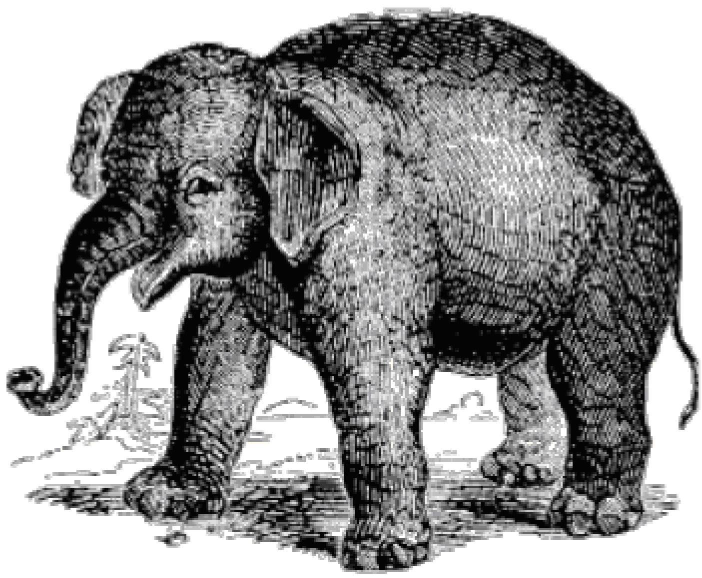
Math 017



Materials With Exercises

July 2012

TABLE OF CONTENTS

"Just a darn minute! — Yesterday you said that X equals **two**!"

<u>Help available</u>: You can find help in the **Learning Lab for math in room B2-36** weekdays and in room B1-28 Monday- Thursday evenings and Saturdays on Main Campus. During fall and spring semesters, free, peer tutoring is available beginning with the second week of classes for all current CCP students and free, weekly workshops begin in the third week of the semester. The peer tutors are experienced CCP students who have taken many of the courses in which they tutor. Math specialists also tutor as well as lead workshops. Check at Regional Centers for days and times of services. Also, during summer sessions, offerings may vary.

Lesson 1

Topics: Variables and algebraic expressions; Evaluation of algebraic expressions.

Variables and algebraic expressions as symbolic representations of numbers

Suppose that you thought of a number but you did not tell me what it was. I can think about your number as a number x. The symbol x is an example of a variable.

Variable **A variable is a symbol that represents an unknown number.**

The choice of the name of a variable is arbitrary. One can as well call it n, m or Ψ. **We treat variables as if they were numbers.** We can, for example, add numbers to variables: $m+3$, or subtract other variables from them: $m-\Psi$. We can multiply them: $4\cdot m$; divide: $\frac{\Psi}{m}$; raise to any given power: m^2; and then, if we wish, add all of the expressions together: $4\cdot m+\frac{\Psi}{m}+m^2$. The resulting expressions are called algebraic expressions.

Algebraic Expression **An algebraic expression is a number, variable or combination of the two connected by some mathematical operations like addition, subtraction, multiplication, division, or exponentiation.**

Notice that numbers and variables are also examples of algebraic expressions. We can refer to $3, x,$ or y as algebraic expressions. Just like $4\cdot 5,\ 2-5,$ or 3^2-1 are numbers (written in a 'complicated' way, but numbers), algebraic expressions $4\cdot m,\ x-y$ or a^n-b are symbolic representations of numbers. Both **variables and algebraic expressions can be thought of as unknown numbers.**

Correct language and conventions used when forming algebraic expressions

Algebraic expressions are read using the same terminology as in arithmetic. For example, $A+5$ can be read as "A plus 5" or "the sum of A and 5"; y^2 can be read as "y raised to the second power" or "y squared"; $-x$ is read "minus x" or "the opposite of x". The following convention is commonly adopted to indicate multiplication of a number and a variable, or multiplication of variables.

> To denote the operation of multiplication, the sign of multiplication between a number and a variable or between two variables or expressions does not have to be explicitly displayed, so for example,
>
> **2*A* means 2 times *A***
>
> ***xy* means *x* times *y*,**
>
> ***y*(*a*+*b*) means *y* times the quantity *a*+*b***

According to the above convention the following is true.

$$\boldsymbol{x = 1\cdot x = 1x}$$

Although $x = 1x$, it is **customary to write *x* instead of 1*x*** (just like any time we want to write 4, we just write 4 not $1\cdot 4$). The following is also true.

$$\boldsymbol{-x = -1\cdot x = -1x}$$

Again, it is **customary to write – *x* instead of –1*x*.**

When forming algebraic expressions, we place parentheses according to the notational convention adopted in arithmetic.

- **Any time two operation signs are next to each other, parentheses are needed.**

  For example, we write: $a-(-b)$, $a\div(-b)$, $a+(-b)$; we *do not* write: $a--b$, $a\div-b$, $a+-b$

we write:	we *do not* write:
$a-(-b)$	$a--b$,
$a\div(-b)$	$a\div-b$
$a+(-b)$	$a+-b$

  Likewise, when multiplying a and $-b$, we write $a(-b)$**.** The parentheses are needed even if the multiplication sign is not explicitly displayed. Notice that if the parentheses are omitted, the expression changes its meaning from multiplication of a and $-b$ to subtraction $a-b$.

- Exponents pertain only to 'the closest' expression. You might, for example, recall that to indicate that the entire fraction is raised to a given power, we use parentheses. We write $\left(\frac{2}{5}\right)^2$.

  Similarly, **whenever you exponentiate any algebraic expression that is not represented by a single symbol, you must use parentheses**: $\left(\frac{x}{y}\right)^2$, $(-x)^2$, $(y-x)^2$. On the other hand 5^2, x^2, y^2 do not require parentheses**.**

Example 1.1 How are the following expressions read?

a) x^2 b) xz c) x^3

d) x^m e) $-x$ f) $\frac{x}{y}$

Solution:

a) 'x raised to the second power' or 'x squared'
b) 'x times z', or 'the product of x and z', or 'x multiplied by z'
c) 'x raised to the third power' or 'x cubed'
d) 'x raised to the *m*-th power'

e) ‘minus x’ or ‘the opposite of x’
f) ‘x divided by y’ or ‘the quotient of x and y’

Example 1.2 In the following expressions parentheses are needed. Explain why they are needed.

a) $a \div (-c)$ b) $(-a)^n$

Solution:

a) Any time two operation signs are next to each other, parentheses are needed. In this case the “$\div$” sign is followed by “$-$” sign.

b) Without the parentheses, only a would be raised to the $n-$th power. With parentheses, we raise $-a$ to the n-th power. The two statements have different meaning, thus parentheses are needed.

Algebraic expressions allow us to express mathematical ideas in a general form

Algebraic expressions allow us to write mathematical ideas in symbols, without using specific numbers. For example, the area of a square is equal to the square of the length of its side. Not every square is going to have the same size, so we use a variable to represent the length of a side. If we denote s to be a side of a square, then the area of the square can be expressed as $s \cdot s = s^2$

Example 1.3 Let x and y denote two different numbers. Express the following statements using algebraic symbols.

a) The sum of x and y
b) The difference between x and y
c) The product of x and y
d) The quotient of x and y

Solution:

a) $x + y$
b) $x - y$
c) xy
d) $x \div y$ or equivalently $\frac{x}{y}$

Example 1.4 Find the algebraic expressions representing the opposite of the following expressions. Do not simplify.

a) x b) $-x$

Solution:

Recall that to find the opposite of a number, we must multiply the number by -1, thus

a) The opposite of x is $-1 \cdot x = -x$

b) The opposite of $-x$ is $-1 \cdot (-x) = -(-x)$. Please, notice that since the minus sign is directly followed by another minus sign, the parentheses are needed.

Example 1.5 Use the letter x to represent a number and write the following statements as algebraic expressions.

a) Double a number
b) Two thirds of a number
c) A quantity increased by 3

Solution:

a) $2x$
b) $\frac{2}{3}x$
c) $x+3$

Example 1.6 Write the following statements as algebraic expressions.

a) x subtracted from A
b) $-x$ added to $-A$
c) $-x$ multiplied by $-y$
d) $\frac{a}{b}$ raised to the sixteenth power

Solution:

a) $A-x$ Notice 'the reversed order' of variables (if you were asked to subtract 3 from 6, you would write $6-3$, similarly we write $A-x$).

b) $-x+(-A)$ Notice the use of parentheses: a plus sign, followed by a minus sign, requires parentheses.

c) $(-x)(-y)$ or $-x(-y)$ Notice the use of parentheses. A multiplication sign (even if not explicitly displayed), followed by a minus sign, requires parentheses.

d) $\left(\frac{a}{b}\right)^{16}$ Parentheses are needed to indicate that the entire expression $\frac{a}{b}$ is raised to the sixteenth power.

Evaluation of algebraic expressions

As mentioned, we can use variables and algebraic expressions to describe certain quantitative relationships without information about their specific values. For example, suppose that in a certain store, a cake costs 5 dollars. Let x be a variable that represents the number of cakes we plan to buy in that store. To calculate how much we will pay for such a purchase, we multiply the price of one cake, 5 dollars, by the number of cakes we buy, x. Thus, we can express the cost as $5 \cdot x = 5x$. The algebraic expression $5x$ represents the cost of x cakes bought at 5 dollars each. From now on, any time we know the number of cakes we wish to buy, i.e. we know the value of x, we can find the price by **evaluating** the expression $5x$.

To Evaluate an Expression **To evaluate an algebraic expression means to find its value once we know the values of its variables. Each variable has to be replaced by its value and the resulting numerical expression has to be calculated.**

For example,
Evaluate $5x$ when $x = 10$ (in the above example it would mean finding the price of 10 cakes at 5 dollars each).

$$5x = 5 \times 10 = 50$$

(The price of 10 cakes is 50 dollars.)

Notice, that what we did was to substitute 10 for x in the expression $5x$. We could do that because x is equal to 10. The following fundamental principle underlies the process of evaluation.

