	
The square of a number	
The cube of a number	
Ninety nine percent of the number	
The product of a number and 89.	
The product of a number and t.	
The square of the sum of a number and 7	
The sum of the squares of a number and 7	

21 Problem Assume t is an even integer. Complete the following table.

The odd integer preceding t	
The even integer preceding t	
The even integer succeeding t	
The odd integer succeeding t	

22 Problem Billy's Father is three times as old as Billy. Billy's grandmother is 25 years older than Billy's father. If Billy's current age is y, complete the following table.

	Age 5 years ago	Current age	Age in 10 years
Billy		y	
Billy's Father		$3y$	
Billy's Grandmother		$3y + 25$	

23 Problem Beer costs B riyals. A loaf read costs half the price of beer. A glass of wine costs 20 riyals more than a loaf of bread. A pound of camel meat costs twice as much as beer. A pound of dates costs 3 riyals less than beer. What is the cost of these amounts, in riyals? Express your answers in terms of B.

The price of beer	
The price of bread	
The price of a glass of wine	
The price of a pound of camel meat	
The price of a pound of dates	

24 Problem Paul's age is P. Mary is twice Paul's age, Peter is 5 years younger than Mary, Judas is 10 years older than Paul, and Jesus is half Peter's age. Complete the following table, writing all the data in terms of P.

Paul's age	
Mary's age	
Peter's age	
Judas' age	
Jesus' age	

4 Collecting Like Terms

25 Problem Collect like terms: $(a + b) + (a + b) + (a + b)$.

26 Problem Collect like terms: $(5\clubsuit - 7\spadesuit) + (-5\clubsuit - 7\spadesuit)$.

27 Problem Collect like terms: $(x^2 - 2x + 1) - (x^2 + 2x + 1)$.

28 Problem Collect like terms: $-3(2x-5y)+2(3x-2y)$.

29 Problem Collect like terms: $-\frac{2}{3}\left(x-\frac{3y}{2}\right)+2\left(\frac{x}{2}-2y\right)$.

5 One-Variable Equations

30 Problem Solve for x: $2(3x-4)-4(2-3x)=1$.

31 Problem Solve for x: $x-\frac{x}{2}-\frac{x}{3}=1$.

32 Problem Solve for x: $\frac{x-2}{2}=\frac{3-x}{3}$.

33 Problem Solve for x: $\frac{x}{a}-1=2$.

34 Problem Solve for x: $\frac{ax}{b}=a$.

35 Problem Solve for x: $ax+b=c$.

36 Problem Solve for x: $\frac{x+a}{2}=2x+1$.

37 Problem Solve for x: $\frac{x+1}{2}-\frac{x+2}{3}=\frac{x-1}{4}$.

38 Problem Solve for x: $\frac{a}{x}=b$.

39 Problem Solve for x: $\frac{ab}{cx}=d$.

40 Problem Solve for x: $\frac{3}{x-2}=1$.

41 Problem Solve for x: $\frac{3}{x-2}=\frac{2}{x+3}$.

6 Word Problems (One-Variable)

42 Problem A knitted scarf uses three balls of wool. I start the day with b balls of wool and knit s scarves. How many balls of wool do I have at the end of the day?

43 Problem Find two numbers whose sum is 28, and whose difference is 4.

44 Problem Divide $\$47$ between Peter, Paul, and Mary, so that Peter may have $\$10$ more than Paul, and Paul $\$8$ more than Mary.

45 Problem The sum of three consecutive odd integers is **609**. Find the numbers.

46 Problem A glass of beer costs **40** cents more than a loaf of bread but 50 cents less than a glass of wine. If the cost of the three items, in cents, is **730**, what is the price of each item, in cents?

47 Problem Currently, the age of a father is four times the age of his son, but in **24** years from now it will only be double. Find their ages.

48 Problem Six times a number increased by **11** is equal to **65**. Find it.

49 Problem Find a number which when multiplied by **11** and then diminished by **18** is equal to **15**.

50 Problem If **3** is added to a number, and the sum multiplied by **12**, the result is **84**. Find the number.

51 Problem One number exceeds another by **3**, and their sum is **27**; find them.

52 Problem Find two numbers whose sum is **19**, such that one shall exceed twice the other by **1**.

53 Problem Split $380 among Peter, Paul and Mary, so that Paul has $30 more than Peter, and Mary has $20 more than Paul.

54 Problem Jane's age is twice Bob's age increased by **3**. Bill's age is Bob's age decreased by **4**. If the sum of their ages is **27**, how old is Bill?