If two quantities are equal, you can always substitute one for the other. **"Equals can be substituted for equals"**

Can any algebraic expression be evaluated? Let us consider evaluation of $\frac{1}{a}$ when $a = 0$. If we replace a by its value 0, the operation we would have to perform is division by 0, but **division by zero is not defined**, so $\frac{1}{a}$ cannot be evaluated if $a = 0$. Expressions like $\frac{x}{0}$, or $x \div 0$ are undefined.

<u>Example 1.7</u> Rewrite each expression replacing variables with their values and then evaluate, if possible. If evaluation is not possible, explain why it is not possible.

a) $x - 6$, if $x = 5$
b) $-C$, if $C = -2$
c) $2x$, if $x = -3$
d) 2^n, if $n = 3$
e) $y - x$, if $x = -1$ and $y = \frac{1}{3}$
f) $\frac{1}{x+2}$, if $x = -2$

Solution:
Each variable should be replaced by its assumed value and the obtained numerical sentence has to be evaluated. Please, pay attention to the way parentheses are used.

a) $x - 6 = 5 - 6 = -1$
b) $-C = -(-2) = 2$
c) $2x = 2(-3) = -6$ Remember that $2x$ means multiplication of 2 and x.
d) $2^n = 2^3 = 8$
e) $y - x = \frac{1}{3} - (-1) = \frac{1}{3} + 1 = 1\frac{1}{3}$
f) $\frac{1}{x+2} = \frac{1}{-2+2} = \frac{1}{0}$ The expression cannot be evaluated, since division by zero is undefined.

Common mistakes and misconceptions

Mistake 1.1

When $x = -1$, and you are asked to evaluate $-x$, you must be careful to write $-x = -(-1)$. Do not forget to recopy the minus sign before -1. It is incorrect to evaluate $-x$ by simply writing $-x = -1$.

Mistake 1.2

One should NOT think that $-x$ always represents a negative quantity. It depends on the value of x. If, for example, $x = -1$, then $-x = -(-1) = 1$. So we see that if x is a negative value, $-x$ actually represents a positive quantity.

Mistake 1.3

When writing a^m, do NOT place m at the same level as a but slightly higher. Otherwise, a^m becomes am. These two expressions have different meanings.

Mistake 1.4

The expression x^2 is NOT read as 'x two' or 'two x'. Instead, read it as 'x raised to the second power' or 'x squared'.

Exercises with Answers (For answers see Appendix A)

Exercises 20-36 will help to review basic arithmetic operations using integers, rational numbers (fractions), decimals.

Ex.1 Fill in blanks using the words: 'variable(s)', 'algebraic expression(s)', and'number(s)' as appropriate.

$3x+2$, y^2, $\dfrac{a+bc}{2}$, $(-2a+1)^3$ are examples of ____________________ .

Ψ x, y, a, b, c are examples of _________________ but also examples of ___________________ .

Variables represent _________________.

If we know the value of x, we can evaluate $3x+2$, and as a result we get a _______________ .

Ex.2 How are the following expressions read?

a) a^2 b) a^3 c) a^{12} d) 2^m

e) $-y$ f) cd g) $a-b$ h) $\dfrac{2}{5}x$

Ex.3 Rewrite the following expressions, inserting a multiplication sign whenever multiplication is implied. Whenever there is no operation of multiplication, clearly say so using the phrase "there is no multiplication performed in this expression".

a) $7n$ b) $-5km$ c) $-x-y$

d) $-x(-y)$ e) $\frac{3x}{2}$ f) $2x-yz+w(-t)$

Ex.4 The operation that is indicated in the algebraic expression $a+b$ is, of course, addition. Name the operation that is to be performed in the following algebraic expressions.

a) ab b) $\frac{q}{s}$ c) x^5

d) $3\div x$ e) $3-x$ f) $3(-x)$

Ex.5 In the following expressions parentheses are needed. Explain why they are needed.

a) $x+(-b)$ b) $\left(\frac{m}{n}\right)^8$ c) $3b(-c)$

d) $(-a)^4$ e) $y(-x)$ f) $a\div(-b)$

Ex.6 Determine which expression is raised to the *n*-th power.

a) $(-s)^n$ b) $-s^n$ c) $(st)^n$

d) st^n e) $\frac{x^n}{y}$ f) $\left(\frac{x}{y}\right)^n$

g) $x(st)^n$ h) $xy-s^n$ i) $x(y-s)^n$

Ex.7 Fill in the blanks.

a) It is customary to write ________ instead of $1\cdot x$.
b) It is customary to write ________ instead of $-1\cdot x$.
c) When one writes ab, it is understood that the operation that is to be performed is ___________ .

Ex.8 Fill in the blanks with numbers to make the statement true.

a) ______ $\cdot x = x$
b) ______ $\cdot x = -x$
c) ______ $\cdot x = 0$

Ex.9 Write an algebraic expression representing the opposite number of (do not remove parentheses).

a) $-x$ b) $\frac{x^3}{y}$

c) $-\frac{x^3}{y}$ d) $\frac{-x^3}{-y}$

Ex.10 Use the letter y to represent a number and write the following phrases as algebraic expressions.

a) Half of a number
b) Three fourths of a number
c) A quantity increased by 5

d) A number subtracted from v
e) A quantity squared
f) Three more than a number
g) A number decreased by x
h) The product of x and a number
i) A number doubled

Ex.11 Write the following phrases as algebraic expressions. Remember to place parentheses when needed (place them only when needed). Do not simplify.

a) The sum of a and $-b$
b) The difference of a and $-b$
c) The product of a and $-b$
d) The opposite of C
f) The opposite of $-C$
g) The opposite of $\frac{-a}{-b}$
h) The product of v, $-t$, and $-p$
i) The quotient of c and $-B$
j) $-x$ raised to m-th power
k) $\frac{x}{y}$ raised to m-th power

Ex.12 Give your answer in the form of an algebraic expression.

a) Carlos is x years old at this moment. How old will Carlos be in 10 years?
b) An item in a store costs x dollars. What is the price of the item, if after a discount, its price was reduced to two thirds of the original one?
c) You have x dollars to divide equally among 3 kids. How much will each child get?
d) You have $100 to divide equally among x kids. How much will each child get?
e) Charles bought 2 more lamps for his apartment today. If there are x lamps in Charles' apartment now, how many lamps were in his apartment before the purchase?
f) There are 30 books on each shelf. How many books are on x shelves?
g) There are x students in a classroom. How many students are still in the classroom, if 3 students leave?
h) John is 5 years older than Tom. If John is x years old, how old is Tom?

Ex.13 Let *d* be a variable representing the distance driven by a car, and let *t* represent the time it took to drive that distance. Write the following phrase as an algebraic expression: The distance divided by time.

Ex.14 Let *m* be a variable representing the mass of a given body, and let *a* represent its acceleration. Use *m* and *a* to write the following phrase as an algebraic expression: The product of the mass of a body and its acceleration.

Ex.15 Let *h* be a variable representing the height of a triangle, and *b* represent the base of the triangle. Use *h* and *b* to write the following statement as an algebraic expression: One half of the product of the base of a triangle and its height.

Ex.16 Let $x=3$. Rewrite the expression replacing the variable with its value and evaluate, if possible. If evaluation is not possible, explain why it is not possible.

a) $x+5$
b) $x-2$
c) $\frac{x}{3}$
d) $4x$
e) x^2
f) $\frac{6}{x}$

Ex.17 If $x=0$ the expression $\frac{1}{x}$ cannot be evaluated. Why not? Can $\frac{1}{x-5}$ be evaluated when $x=0$? What if $x=5$? Find another example of an algebraic expression and a value of a variable(s) for which evaluation is not possible.

Ex.18 Let $x=0$. Rewrite the expression replacing the variable with its value and evaluate, if possible. If evaluation is not possible, explain why it is not possible.

a) $3x$
b) $x-2$
c) $\frac{4}{x}$
d) $\frac{x}{7}$
e) $\frac{2}{x-3}$
f) $\frac{0}{x}$

Ex.19 Let $x=2$. Rewrite the expression replacing the variable with its value and evaluate, if possible. Otherwise, write "undefined".

a) 3^x
b) x^3
c) x^x

Ex.20 Evaluate $-A$, if

a) $A=2$
b) $A=-2$

Ex.21 Substitute $x=6$ in the following expressions and then evaluate, if possible. Otherwise, write "undefined".

a) $x-8$
b) $-10-x$
c) $-4+x$
d) $x-6$
e) $-2+x-6$

Ex.22 Substitute $x=-2$ in the following expressions and then evaluate, if possible. Otherwise, write "undefined".

a) $2+x$
b) $2-x$
c) $-2-x$
d) $-5-x+4$
e) $6+x-10-x$

Ex.23 Substitute $x=10$ in the following expressions and then evaluate, if possible. Otherwise, write "undefined".

a) $3x$

b) $-5x$

c) $\frac{-200}{x}$

d) $-\frac{x}{2}$

e) $-5 \div x$

f) x^4

Ex.24 Substitute $x=-12$ in the following expressions and then evaluate, if possible. Otherwise, write "undefined".

a) $-1000x$

b) $\frac{x}{6}$

c) $-5x$

d) $\frac{6}{x+12}$

e) $-24 \div x$

f) x^2

Ex.25 Substitute $x=\frac{2}{3}$ in the following expressions and then evaluate, if possible. Otherwise, write "undefined".

a) $\frac{5}{3}+x$

b) $x+\frac{1}{5}$

c) $-x+\frac{2}{7}$

d) $-\frac{5}{12}-x$

e) $2+x$

f) $-x-3$

Ex.26 Substitute $x=-\frac{3}{5}$ in the following expressions and then evaluate, if possible. Otherwise, write "undefined".

a) $\frac{3}{10}-x$

b) $-\frac{1}{7}-x$

c) $2\frac{1}{5}+x$

d) $-1\frac{1}{4}-x$

e) $-x-3\frac{1}{2}$

Ex.27 Substitute $x=\frac{2}{7}$ in the following expressions and then evaluate, if possible. Otherwise, write "undefined".

a) $2x$

b) $-7x$

c) $-\frac{14}{3}x$

d) $\frac{5}{28} \div x$

e) $\dfrac{5}{x}$

f) $\dfrac{-x}{2}$

Ex.28 Substitute $x=-\dfrac{3}{4}$ in the following expressions and then evaluate, if possible. Otherwise, write "undefined".

a) x^2

b) $\dfrac{4}{3}x$

c) $\dfrac{-x}{1\frac{2}{3}}$

d) $-1\dfrac{1}{8}\div x$

e) $\dfrac{x}{-3}$

f) $\dfrac{0}{x}$

Ex.29 Substitute $x=0.2$ in the following expressions and then evaluate, if possible. Otherwise, write "undefined".

a) $x+3.21$

b) $35.01-x$

c) $\dfrac{x}{4}$

d) $-40x$

e) $0.3x$

f) $\dfrac{-x}{0.04}$

Ex.30 Substitute $x=-0.6$ in the following expressions and then evaluate, if possible. Otherwise, write "undefined".

a) $-x-4.5$

b) $-2.7-x$

c) $\dfrac{1.2}{-x}$

d) $-600x$

e) $0.001x$

f) x^3

Ex 31 Substitute $x=-1.5$ in the following expressions and then evaluate, if possible. Otherwise, write "undefined".

a) $x-0.08$
b) $-3-x+0.4$
c) $x\div 0.15$
d) $-0.2x$
e) $\dfrac{-30}{x}$

Ex.32 If possible, evaluate $x+y$ when

a) $x=\dfrac{3}{5},\ y=\dfrac{2}{3}$

b) $x=\dfrac{2}{7},\ y=-\dfrac{9}{14}$

c) $x=-0.2,\ y=-1.08$

Ex.33 If possible, evaluate $x - y$ when

a) $x = \frac{3}{5}$, $y = \frac{2}{3}$

b) $x = \frac{2}{7}$, $y = -\frac{9}{14}$

c) $x = -0.2$, $y = -1.08$

Ex.34 If possible, evaluate xy when

a) $x = \frac{2}{11}$, $y = \frac{22}{9}$

b) $x = -4$, $y = -\frac{9}{10}$

c) $x = -0.2$, $y = 0.01$

Ex.35 If possible, evaluate $\frac{x}{y}$ when

a) $x = \frac{2}{11}$, $y = \frac{22}{9}$

b) $x = -4$, $y = -\frac{9}{10}$

c) $x = -0.2$, $y = 0.01$

Ex.36 If possible, evaluate $(-x)^m$ when

a) $x = 10$, $m = 7$

b) $x = -2$, $m = 4$

c) $x = \frac{1}{2}$, $m = 3$

d) $x = -0.1$, $m = 5$

Ex.37 Use the letter x to represent a number and write the following statements as algebraic expressions. Then evaluate each expression when $x = -\frac{1}{2}$.

a) A number doubled

b) Three fourths of a number

c) A number raised to the second power

Ex.38 Evaluate $-t$, when $t = 1$, $t = -1$

Based on your results, which of the following are true?

a) $-t$ is always positive

b) $-t$ is always negative

c) $-t$ may be positive or negative depending on the value of t

Lesson 2

Topics: Algebraic expressions and their evaluations; Order of operations.

The order of operations

Recall the order of operations.

> **When evaluating arithmetic expressions, the order of operations is**
> 1) **Perform all operations in parentheses first.**
> 2) **Do any work with exponents.**
> 3) **Perform all multiplications and divisions, working from left to right.**
> 4) **Perform all additions and subtractions, working from left to right.**
>
> **If a numerical expression includes a fraction bar, perform all calculations above and below the fraction bar before dividing the top by the bottom number.**

Let x be an unknown number. If we increase the number by 1, the resulting number can be represented by $x+1$. Now, suppose that after adding 1, we multiply some other number, called y, by the result of the addition. We may now write the expression $y(x+1)$. Let us analyze why we must place parentheses around $x+1$. **If an algebraic expression involves more than one mathematical operation, then the order of operations is followed.** Thus, if we simply write $yx+1$ (without using parentheses), we would only be multiplying y and x, rather than y times the entire quantity $x+1$. According to the order of operations, using parentheses in $y(x+1)$ indicates that we add 1 to x first, and then multiply the result by y. Expressions $yx+1$ and $y(x+1)$ have entirely different meanings, and only $y(x+1)$ correctly represents the result of operations performed in this example.

The order of operations is used when algebraic expressions are evaluated.
For example, let us evaluate $2x^2 - x$ when $x = 3$.

$2x^2 - x =$	replace each x with 3
$2\times 3^2 - 3 =$	raise 3 to the second power
$2\times 9 - 3 =$	multiply it by 2
$18 - 3 =$	subtract 3 from the result
15	

Example 2.1 Rewrite the following expressions and circle the arithmetic operation together with its operands that has to be performed first. Write the name of the operation next to your expression.

a) $3-5y$ b) $2x^4$

Solution:

a) We perform multiplication before subtraction

$$3-5y \quad \text{multiplication}$$

b) We exponentiate before multiplication

$$2x^4 \quad \text{exponentiation}$$

Example 2.2 List, according to the order of operations, all the operations together with operands that are to be performed in the following expressions.

a) $4+3x$ b) $4\div 8a$

Solution:

a) There are two operations in $4+3x$, multiplication and addition. According to the order of operations, multiplication should be performed first. Thus, the answer is: Multiply 3 by x, then add 4.

b) There are two operations in $4\div 8a$, division and multiplication. According to the order of operations, they should be performed as they appear reading from the left to the right. Thus, the answer is: Divide 4 by 8 first, and then multiply by a.

Example 2.3 Write the algebraic expressions representing the following

a) $a+b$ raised to the seventh power

b) $a+b$ subtracted from y

c) 8 times the quantity $a+b$

d) the opposite of $a+b$

Solution:

Parentheses must be used in each of these examples.

a) $(a+b)^7$ We learned that the exponent pertains only to the closest expression, so if we write $a+b^7$ instead of $(a+b)^7$, only b would be raised to the power 7. This rule can be viewed as a consequence of the order of operations. Exponentiation should be performed before addition, hence in $a+b^7$, b is raised to the seventh power, and then the result is added to a. To ensure that b is first added to a, we need to use parentheses, and only then the sum is raised to the seventh power.

b) $y-(a+b)$ Notice the order of expressions: “subtract from y” indicates that y is written first and we subtract from it.

c) $8(a+b)$ Thanks to parentheses, we add first, and then multiply.

d) $-(a+b)$ Taking the opposite means multiplication by -1. To ensure addition first, we need parentheses.

Example 2.4 Use the letter x to represent a number and write the following as algebraic expressions.

a) Add three to a number, and then divide it by z

b) Seven more than one third of a number

c) A quantity decreased by 9, and then multiplied by A

d) A number cubed, and then decreased by y

Solution:
Please, notice the use of parentheses in the examples below.

a) $(x+3) \div z$ or $\dfrac{x+3}{z}$

b) $\dfrac{1}{3}x+7$

c) $(x-9)A$

d) $x^3 - y$

Example 2.5 Determine if, in the following algebraic expressions, parentheses are necessary, i.e. they change or do not change the order of operations. To this end determine if the first operation that should be performed is the same as if the expression were written without any parentheses.

a) $(a+b)-c$ b) $(a+b)c$

Solution:

a) In the expression $(a+b)-c$ the first operation that should be performed is the addition of a and b. If we rewrite the same expression but without parentheses, we get $a+b-c$ and the operation of addition of a and b is the first operation as well (since addition and subtraction are always performed in the order as they appear from the left to the right). Thus, we conclude, that parentheses do not change the order of operations in this case.

b) In the expression $(a+b)c$ the addition of a and b should be performed first, but if we "drop" the parentheses the resulting expression is $a+bc$ and thus we would first multiply b and c, and then add a. Parentheses are needed since they do change the order of operations.

Example 2.6 Rewrite the expression replacing variables with their values. Then, evaluate, if possible. If evaluation is not possible, explain why it is not possible.

a) $\dfrac{x-1}{x+2}$ if $x=-2$

b) $3y-x$ if $x=-1$ and $y=\dfrac{1}{3}$

c) $(n+3)^m$ if $m=2$ and $n=-5$

Solution:
Each variable has to be replaced by its given value and the resulting numerical expression has to be evaluated. Please, pay attention to the way parentheses are used.

a) $\dfrac{x-1}{x+2} = \dfrac{-2-1}{-2+2} = \dfrac{-3}{0}$, but division by zero is not defined. The expression cannot be evaluated. The operation is "undefined".

b) $3y-x = 3\cdot\dfrac{1}{3}-(-1) = 1+1 = 2$

c) $(n+3)^m = (-5+3)^2 = (-2)^2 = 4$

Common mistakes and misconceptions

Mistake 2.1

When evaluating $2 \times 10 + 1$, it is INCORRECT to write

$$2 \times 10 + 1 = 20 = 21 \qquad (20 \neq 21)$$

Instead, one should write

$$2 \times 10 + 1 = 20 + 1 = 21$$

Numbers or expressions not involved in the operation that is being carried out *must always be rewritten*. An equal sign means that the quantities on either side are equal.