55 Problem John and Mary are betting and they start with **$60** among themselves. If they play until John's amount is twice of that won by Mary, how much money does each one have at the end?

56 Problem Bill currently has five times as much money as Bob. If he gives $20 to Bob, then Bill will only have four times as much. Find their current amounts.

57 Problem Find a number so that six sevenths of it exceed four fifths of it by **2**.

58 Problem The difference between two numbers is **8**. If we add **2** to the largest we obtain **3** times the smaller one. Find the numbers.

59 Problem Find two numbers whose difference is **10**, and whose sum equals twice their difference.

60 Problem I bought a certain amount of avocados at four for $2; I kept a fifth of them, and then sold the rest at three for $2. If I made a profit of $2, how many avocados did I originally buy?

61 Problem Find a number whose fourth, sixth, and eighth add up to **13**.

62 Problem A fifth of the larger of two consecutive integers exceeds a seventh of the smaller by **3**. Find the integers.

63 Problem I bought a certain number of oranges at three for a dollar and five sixths of that number at four for a dollar. If I sold all my oranges at sixteen for six dollars, I would make a profit of three and a half dollars. How many oranges did I buy?

7 Laws of Exponents

64 Problem Simplify and express with positive exponents only: $\dfrac{x^6}{y^9} \div \dfrac{x^{-2}}{y^{-3}}$

65 Problem What is the exact numerical value of $\dfrac{2^{20}6^{10}}{4^{11}3^{10}}$?

66 Problem What is the exact numerical value of $\left(\left(\dfrac{2}{5}\right)^{-1} - \left(\dfrac{1}{2}\right)^{-1}\right)^{-2}$?

8 Operations with Polynomials

67 Problem Expand and collect like terms: $(x+2)(x+3) - (x-2)(x-3)$.

68 Problem Expand and collect like terms: $(x-1)(x^2+x+1) - (x+1)(x^2-x+1)$.

69 Problem Expand and collect like terms: $\dfrac{x^3-8}{x-2} - x^2$.

70 Problem Expand and collect like terms: $\dfrac{6x^3-x^2-4x-1}{2x+1} - \dfrac{3x^3+3x-x^2-1}{3x-1}$.

71 Problem *Without using a calculator*, what is the exact numerical value of $(123456789)^2 - (123456787)(123456791)$?

72 Problem *Without using a calculator*, what is the exact numerical value of $(666\,666\,666)^2 - (333\,333\,333)^2$?

9 Factoring and Equations of Higher Order

73 Problem Resolve into factors: $5x^2+17x+6$.

74 Problem Solve for x: $5x^2+17x+6=0$.

75 Problem Resolve into factors: $14x^2+29x-15$.

76 Problem Solve for x: $14x^2+29x-15=0$.

77 Problem Resolve into factors: x^4-16.

78 Problem Resolve into factors: $(a+b)^2-c^2$

79 Problem Resolve into factors: x^3-x.

80 Problem Solve for x: $x^3-x=0$.

81 Problem Resolve into factors: x^3-x^2.

82 Problem Solve for x: $x^3 - x^2 = 0$.

10 The Plane

83 Problem A drunk ambulates on the plane, starting at $(-3, 2)$. She goes, successively, 16 units up, 8 units right, 4 units down, 2 units left, and finally, 1 unit up again. Where are her final coordinates?

84 Problem If the point $(a - 2, a + 1)$ lies on the line $2x - 3y = 1$, find the value of a.

85 Problem Find (x, y): $2x + 3y = 1$, $3x + 2y = -1$.

86 Problem Find (x, y): $x = 3y + 1$, $y = 3x + 1$.

87 Problem Draw the line $y = -3x + 2$.

88 Problem Draw the line $2x - 3y = 6$.

89 Problem Draw the lines and find their intersection: $L : x = 2y$, $L' : x + y = -3$.

11 Some Answers, Solutions, and Hints

1 The frog will escape after seven days. At the end of the sixth day, the frog has leaped 6 feet. Then at the beginning of the seventh day, the frog leaps 5 more feet and is out of the well.