Exercises with Answers (For answers see Appendix A)

Ex.1 Write the algebraic expression representing the following.

a) $m - 2n$ subtracted from x
b) the opposite of $m - 2n$
c) $m - 2n$ multiplied by 7
d) $3a$ subtracted from $m - 2n$
e) the opposite of $k^2 - 3k + 1$
f) 4 divided by $-4x + y$ (use the "$\div$" symbol in your answer)

Ex.2 Write the following phrases as algebraic expressions. Remember to place parentheses where needed (please, place them only when needed).

a) Multiply 3 by x, and then add y
b) Multiply the sum of a and b by 4
c) The opposite of x, then raise it to the sixth power
d) Subtract 3 from y, and then multiply the result by z
e) Raise x to the third power, and then multiply the result by 9
f) Multiply x by 9, and then raise the result to the third power
g) The difference of a and b, then divided by c
h) Divide 3 by y, and then add x
i) The opposite of the sum of M and 3
j) Raise $-x$ to the third power, raise y to the seventh power, and then add them together

Ex.3 Use the letter x to represent a number and write the following as an algebraic expression.

a) A number decreased by 7, and then doubled
b) Add c to a number, and then take two thirds of the sum
c) Take one fourth of a number, and then subtract 5 from it
d) Multiply a number by 9, and then subtract it from c
e) A number, first divided by 2, and then raised to the third power.
f) The opposite of a number, then multiplied by 4
g) A quantity raised to the third power, and then increased by 6
h) A number decreased by 4, and then the result multiplied by y
i) Subtract a number from y and then take the opposite of the result

j) A number multiplied by the sum of the same number and 5
k) The opposite of a number, then raised to one hundred and twenty first power
l) Square a number, and then take the opposite of it

Ex.4 Let C be a variable representing the temperature in Celsius. Write the following phrase as an algebraic expression: Nine fifths of the Celsius temperature plus 32.

Ex.5 Let *L* be a variable representing the length of a rectangle, and *W* its width. Use *L* and *W* to write the following phrase as an algebraic expression: The sum of the length of a rectangle and its width, then multiplied by 2.

Ex.6 Let m represent mass and c the speed of light. Use m and c to write the following phrase as an algebraic expression: The product of mass and the square of the speed of light.

Ex.7 In the following expressions circle the arithmetic operation, together with its operands, that has to be performed first. Write the name of the operation next to your expression. For example, in $4+3x$, multiplication of 3 and x has to be performed first, thus the answer is

$4 \quad + 3x$ multiplication

a) $a+b^5$ b) $(a+b)^5$

c) $-x^8$ d) $(-x)^8$

e) $\dfrac{a-b}{c}$ f) $a \div b \times c$

g) $4-7y$ h) $3+a \div b$

Ex.8 There are two operations in the algebraic expression $a+3b$, addition and multiplication. In order to evaluate $a+3b$, we would have to perform them according to the order of operations. First multiply 3 and *b,* and then add a. List, according to the order of operations, the operations that are in the following algebraic expressions.

a) $4x-y$ b) $\dfrac{a+3}{x}$

c) $(x+3)y$ d) $\dfrac{s}{t}+2$

e) $3x^2$ f) $(3x)^2$

g) $(a+c)^4$ h) $a+c^4$

Ex.9 Determine if, in the following algebraic expressions, parentheses are necessary, i.e. they change or do not change the order of operations. To this end, determine if the first operation that should be performed is the same as if the expression were written without any parentheses. If the operation is different, write "parentheses are needed', otherwise rewrite the expression without any changes.

a) $(2-a)x$ b) $(c-3)-a$

c) $(3a)+x$ d) $a-(c+b)$

e) $x \div (2ab)$ f) $\dfrac{(-c+d)}{a}$

g) $(a+2)^4$ h) $y(x)^8$

i) $(ab)^4$ j) $(a+d)\div c$

Ex.10 Evaluate, if possible.

a) $2x+1$, if $x=\frac{1}{2}$

b) $2a+1$, if $a=\frac{1}{2}$

c) $2y+1$, if $y=\frac{1}{2}$

d) Did you get the same answer for a, b, and c? Can you explain why it is so?

e) If $\frac{4x^3+x}{x-1}=-2$ when $x=\frac{1}{2}$, evaluate $\frac{4a^3+a}{a-1}$ when $a=\frac{1}{2}$. You should be able to arrive at your answer without performing any evaluation.

Ex.11 Let $x=3$. Rewrite the expression replacing the variable with its value and evaluate, if possible. Otherwise, write "undefined".

a) $-2x-5$

b) $-4+x^2$

c) $\frac{x}{x-3}$

d) $(-x)^2$

e) $-x^2$

f) $\frac{3-x}{4+x}$

g) x^x

Ex.12 Let $x=4$. Rewrite the expression replacing the variable with its value and evaluate, if possible. Otherwise, write "undefined".

a) -2^x

b) $(-2)^x$

c) $(-x)^2$

d) $-x^2$

Ex.13 Let $x=-1$. Rewrite the expression replacing the variable with its value and evaluate, if possible. Otherwise, write "undefined".

a) $-x+x$

b) $-x-x$

c) $(-x)(-x)$

d) x^2

e) $-x^2$

Ex.14 Substitute $A = \frac{1}{2}$ and then evaluate the following expressions, if possible. Otherwise, write "undefined".

a) $\frac{1}{A}$

b) $\frac{1}{A} + A$

c) $(-A)^2$

d) $-A^2$

Ex.15 Let $x = -0.3$. Rewrite each expression replacing the variable with its value and evaluate, if possible. Otherwise, write "undefined".

a) $x^2 - x$

b) $\frac{x}{0.1} - 2$

c) $\frac{0.3 - x}{x - 0.3}$

d) $1000x - 100x + 10x$

Ex.16 The expression $\frac{3 - x}{y - 5}$ cannot be evaluated for which of the following values of x and y? Explain why.

a) $x = 3, \quad y = -5$

b) $x = -3, \quad y = 5$

c) $x = 3, \quad y = 5$

d) $x = -3, \quad y = -5$

e) $x = 3, \quad y = 0$

f) $x = 0, \quad y = 5$

Ex.17 If possible, evaluate when $m = -2, \quad n = 5$. Otherwise, write "undefined". Before evaluating, rewrite the expressions substituting the numerical values of m and n.

a) $2m - 3n$

b) $2m(-3n)$

c) $2(m - 3n)$

d) $(2m - 3)n$

e) $2(m - 3)n$

Ex.18 If possible, evaluate when $m = -\frac{1}{8}, n = \frac{4}{5}$. Otherwise, write "undefined". Before evaluating, rewrite the expressions substituting the numerical values of m and n.

a) $8m - 10n$

b) $10mn$

c) $2(n - m)$

d) $-8m^2 + n$

e) $n \div (\frac{1}{8} + m)$

f) $n \div \frac{3}{10} + m$

Ex.19 If possible, evaluate when $A=\frac{1}{3}, \quad B=-\frac{2}{3}$. Otherwise, write "undefined". Before evaluating, rewrite the expressions substituting the numerical values of A and B.

a) $2A^4$
b) B^4
c) $-B^4$
d) $\frac{A+B}{A-B}$
e) $\frac{A(-B)}{A\div B}$

Ex.20 Evaluate the following expressions: $a^3, \quad 4^n, \quad ab^2, \quad (ab)^2, \quad -a^n, \quad a^{n+m}$ if $a=-1, \; b=\frac{1}{3}$, $n=3, \quad m=2$. If evaluation is not defined, write "undefined".

Ex.21 Let $x=2$, $y=-0.1$, and $z=-1$. If possible, evaluate the following expressions. Otherwise, write "undefined".

a) $x(z+y)$
b) $xz+y$

Ex.22 Let $a=0.1, \quad b=-0.2, \quad c=-1$. If possible (otherwise, write "undefined"), evaluate the following expressions. Before evaluating, rewrite the expressions substituting the numerical values of variables.

a) $a-bc$
b) a^{-c}
c) b^{10a}

Ex.23 Find the value of $2a^2-(2a)^2$ if

a) $a=1$
b) $a=-1$

Ex.24 Find the value of $2A-B$, if

a) $A=-1, B=3$
b) $A=-2, B=-4$
c) $A=0.3, B=-0.7$
d) $A=\frac{2}{8}, B=-1$
e) $A=1\frac{5}{6}, B=\frac{4}{5}$
f) $A=-\frac{3}{10}, B=-\frac{5}{7}$

Ex.25 Find the value of $-(A+3B)$, if

a) $A=-1, B=-1$
b) $A=2, B=-3$
c) $A=0.1, B=-0.2$

d) $A = -2,\ B = -1\frac{2}{3}$

e) $A = \frac{2}{7},\ B = -\frac{1}{6}$

f) $A = -4,\ B = -\frac{5}{9}$

Ex.26 Find the value of $\frac{0.1x}{y}$, if

a) $x = 2,\ y = 0.02$
b) $x = -200,\ y = 0.4$
c) $x = 0.1,\ y = -0.2$

Ex.27 Evaluate the following expressions, if $m = -1,\quad n = 2,$ and $p = -3$. Before evaluating, rewrite the expressions substituting the numerical values of variables.

a) $m - (n + p)$
b) $m - n + p$

Ex.28 In the following expressions, identify the first operation that should be performed according to the order of operations and anytime it is a numerical operation, perform it.

a) $3 + 4 + x$
b) $3 + 4x$
c) $3(4)x$
d) $(3 + 4)x$
e) $x \cdot 2^3$
f) $\frac{2-3}{x}$

Ex.29 Write the following phrases as algebraic expressions and then evaluate them when $x = -3$.

a) 3 multiplied by x, and then squared
b) -4 subtracted from x, and then divided by 0.2
c) 9 divided by x, and then cubed

Lesson 3

Topics: Equivalent algebraic expressions.

Definition of equivalent algebraic expressions

Suppose we wish to write as an algebraic expression "a number x doubled". Should we write $x \cdot 2$ or $2 \cdot x$? Because of the commutative property of multiplication, both answers are right. Both have the same meaning, although they appear to be different. We encounter a similar idea in arithmetic. The fractions $\frac{2}{4}$ and $\frac{1}{2}$ are equivalent, which means that they represent the same number although they 'do not look the same'. Similarly, we would say that $x \cdot 2$ and $2 \cdot x$ are equivalent (we often say equal) and write $x \cdot 2 = 2 \cdot x$.