2 We work backwards. He obtained 48 from $48 + 4 = 52$. This means that he should have performed $52 \div 4 = 13$.

3 We work backwards as follows. We obtained 37 by adding 16 to $37 - 16 = 21$. We obtained this 21 by multiplying by 3 the number $21 \div 3 = 7$. Thus the original number was a 7.

4 Notice that I always choose my number so that when I add it to your number I get 99, thus, I end up adding 99 three times and $3 \times 99 = 297$.

5 Pair up the numbers into the fifty pairs[1]

$$(100 + 1) = (99 + 2) = (98 + 3) = \cdots = (50 + 51).$$

Thus we have 50 pairs that add up to 101 and so the desired sum is $101 \times 50 = 5050$.

6 No. When we add an even integer to another even integer the result is an even integer. Thus the sum of five even integers is even, but 25 is odd.

7 27th floor.

10 $-\frac{1}{2}$.

11 $\frac{32}{105}$

12 $\frac{14}{45}$

13 $\frac{51}{250}$

[1] This trick is known as *Gauß' trick*, after the German mathematician Karl Friedrich Gauß (1777-1855). Presumably, when Gauß was in first grade, his teacher gave this sum to the pupils in order to keep them busy. To the amazement of the teacher, Gauß came up with the answer almost instantaneously.

14 $0.\overline{81}$

15 171

16 1.6

17 Let the number be x. Multiplying the number by 3 we obtain $3x$. When we add 45 we obtain $3x+45$. When we divide this result by 3 we obtain $\frac{3x+45}{3}$. If to this we subtract the original number we obtain $\frac{3x+45}{3}-x$. All we are saying is that

$$\frac{3x+45}{3}-x=15.$$

18 Notice that the right hand side consists of the last number on the left times its successor, and this is then divided by 2. Thus we are asserting that

$$1+2+3+\cdots+(n-1)+n=\frac{(n)(n+1)}{2}.$$

19 First observe that

$$3=2\times 1+1,$$
$$5=2\times 2+1,$$
$$7=2\times 3+1,$$

etc. Then it becomes clear that if the last number on the left is n, then the right hand side is $n(n+1)(2n+1)$ divided by 6. Hence we are asserting that

$$1^2+2^2+3^2+\cdots+(n-1)^2+n^2=\frac{(n)(n+1)(2n+1)}{6}.$$

25 $3a+3b$

26 $-14\spadesuit$

27 $-4x$

28 $11y$.

29 $-\frac{x}{3}-3y$.

30 $x=\frac{17}{18}$

31 $x=6$

32 $x=\frac{12}{5}$

33 $x=3a$

34 $x=b$

35 $x=\frac{c-b}{a}$

36 $x=\frac{a-2}{3}$

37 $x=1$

38 $x=\frac{a}{b}$

39 $x=\frac{cd}{ab}$

40 $x=5$

41 $x = -13$

42 $b - 3s$

43 Let x be one of the numbers, then the other number is $28 - x$. Then we have

$$\begin{aligned} x - (28 - x) = 4 &\implies 2x - 28 = 4 \\ &\implies x = 16. \end{aligned}$$

The numbers are $x = 16$ and $28 - x = 28 - 16 = 12$.

44 Let p be Paul's amount in dollars. Then Peter has $p + 10$ dollars and Mary has $p - 8$ dollars. Then we have

$$\begin{aligned} p + (p + 10) + (p - 8) = 47 &\implies 3p + 2 = 47 \\ &\implies 3p = 45 \\ &\implies p = 15. \end{aligned}$$

Thus Paul has $\$15$, Peter has $\$25$ and Mary has $\$7$.

45 Let the numbers be $x - 2, x, x + 2$. Then

$$\begin{aligned} (x - 2) + x + (x + 2) = 609 &\implies 3x = 609 \\ &\implies x = 203 \end{aligned}$$

The numbers are $x - 2 = 201$, $x = 203$, and $x + 2 = 205$.

46 Let b be the price of a glass of beer in cents. Then bread costs $b - 40$ cents and wine costs $b + 50$ cents. This gives

$$b + (b - 40) + (b + 50) = 730 \implies 3b + 10 = 730 \implies b = 240.$$

Thus beer costs 240 cents, bread costs 200 cents and wine costs 290 cents.