Equivalent Algebraic Expressions — **Two algebraic expressions are equivalent if, when evaluated, they have the same value for all replacements of the variables.**

Suppose that two algebraic expressions are equivalent, like the two mentioned above, $x \cdot 2$ and $2 \cdot x$. What it means, according to the definition, is that if we choose any value of x, let's say $x = 1$, and evaluate $x \cdot 2 = 1 \cdot 2 = 2$, and then evaluate $2 \cdot x = 2 \cdot 1 = 2$, the results must be the same. If we change the value of x, for example to 4, again two results are equal ($x \cdot 2 = 4 \cdot 2 = 8$, and $2 \cdot x = 2 \cdot 4 = 8$). No matter what the value of x, the two results are *always* going to be equal. Thus, to determine that two expressions are equivalent one would have to evaluate them for *all* possible sets of values of variables. Since we cannot check *all,* we cannot prove equivalence by performing evaluation (make sure that you understand that **even if we determine that two expressions assume the same value for many sets of values of variables, we still cannot claim that the two expressions are equivalent**). To prove the equivalence of algebraic expressions, some general rules must be employed.

Terms and factors

In arithmetic, we often refer to numbers that are being added as terms, and to numbers multiplied as factors. For example, 3 and 4 are terms of addition $3 + 4 = 7$, while 3 and 5 are called factors of 15, since $3 \times 5 = 15$. We will now generalize the notions of terms and factors.

Terms — **Algebraic expressions that are added (or subtracted) are called terms. Each sign, + or –, is a part of the term that follows the sign.**

In other words, the addition and subtraction signs break the expression into smaller parts, called terms, and so, in $3x + 2xy - y$ there are three terms: $3x$, $2xy$, $-y$. Notice that because y is preceded by a minus sign, the minus sign is a part of the term: $-y$. The expressions a and $\frac{5a}{d}$ are terms in $a + \frac{5a}{d}$. Some expressions have just one term. For example, both $3xy$ and $x^3 y$ have only one term.

Factors **Algebraic expressions that are multiplied are called factors.**

The expression $4mn$ has factors 1, 4, *m*, *n* and any combination of those, like $4m$, $4n$, and of course, $4mn$. In ab^2, the expressions a and b^2 are called factors but 1, *b*, b^2, *ab*, ab^2 are also factors of this expression. During our study we will be talking about **explicit factors**. Explicit factors of $4mn$ are 4, *m* and *n*, i.e. the factors that are separated by the multiplication sign (displayed or not displayed) of an expression. Explicit factors of ab^2 are a and b^2. The expressions $2x-3$ and $x+4$ are explicit factors in $(2x-3)(x+4)$.

Example 3.1 List all terms of the following expressions.

a) $3-y+z$ b) $-3x^2+4(z+1)-\dfrac{x}{2}$

Solution:

a) The terms are $3, \quad -y, \quad z$. Remember that signs are always a part of terms. We list $-y$ as a term (*y* is preceded by the minus sign).

b) The terms are $-3x^2, 4(z+1), -\dfrac{x}{2}$. Notice the minus sign in $-3x^2$ and $-\dfrac{x}{2}$, and that the expressions $4(x+1)$ should be viewed as one term.

Example 3.2 List all explicit factors of the following multiplication.

a) $5ab$ b) $-3(a-1)x$

Solution:

a) Since $5ab = 5 \cdot a \cdot b$, the explicit factors are $5, \quad a, \quad b$.

b) Since $-3(a-1)x = -3 \cdot (a-1) \cdot b$, the factors are; -3, $(a-1)$ and x. Notice that when listing factors, the expressions in parentheses are treated as one "unsplitable" expression.

Algebra is an abstract generalization of arithmetic, where numbers are 'replaced' with variables. ***The laws that are true for numbers also hold for algebraic expressions*** *(recall, algebraic expressions are merely symbolic representations of numbers).*

We will discuss some of the laws in the context of equivalent expressions.

Commutative property of addition: rearranging terms results in equivalent expressions

We know that $3+5$ and $5+3$ are both equal to the same number, 8. It is because the result of addition does not depend on the order of numbers that are being added. This property is called the commutative property of addition. Remember, subtraction does not have this property: $5-3 \neq 3-5$. But, if we view subtraction as the addition of the opposite number, we get $5-3 = 5+(-3) = -3+5$. With the use of variables, we can express the above ideas in a general form (without the use of specific numbers). For any value of x and *y*,

Commutative Property of Addition

$$x + y = y + x$$

Also, since $x - y = x + (-y) = -y + x$, we have

Consequence of Commutative Property of Addition

$$x - y = -y + x$$

Equivalently, we can say that, **changing the order of terms results in an equivalent expression** (an expression that looks different, but 'means' the same). For example,

The terms of $a + 3$ are a and 3. If we reverse the order of terms, we obtain $a + 3 = 3 + a$.

The terms of $a^2 - b$ are a^2 and $-b$. Reversing their order gives us $a^2 - b = -b + a^2$.

The terms of $-c - 2d$ are $-c$ and $-2d$. Reversing their order gives us $-c - 2d = -2d - c$.

Similarly, if an algebraic expression consists of more than two terms, we can rearrange them in any order. For instance, the terms of $3 - B + C$ are 3, $-B$, and C. Thus, we can rewrite the expression $3 - B + C$ as $-B + C + 3$ or $3 - B + C$:

$3 - B + C = -B + C - 3 = 3 - B + C$ (there are more possible rearrangements).

<u>Example 3.3</u> Using the fact that changing the order of terms results in an equivalent expression, rewrite the following expressions in their equivalent form by rearranging the terms. Use the equal sign to indicate that the resulting expressions are equivalent.

a) $7a - b$ b) $-(b + c)^2 + 3a$

Solution:

a) The terms are $7a$ and $-b$. We get $7a - b = -b + 7a$

b) The terms of $-(b + c)^2 + 3a$ are $-(b + c)^2$ and $3a$. If we reverse the order, we get $-(b + c)^2 + 3a = 3a - (b + c)^2$

<u>Example 3.4</u> Determine which of the following expressions are equal to $2x - 5y + 4z$.

a) $2x + 4z - 5y$ b) $-5y + 4z + 2x$ c) $-5y + 2x + 4z$

Solution:

All of them are. The terms of $2x - 5y + 4z$ are $2x, -5y$ and $4z$. As long as the sign that is in front of an expression is not altered, we can rearrange terms. The sign that goes before $2x$ is plus, and in all expressions (a)-(c) $2x$ is also preceded by a plus sign. $5y$ follows the minus sign and the same is true for all expressions (a)-(c). Finally, $4z$ is preceded by a plus sign in all these expressions.

Example 3.5 Replace Ψ with expressions such that the resulting statements are true. Use parentheses when needed.

a) $2x - y + 5z = \Psi + 2x$ b) $-xyz = z \cdot \Psi$

Solution:

a) $2x - y + 5z = -y + 5z + 2x$; $\Psi = -y + 5z$

b) $-xyz = z(-x)y$; $\Psi = (-x)y$

Commutative property of multiplication. Rearranging factors results in equivalent expressions

Multiplication, like addition, is commutative. the result of multiplication does not depend on the order, $3 \times 5 = 5 \times 3$. In general, we have

Commutative Property of Multiplication

$$xy = yx \text{ or } x \cdot y = y \cdot x$$

This means that, **the rearrangement of the order of factors results in an equivalent expression**. For example,

$2a = a \cdot 2$, $x^3 y = yx^3$ or $(a+b)x = x(a+b)$.

Factors can also be rearranged if we have more than two factors. For example,

$3ab = ba \cdot 3$,

$cdef = dcef = efdc = fedc$ (or any other order of factors of *c*, *d*, *e*, and *f*)

Example 3.6 Rewrite the expression $y \cdot x \cdot \dfrac{a+1}{b}$ in two equivalent forms by multiplying its factors in a different order.

Solution:

We can rewrite the above expression in more than two equivalent forms. For example, $x \cdot y \cdot \dfrac{a+1}{b}$, $x \cdot \dfrac{a+1}{b} \cdot y$, $\dfrac{a+1}{b} \cdot yx$, $\dfrac{a+1}{b} \cdot x \cdot y$, or $y \cdot \dfrac{a+1}{b} \cdot x$. Any two would be the correct answer.

Applying rules of operations on fractions results in an equivalent expression

The rules for addition and subtraction of fractions with common denominators are

Rule for Addition and Subtraction of Fractions with Common Denominators

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}, \qquad c \neq 0$$

For example, $\frac{2}{7}+\frac{x}{7}=\frac{2+x}{7}$ and thus also $\frac{2+x}{7}=\frac{2}{7}+\frac{x}{7}$ (notice, that we switched the left side with the right one of the equation, so both equations are true, just like if $2+3=5$ then $5=2+3$). It is also true that $\frac{a-3}{x}=\frac{a}{x}-\frac{3}{x}$ and $\frac{2}{x-3}-\frac{y}{x-3}=\frac{2-y}{x-3}$.