47 Let s be the current age of the son. Then the current age of the father is $4s$. In 24 years the son will be $s + 24$ and the father will be $4s + 24$ and we will have

$$4s + 24 = 2(s + 24).$$

This give

$$4s + 24 = 2s + 48 \implies 4s - 2s = 48 - 24 \implies s = 12.$$

Thus the son is currently 12 years-old and the father is currently 48 years-old.

48 If x is the number, then

$$6x + 11 = 65 \implies 6x = 54 \implies x = 9,$$

so the number is 9.

49 If x is the number, then

$$11x - 18 = 15 \implies 11x = 33 \implies x = 3,$$

so the number is 3.

50 If x is the number, then

$$12(x + 3) = 84 \implies 12x + 36 = 84 \implies 12x = 48 \implies x = 4,$$

so the number is 4.

52 If x is one number, the other is $19 - x$. If x exceeds twice $19 - x$ by 1, then

$$x - 2(19 - x) = 1 \implies 3x - 38 = 1 \implies 3x = 39 \implies x = 13.$$

Thus one number is 13 and the other is 6.

53 If Paul has x dollars, then Mary has $x + 20$ and Peter has $x - 30$. Therefore

$$x + x + 20 + x - 30 = 380 \implies 3x - 10 = 380 \implies 3x = 390 \implies x = 130.$$

Hence Paul has \$130, Mary \$150 and Peter \$100.

54 If Bob's age is x, then Bill's age is $x - 4$ and Jane's $2x + 3$. Therefore

$$2x + 3 + x - 4 + x = 27 \implies 4x - 1 = 27 \implies 4x = 28 \implies x = 7.$$

Thus Bob is 7, Bill is 3, and Jane is 17.

55 If John's amount at the beginning is x, then Mary's amount is $60 - x$. At the end we have

$$x = 2(60 - x) \implies x = 120 - 2x \implies 3x = 120 \implies x = 40.$$

Hence John has \$40 at the beginning and Mary has \$20 at the beginning.

56 If Bob currently has b dollars, then Bill has $5b$ dollars. After giving \$20 to Bob, Bill now has $5b - 20$ and Bob now has $b + 20$. We are given that

$$5b - 20 = 4(b + 20) \implies 5b - 20 = 4b + 80 \implies b = 100.$$

So currently, Bill has \$500 and Bob has \$100.

57 If x is the number then

$$\frac{4x}{5} - \frac{6x}{7} = 2.$$

Multiplying both sides of this equation by 35 we obtain

$$35\left(\frac{4x}{5} - \frac{6x}{7}\right) = 2(35) \implies 28x - 30x = 70 \implies -2x = 70 \implies x = -35.$$

58 If one number is x, the larger is $x + 8$. Thus

$$x + 8 + 2 = 3x \implies x + 10 = 3x \implies 10 = 2x \implies 5 = x,$$

so the smallest number is 5 and the larger is 13.

59 If one of the numbers is x, the other is $x + 10$. Thus

$$x + x + 10 = 2(10) \implies 2x + 10 = 20 \implies 2x = 10 \implies x = 5.$$

The smaller number is 5 and the larger is 15.

60 If x is the amount originally bought, then I spent $2\left(\frac{x}{4}\right) = \frac{x}{2}$ dollars. Since I kept $\frac{x}{5}$, I must have sold $\frac{4x}{5}$ of them, making $2\left(\frac{\frac{4x}{5}}{3}\right) = \frac{8x}{15}$ dollars on this sale. My net gain is thus

$$\frac{8x}{15} - \frac{x}{2} = 2.$$

Multiplying both sides by 30, we have

$$30\left(\frac{8x}{15} - \frac{x}{2}\right) = 2(30) \implies 16x - 15x = 60.$$

I originally bought sixty avocados.

61 If x is the number, then

$$\frac{x}{4}+\frac{x}{6}+\frac{x}{8}=13.$$

Multiplying both sides by 24, we have

$$24\left(\frac{x}{4}+\frac{x}{6}+\frac{x}{8}\right)=13(24) \implies 6x+4x+3x=312 \implies 13x=312 \implies x=24,$$

whence the number is 24.

62 Let x and $x+1$ be the integers. Then

$$\frac{x+1}{5}-\frac{x}{7}=3 \implies 35\left(\frac{x+1}{5}-\frac{x}{7}\right)=3(35) \implies 3(x+1)-5x=105 \implies 3-2x=105 \implies 2x=102 \implies x=51.$$

The integers are 51 and 52.