The rules for multiplication and division of fractions:

Rule for Multiplication and Division of Fractions

$$a\cdot\frac{x}{y}=\frac{ax}{y}=\frac{a}{y}\cdot x \qquad x\div y=x\cdot\frac{1}{y}=\frac{x}{y}, \qquad y\neq 0$$

Thus, we should recognize that the following are true:

$$\frac{x}{2}=\frac{1}{2}x, \qquad \frac{3x}{5}=\frac{3}{5}x, \qquad \text{and} \qquad (a+b)\frac{1}{3}=\frac{a+b}{3}$$

Finally, you might recall that $-\frac{1}{2}=\frac{-1}{2}=\frac{1}{-2}$. Similarly, one may show (ask your instructor, if you would like to see how to do it) that

Rule for Negative Signs in Fractions

$$-\frac{x}{y}=\frac{-x}{y}=\frac{x}{-y}, \qquad y\neq 0$$

<u>Example 3.7</u> Rewrite each of the following expressions as a sum or a difference of two expressions. Use $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$ or $\frac{a-b}{c}=\frac{a}{c}-\frac{b}{c}$ (we assume that $c\neq 0$). Indicate with the equal sign that the resulting expressions are equivalent.

a) $\frac{2+s}{5}$

b) $\frac{a^2-2}{5+x}$

Solution:

a) $\frac{2+s}{5}=\frac{2}{5}+\frac{s}{5}$

b) $\frac{a^2-2}{5+x}=\frac{a^2}{5+x}-\frac{2}{5+x}$

Example 3.8 Use the fact that $\frac{ax}{y} = \frac{a}{y}x$ to rewrite the following expressions as a product of a numerical factor and an algebraic expression.

a) $\frac{2b}{5}$ b) $\frac{x+b}{5}$

Solution:

a) $\frac{2b}{5} = \frac{2}{5}b$; $\frac{2}{5}$ is a numerical factor.

b) $\frac{x+b}{5} = \frac{1}{5}(x+b)$ Notice the use of parentheses. The entire expression in the numerator should be multiplied by $\frac{1}{5}$. Had we omitted the parentheses, only x would have been multiplied by $\frac{1}{5}$.

Example 3.9 Rewrite each of the following expressions in their equivalent form as a single fraction.

a) $\frac{6}{11}a$ b) $2\frac{x}{y}$

c) $\frac{1}{3}(a^2+1)$ d) $\frac{a}{x-2}+\frac{2b}{x-2}$

e) $-\frac{a}{4s}-\frac{1}{4s}a^2$ f) $2\frac{a-1}{x}-\frac{b+c}{x}$

Solution:

a) $\frac{6}{11}a = \frac{6a}{11}$

b) $2\frac{x}{y} = \frac{2x}{y}$

c) $\frac{1}{3}(a^2+1) = \frac{1(a^2+1)}{3} = \frac{a^2+1}{3}$ Write a^2+1 instead of $1(a^2+1)$!

d) $\frac{a}{x-2}+\frac{2b}{x-2} = \frac{a+2b}{x-2}$

e) $-\frac{a}{4s}-\frac{1}{4s}a^2 = \frac{-a}{4s}-\frac{a^2}{4s} = \frac{-a-a^2}{4s}$ First replace $-\frac{a}{4s}$ by $\frac{-a}{4s}$, and $\frac{1}{4s}a^2$ by $\frac{a^2}{4s}$ (equals can be substituted by equals). Then subtract, just as you subtract fractions.

f) $2\frac{a-1}{x}-\frac{b+c}{x} = \frac{2(a-1)}{x}-\frac{b+c}{x} = \frac{2(a-1)-(b+c)}{x}$ Notice the use of parentheses.

Example 3.10 Determine which of the following expressions are equal to $\frac{2-b}{c}$.

$$\frac{b-2}{c}, \qquad \frac{2}{c}-\frac{b}{c}, \qquad \frac{1}{c}(2-b), \qquad \frac{-b+2}{c}$$

Solution:
The only expression that is not equivalent is $\frac{b-2}{c}$ (since $b-2 \neq 2-b$). To demonstrate that, let us set $b=0$, $c=1$ and evaluate $\frac{2-b}{c}=\frac{2-0}{1}=2$, $\frac{b-2}{c}=\frac{0-2}{1}=-2$. Since $-2 \neq 2$, we conclude that $\frac{2-b}{c}$ is not equivalent to $\frac{b-2}{c}$. The expression $\frac{2}{c}-\frac{b}{c}$ is equivalent because of the rules for addition of fractions. $\frac{1}{c}(2-b)$ is equivalent because of the rules for multiplication of fractions. $\frac{-b+2}{c}$ is equivalent because we can replace $2-b$ by its equivalent expression $-b+2$.

Performing numerical operations results in equivalent expressions

There are many operations one can perform on an algebraic expression to obtain an equivalent one. One of them is performing numerical operations according to the order of operations. For example,

$$2+3+x=5+x$$
$$2\cdot 3x=6x$$

Example 3.11 When possible, perform a numerical operation to create an equivalent expression. If no numerical operation can be performed, clearly indicate so.

a) $-2+x+1$ b) $\frac{4x}{8}$ c) $12+3x$

d) $(2\times 3)^m$ e) 2×3^m f) $2x(-3)y$

g) $-(-x)$ h) $6\cdot\frac{x}{3}$ i) $(3m)(-2n)$

Solution:

a) $-2+x+1=-2+1+x=-1+x$

b) $\frac{4x}{8}=\frac{x}{2}$ Divide the numerator and denominator by 4 to reduce the fraction.

c) $12+3x$ Since multiplication of 3 and x has to be performed before addition, no numerical operation can be performed.

d) $(2\times 3)^m=6^m$

e) 2×3^m Since 3 must be first raised to m-th power, no numerical operation can be performed.

f) $2x(-3)y = 2(-3)xy = -6xy$
g) $-(-x) = (-1)(-1)x = x$
h) $6 \cdot \frac{x}{3} = 2x$ We can cancel 3 in the denominator with the factor of 3 in $6 = 3 \cdot 2$.
i) $(3m)(-2n) = 3m(-2)n = 3(-2)mn = -6mn$

How to show that two expressions are not equivalent

Two algebraic expressions are equivalent, if for all values of variables they assume the same value. Thus, **if we can find just one set of values of variables for which the expressions do not assume the same value, it is enough to conclude that they are not equivalent.**

As an illustration, we will demonstrate that x^2 is not equivalent to $2x$. To this end, we must find some value of x that when evaluated, the two expressions assume different values. We will use $x = 5$ (the choice of the value of x is arbitrary). We evaluate both algebraic expressions.

$$x^2 = 5^2 = 25$$

and $$2x = 2 \times 5 = 10.$$

Since $25 \neq 10$, we conclude that $2x$ is not equivalent to x^2.
Notice that there are other values of x for which $2x$ is not equal to x^2, but since **we only need one such value**, we already proved that x^2 in not equivalent to $2x$.

Example 3.12 Show that the following two expressions $(2a)^3$ and $2a^3$ are not equivalent by evaluating them when $a = 1$ and demonstrating that the values of the two expressions are not equal.

Solution:
$(2a)^3 = (2 \times 1)^3 = 2^3 = 8$ but $2a^3 = 2 \times 1^3 = 2 \times 1 = 2$. Since $8 \neq 2$, $(2a)^3$ and $2a^3$ are not equivalent.

Common mistakes and misconceptions

Mistake 3.1

When writing an expression like, for example, $-\frac{x}{m} + \frac{s}{m}$, as a single fraction, make sure that all the symbols are clearly above the fraction bar. You should not write

$$-\frac{x+s}{m}$$

The answer should be written as $\frac{-x+s}{m}$, so the minus sign is included in the numerator.

Exercises with Answers (For answers see Appendix A)

Ex. 1 Write a word to complete each sentence.
In the expression $4x^2 \times 2y$, $4x^2$ and $2y$ are called ________________ .
In the expression $4x^2 + 2y$, $4x^2$ and $2y$ are called ________________ .

Ex.2 List all terms of the following expressions.
a) $3+x$ b) $ab-cd$
c) $\frac{xy}{2}+2y^2-1$ d) $-(2-b)^2+\frac{x}{y}-z$

Ex.3 Is 2+8 equal to 8+2? Is $x+8$ equal to $8+x$? How about $\frac{2a}{b}+\frac{cd}{2}$ and $\frac{cd}{2}+\frac{2a}{b}$? Why?

Ex.4 a) Evaluate $m-n$ and $n-m$ when $m=2$ and $n=3$. Based on this evaluation, can you determine if the two expressions are equivalent?
b) Is it true that $m-n=-n+m$?

Ex.5 For each of the following expressions
- List all its terms
- Using the fact that changing the order of terms results in an equivalent expression, rewrite the following expressions in their equivalent form by rearranging the terms. Use the equal sign to indicate that the resulting expressions are equivalent (for example, the expression $A+9$ should be rewritten as $A+9=9+A$).

a) $2m+z$ b) $x-2$
c) $-3c+2$ d) $-2x^2-\frac{y^3}{2}$
e) $c(d-f)+y^2$ f) $-(x-y)^2+\frac{s+2}{3}$

Ex.6 Fill in the blanks to make a true statement.
a) $x-mn+2=-mn+2$__________ b) $3-(2a-3b)+4x=4x$__________

Ex.7 List all terms, and then, by changing the order of these terms, create two new equivalent expressions for each of the following.
a) $-x^2+x-x^3$ b) $-a^2-2bc+\frac{3x}{2}$

Ex.8 For each of the following expressions (1)-(5) find an expression equivalent to it among expression (A)-(E). Rewrite each matched pair with the equal sign between them to indicate their equivalence.

(1) $s+t+u$ (A) $t-u+s$
(2) $-t+s+u$ (B) $-s-t+u$
(3) $-u+s+t$ (C) $t+s+u$
(4) $u-t-s$ (D) $s-u-t$
(5) $s-t-u$ (E) $s+u-t$

Ex. 9 Rewrite the following expressions placing the multiplication sign '$\times$' whenever (according to the convention) it was omitted. Then, identify all explicit factors.

a) $2a$ b) $3(a+b)$

c) $-3x\dfrac{2}{y}$ d) $4(x+y)(b-c)$

Ex.10 Rewrite each of the following expressions in its equivalent form using $xy = yx$. Use the equal sign to indicate that the resulting expression is equivalent to the original one (for example, the expression $9A$ should be rewritten as $9A = A\cdot 9$). Remember about parentheses.

a) mn b) -5×7

c) $-cd$ d) $-c(a+d)$

Ex.11 a) Rewrite the expression vst in its equivalent form by changing the order of its factors to create three new equivalent expressions. Indicate their equivalence by using the equal sign (for example, one of the answers might be $vst = tsv$).

b) Repeat the above exercise for $v(x-y)t$.

Ex.12 a) Is AB equivalent to BA? How about $-3AB$ and $-3BA$, $-3AB$ and $BA(-3)$? Why?

b) Is $-3AB$ equivalent to $BA-3$? Why? Support your answer by evaluating both expressions when $A=1$ and $B=2$.

Ex.13 Is $ab+2$ equivalent to $2+ab$? How about $ab+2$ and $2+ba$? Why? How about $(mn+4)(a+b)$ and $(b+a)(4+nm)$? Why?

Ex.14 According to the rules for adding and subtracting fractions, we have $\dfrac{a+b}{c} = \dfrac{a}{c}+\dfrac{b}{c}$ and $\dfrac{a-b}{c} = \dfrac{a}{c}-\dfrac{b}{c}$ (assume $c \neq 0$) Rewrite each of the expressions below as a sum or a difference of two expressions. Use equal signs to indicate that the resulting expressions are equivalent to the original ones (for example, the expression $\dfrac{2-t}{3}$ should be rewritten as $\dfrac{2-t}{3} = \dfrac{2}{3}-\dfrac{t}{3}$).

a) $\dfrac{2-5}{7}$ b) $\dfrac{a+6}{3}$

c) $\dfrac{a-2}{a+b}$ d) $\dfrac{b^2+c}{ab^2-c}$

Ex.15 Using the fact that $\dfrac{a+b}{c} = \dfrac{a}{c}+\dfrac{b}{c}$ and $\dfrac{a-b}{c} = \dfrac{a}{c}-\dfrac{b}{c}$ (assume $c \neq 0$), rewrite the following expressions as a single fraction. Do not simplify.

a) $\dfrac{m}{4}+\dfrac{n}{4}$ b) $\dfrac{7m}{4}-\dfrac{n^2}{4}$

c) $\dfrac{5m}{4c-2}-\dfrac{2n^2}{4c-2}$ d) $\dfrac{A}{x}-\dfrac{B+2C}{x}$

Ex.16 Write the following expressions as a single fraction using the fact that $\frac{a}{d}+\frac{b}{d}+\frac{c}{d}=\frac{a+b+c}{d}$

a) $\frac{4}{5}-\frac{7}{5}+\frac{2}{5}$

b) $\frac{7m}{4x}+\frac{n^2}{4x}-\frac{3}{4x}$

c) $\frac{m}{s-1}-\frac{3}{s-1}-\frac{t}{s-1}$

Ex.17 According to the rule for multiplication of fractions the following is true: $\frac{3x}{7}=\frac{3}{7}x$. We can say that the quotient $\frac{3x}{7}$ was written as a product of a numerical factor $\frac{3}{7}$ and an algebraic expression x. Write the following expressions as a product of a numerical factor and an algebraic expression.

a) $\frac{2x}{3}$

b) $\frac{2x^2}{y}$

c) $\frac{-2x^2}{3}$

d) $\frac{-2(a+2b)}{y}$

e) $\frac{x}{3}$

f) $\frac{a+2b}{3}$

Ex.18 Using the rule for multiplication of fractions $a\frac{x}{y}=\frac{ax}{y}$, rewrite each of the following expressions in their equivalent form as a single fraction (for example, $2\cdot\frac{x}{y}=\frac{2x}{y}$). Remember to use parentheses when needed.

a) $3\cdot\frac{m}{n}$

b) $3\cdot\frac{-m}{n}$

c) $a\left(\frac{-b}{4}\right)$

d) $-a\left(\frac{-b}{4}\right)$

e) $(s-4)\frac{t}{n}$

f) $4\cdot\frac{-a}{n-1}$

g) $3\frac{m+n}{t}$

h) $a\cdot\frac{-x+1}{x^2}$

Ex.19 Students were to write an answer to the following problem: Using algebraic symbols**,** write an opposite number to $\frac{s}{t}$.

Student A gave the answer: $-\frac{s}{t}$

Student B gave the answer: $\frac{-s}{t}$

Student C gave the answer: $\dfrac{s}{-t}$

Who was right? Why?

Ex.20 You know that $-\dfrac{x}{y}=\dfrac{-x}{y}=\dfrac{x}{-y}$. By placing the minus sign differently, write each expression in two additional equivalent ways. Use parentheses when needed.

a) $-\dfrac{2a}{b}$

b) $\dfrac{2a+c}{-2d}$

Ex.21 Write each of the following expressions in their equivalent form as a single fraction. Do not simplify.

a) $\dfrac{2}{3}x+\dfrac{4}{3}y$

b) $\dfrac{2}{3}x-\dfrac{1}{3}y$

c) $2x\cdot\dfrac{1}{3}-7\cdot\dfrac{y}{3}$

d) $\dfrac{3}{t}x-\dfrac{1}{t}y$

e) $-2\dfrac{x}{t}+3\dfrac{y}{t}$

f) $-\dfrac{1}{k+t}-2\dfrac{n}{k+t}$

g) $3\dfrac{a-b}{t}+2\dfrac{cd-1}{t}$

h) $-\dfrac{2+a}{xy}-3\dfrac{m-n}{xy}$

Ex.22 Fill in the blanks to make a true statement.

a) $\dfrac{3x}{y}=x\cdot$ ____________

b) $\dfrac{3x}{y}=3\cdot$ __________

c) $\dfrac{3x}{y}=\dfrac{1}{y}\cdot$ ____________

d) $\dfrac{a+b}{y}=\dfrac{1}{y}\cdot$ _________

Ex.23 Perform all numerical operations that are possible. If none are possible, write "not possible".

a) $4+a-8$

b) $4a(-8)$

c) $\dfrac{8a}{4}$

d) 8^2x

e) $8x^2$

f) $4-2x$

g) $(4-2)x$

h) $-(-2x)$

i) $(5-2)^m$

j) $2x^2\left(\dfrac{1}{2}\right)$

k) $-0.1x(10y^2)$

l) $5-2^m$

m) $10\cdot\dfrac{x}{5}$

n) $2\cdot(6-5)x$

o) $4\cdot\dfrac{2}{5}-x$

p) $\dfrac{12y}{2xz}$

q) $-\dfrac{1}{2}-x-\dfrac{1}{2}$

r) $\dfrac{3bd}{9ac}$

s) $(0.2xy)(-0.3z)$ t) $\frac{4x}{-1}$

Ex.24 Is $\left(\left(-\frac{1}{2}\right)^2+\frac{3}{4}\right)x$ equivalent to x? Is $x\left(\left(-\frac{1}{2}\right)^2+\frac{3}{4}\right)$ equivalent to x?

Ex.25 Is $(x+y)(1+a)$ equivalent to $x+y(1+a)$? Explain your answer.

Ex.26 Replace Ψ with expressions such that the resulting statement is true. Use parentheses when needed.

a) $a-2b+c=c-2b+\Psi$ b) $x=\frac{x}{\Psi}\cdot 4$

c) $\frac{xy}{4}=\frac{1}{4}x\Psi$ d) $a\div 2=\frac{\Psi}{2}$

e) $\frac{x+y}{3}=\frac{x}{3}+\Psi$ f) $z-c=-c+\Psi$

g) $-xyz=yz\Psi$ h) $\frac{ab}{2}=a\cdot\Psi$

Ex.27 Determine which of the following expressions are equivalent to $-m$

$\frac{m}{-1}$, $-\frac{m}{1}$, $m(-1)$, $m-1$, $-\frac{1}{m}$

Ex.28 Determine which of the following expressions are equivalent to $m-n$:

$n-m$, $m(-n)$, $m-(n)$, $-nm$, $-n+m$, $(-1)n-m$

Ex.29 Determine which of the following are equivalent to $-a-c+b+d$.

$-a+b-c+d,$ $-a+b-c-d,$ $d+b-c-a,$ $a+c-b-d$

Ex.30 Determine which of the following are equivalent to $-\frac{a}{b}$.

$\frac{-a}{b}$, $-a\cdot\frac{1}{b}$, $-b\cdot\frac{1}{a}$, $-\frac{2a}{2b}$, $\frac{a}{-b}$, $\frac{-a}{-b}$

Ex.31 Determine which of the following expressions are equivalent to $\frac{5x}{6}$.

$5\cdot\frac{x}{6}$, $\frac{5}{6}x,$ $x\cdot\frac{5}{6}$, $\frac{5}{6x}$, $\frac{10x}{12}$, $\frac{1}{6}\cdot 5x$

Ex.32 Determine which of the following expressions are equivalent to $x\div 2$.

$\frac{x}{2}$, $2x$, $2\div x$, x^2, $\frac{4x}{8}$, $\frac{-x}{-2}$

Ex.33 Determine which of the following expressions are equivalent to $3+8a$.

$11a$, $8a+3$, $3+a\cdot 8$, $11+a$, $2\cdot\dfrac{3+8a}{2}$

Ex.34 Determine which of the following expressions are equivalent to $\dfrac{3a-b}{6}$.

$\dfrac{b-3a}{6}$, $\dfrac{1}{6}(3a-b)$, $\dfrac{3a}{6}-b$, $\dfrac{3a}{6}-\dfrac{b}{6}$, $\dfrac{-b+3a}{6}$, $(3a-b)\dfrac{1}{6}$

Ex.35 Determine which of the following are equivalent to $\dfrac{m}{3}-\dfrac{n}{3}$.

$\dfrac{n}{3}-\dfrac{m}{3}$, $\dfrac{m-n}{3}$, $\dfrac{1}{3}\cdot m-\dfrac{1}{3}n$, $(m-n)\cdot\dfrac{1}{3}$, $\dfrac{-n+m}{3}$

Ex.36. The correct answer to a problem is $\dfrac{vt^2}{2}$. John's answer is $\dfrac{1}{2}vt^2$. Is John right? How about Mary whose answer is $\dfrac{t^2}{2}\cdot v$?