63 Let x be total amount of oranges bought at three for a dollar. On these I spent $\frac{1}{3}\cdot x\cdot 1=\frac{x}{3}$ dollars. I bought $\frac{5x}{6}$ oranges at four for a dollar, thus spending $\frac{1}{4}\cdot\left(\frac{5x}{6}\right)\cdot 1=\frac{5x}{24}$ dollars. Notice that I have bought a total of $x+\frac{5x}{6}=\frac{11x}{6}$ oranges. If I sell all of them at sixteen for six dollars, I make $6\cdot\frac{11x}{6}\cdot\frac{1}{16}=\frac{11x}{16}$ dollars. Thus

$$\frac{11x}{16}-\left(\frac{x}{3}+\frac{5x}{24}\right)=\frac{7}{2} \implies x=24.$$

Hence I bought $\left(\frac{11}{6}\right)24=44$ oranges.

64 $\frac{x^8}{y^{12}}$

65 $2^8=128$

66 4

67 $10x$.

68 $2x^3$.

69 $2x+4$.

70 $2x^2-2x-2$.

71 Put $x=123456789$. Then

$$(123456789)^2-(123456787)(123456791)=x^2-(x-2)(x+2)=x^2-(x^2-4)=4.$$

72 Using $x^2-y^2=(x-y)(x+y)$,

$$\begin{aligned}(666\,666\,666)^2-(333\,333\,333)^2 &= (666\,666\,666-333\,333\,333)(666\,666\,666+333\,333\,333)\\ &= (333\,333\,333)(999\,999\,999)\\ &= (333\,333\,333)(10^9-1)\\ &= 333\,333\,333\,000\,000\,000-333\,333\,333\\ &= 333\,333\,332\,666\,666\,667\end{aligned}$$

73 $(5x+2)(x+3)$

74 $x = -\frac{2}{5}$ or $x = -3$.

75 $(7x - 3)(2x + 5)$

76 $x = \frac{3}{7}$ or $x = -\frac{5}{2}$

77 $(x - 2)(x + 2)(x^2 + 4)$

78 $(a + b - c)(a + b + c)$

79 $x(x - 1)(x + 1)$

80 $x = 0$ or $x = 1$ or $x = -1$.

81 $x^2(x - 1)$

82 $x = 0$ or $x = 0$ or $x = 1$.

83 $(-3 + 8 - 2, 2 + 16 - 4 + 1) = (2, 15)$.

84 $a = -8$.

85 $(x, y) = (-1, +1)$

86 $(x, y) = (-\frac{1}{2}, -\frac{1}{2})$

87 The graph appears in figure 1.

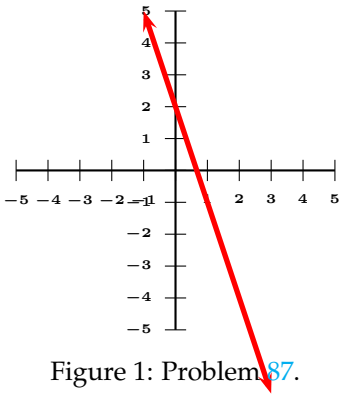

Figure 1: Problem 87.

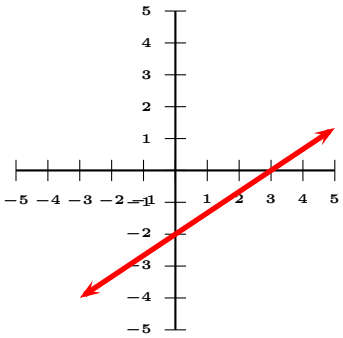

Figure 2: Problem 88.

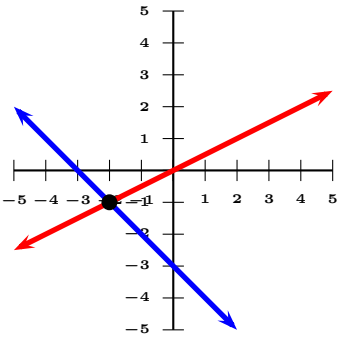

Figure 3: Problem 89.

88 The graph appears in figure 2.

89 The graph appears in figure 3. The intersection is the point $(-2, -1)$.