Ex.37 Show that $(-x)^4$ is not equivalent to $-x^4$ by evaluating both expressions when $x=-1$ and demonstrating that the values are not the same.

Ex.38 Show that x^2+y^2 is not equivalent to $(x+y)^2$ by evaluating both expressions when $x=-1$, $y=2$ and demonstrating that the values are not the same.

Ex.39 Show that $m-n+p$ is not equivalent to $m-(n+p)$ by evaluating both expressions when $m=2$, $n=5$, and $p=1$ and demonstrating that the values are not the same.

Ex.40 Evaluate $(-1)^m$ and -1^m when

a) $m=1$
b) $m=3$
c) $m=5$
d) $m=7$
e) Based on the above, can you determine if $(-1)^m$ and -1^m are equivalent?
f) Evaluate $(-1)^m$ and -1^m when $m=2$. Can you now determine if $(-1)^m$ and -1^m are equivalent?

Lesson 4

Topics: Operations on power expressions with non-negative integer exponents.

Exponential notation

Exponential notation is used to write repeated factors in a compact way. Exponents are another way of writing multiplication. For example,

$$2^5 = \underbrace{2\times 2\times 2\times 2\times 2}_{5-times}, \qquad 3^2 = \underbrace{3\times 3}_{2-times}.$$

We will extend this idea to algebraic expressions.

Exponential Expression **The expression of the form a^n is called an exponential expression. It is defined as follows**

$$a^0 = 1,$$
$$a^1 = a,$$
$$a^2 = a\times a,$$
$$\dots$$
$$a^n = \underbrace{a\times a\times \dots \times a}_{n \quad times}, \qquad \text{for } n = 1, 2, 3, \dots .$$

(a is repeated as a factor n times).
***a* is called the base, *n* is called an exponent or power.**

Notice that, according to the definition any expression raised to the zero-th power is equal to one. Also, a **variable that appears to have no exponent is raised to the first power.**

$$a^1 = a$$

Recall that a^n is read as "a to the n-th power". In the case of a^2, often, instead of "a to the second power" we read it as "a squared"; a^3 is often read as "a cubed".

You should also remember that:
The exponent pertains only to "the closest" number or variable. To apply the exponent to the entire expression we must place parentheses around the expression. The exponent is then placed outside the parentheses.
For example,

in $2b^3$, only b is raised to the third power.
in $(2b)^3$, $2b$ is raised to the third power.

or

in $-x^n$, only x is raised to the n-th power (n pertains only to x not to $-x$)
in $(-x)^n$, $-x$ is raised to the n-th power

The above convention can be interpreted as the consequence of the order of operations. For example, in $(2x)^2$ parentheses indicate that multiplication should be performed first, and only then the result is raised to the second power. This means that the entire $2x$ is raised to the second power. Without parentheses, we first exponentiate, and then multiply by 2, so only x is squared. Notice also, that without this convention, we would have no means to distinguish between, let's say, $2x^2$ and $(2x)^2$, or $(a+b)^3$ and $a+b^3$.

Example 4.1 Expand, that is write without exponential notation.

a) $(5A)^3$ b) $5A^3$

Solution:

a) The exponent pertains to the entire expression $5A$: $(5A)^3 = 5A\cdot 5A\cdot 5A$

b) The exponent pertains only to A: $5A^3 = 5A\cdot A\cdot A$.

Example 4.2 Rewrite using exponential notation whenever it is possible.

a) $7mmm$ b) $-aaaa-aa$

c) $b(c+2d)bb(2d+c)bb$ d) $\frac{y}{x}\cdot y\cdot y$

Solution:

a) $7\underbrace{mmm}_{3-tomes} = 7m^3$

b) $-\underbrace{aaaa}_{4-times}-\underbrace{aa}_{2-times} = -a^4 - a^2$

c) Notice that $c+2d = 2d+c$, thus

$$b(c+2d)bb(2d+c)bb = \underbrace{bbbbb}_{5-times}\underbrace{(c+2d)(c+2d)}_{2-times} = b^5(c+2d)^2$$

d) Notice that $\frac{y}{x}\cdot y\cdot y$ is equivalent to $\frac{yyy}{x}$, thus

$$\frac{y}{x}\cdot y\cdot y\cdot = \frac{yyy}{x} = \frac{y^3}{x}$$

Example 4.3 Evaluate a) $(2x)^0$ b) $2x^0$ c) $4(mn^7)^0$

Solution:

a) Remember that any expression raised to the zero-th power is equal to 1, and that because of parentheses the zero power pertains to the entire expression $2x$, thus $(2x)^0 = 1$.

b) This time only x is raised to the zero-th power, hence $2x^0 = 2\times 1 = 2$.

c) The expression $m\times n^7$ is raised to the zero-th power, so $4(mn^7)^0 = 4\times 1 = 4$

Numerical Coefficient **In a product of a number, variables or algebraic expressions, the numerical factor is called a numerical coefficient.**

For example, in the expression $2x$, the numerical coefficient (often called the coefficient) is equal to 2. The coefficient of $5(a+b)^m$ is 5 and the coefficient of $-4a^5$ is -4. If there is no number in front of a product, it is implied that the coefficient is 1. If there is a negative sign, it is implied that the coefficient is -1. For instance, the coefficient of x^2y is equal to 1 (because $x^2y = 1\cdot x^2y$). The coefficient of $-a$ is -1 (because $-a = -1\cdot a$).

Example 4.4 In the following expressions, identify bases, exponents and numerical coefficients.

a) $3a^{27}$ b) $\left(\dfrac{yz}{k}\right)^9$ c) $-x^4$

Solution:

a) base: a; exponent: 27; numerical coefficient: 3

b) base: $\dfrac{yz}{k}$; exponent: 9; numerical coefficient: 1

c) base: x; exponent: 4; numerical coefficient: -1

Laws of exponents

Consider $a^3\cdot a^2 = (a^3)\cdot(a^2) = (a\cdot a\cdot a)\cdot(a\cdot a) = a\cdot a\cdot a\cdot a\cdot a = a^5$

We have 3 a's and another 2 a's for a total of $3+2=5$ a's (to multiply a^3 and a^2 we added exponents). The idea can be generalized to obtain

Product Rule for Exponents

$$\boxed{a^m a^n = a^{m+n}}$$

When multiplying exponential expressions with like bases, we add the exponents and keep the common base. For example, $x^{10}x^5 = x^{10+5} = x^{15}$.

If we have

$$\frac{a^5}{a^3} = \frac{a\cdot a\cdot a\cdot a\cdot a}{a\cdot a\cdot a} = \frac{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}\cdot a\cdot a}{\cancel{a}\cdot\cancel{a}\cdot\cancel{a}} = a\cdot a = a^2, \qquad a\neq 0$$

Out of 5 repeated a's in the numerator, we canceled 3 a's for a total $5-3=2$ a's (to divide a^5 by a^3 we subtracted exponents). In general one can prove that

Quotient Rule for Exponents

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

When dividing exponential expressions with like bases, subtract the exponents and keep the common base. For example, $\frac{x^{15}}{x^5} = x^{15-5} = x^{10}$

If we have $(a^2)^3 = (a^2)\cdot(a^2)\cdot(a^2) = (a\cdot a)\cdot(a\cdot a)\cdot(a\cdot a) = a\cdot a\cdot a\cdot a\cdot a\cdot a = a^6$
We can think of this as 3 groups of 2 for a total of $2\times 3 = 6$ a's. In general,

Power Rule for Exponents

$$(a^m)^n = a^{mn}$$

When raising an exponential expression to another power, keep the same base, and multiply the exponents. For example, $(x^{10})^5 = x^{10\times 5} = x^{50}$.

Now, consider $(ab)^3 = (ab)\cdot(ab)\cdot(ab) = (a\cdot a\cdot a)\cdot(b\cdot b\cdot b) = a^3b^3$
This can be extended to

Product to Powers Rule for Exponents

$$(ab)^n = a^n b^n$$

An exponent outside the parentheses applies to all parts of a product inside the parentheses and thus to raise a product to a power, one can equivalently raise each factor to that power.
For example, $(2x)^5 = 2^5 x^5$.

Finally, $\left(\frac{a}{b}\right)^3 = \frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b} = \frac{a\cdot a\cdot a}{b\cdot b\cdot b} = \frac{a^3}{b^3}, \qquad b \neq 0$
In general,

Quotient to Powers Rule for Exponents

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \quad b \neq 0$$

An exponent outside the parentheses applies to all parts of a quotient inside the parentheses. Thus, to raise a quotient to a power, one can equivalently raise both numerator and denominator to that power. For example, $\left(\frac{x}{3}\right)^5 = \frac{x^5}{3^5}$.

For your convenience, we will display all the rules together.

Laws of Exponents			
1. $a^0 = 1$	2. $a^1 = a$	3. $a^m a^n = a^{m+n}$	4. $\dfrac{a^m}{a^n} = a^{m-n}$, $a \neq 0$
5. $(a^m)^n = a^{mn}$	6. $(ab)^n = a^n b^n$	7. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$, $b \neq 0$	

Example 4.5 Perform the indicated operations and simplify.

a) $3x(-2)(-4x)$ b) $\dfrac{8x^5}{6x^2}$ c) $(2x^6)^3$

d) $\dfrac{-y^5}{y^8} \cdot y^3$ e) $(2a^2b)(-a^3b^2)$ f) $\dfrac{(a+b)^7}{2(a+b)^3}$

Solution:

a) $3x(-2)(-4x) = 3(-2)(-4)xx = 24x^2$

b) $\dfrac{8x^5}{6x^2} = \dfrac{4}{3}x^{5-2} = \dfrac{4}{3}x^3$

c) $(2x^6)^3 = 2^3(x^6)^3 = 8x^{6\times 3} = 8x^{18}$

d) $\dfrac{-y^5}{y^8} \cdot y^3 = \dfrac{-y^5y^3}{y^8} = \dfrac{-y^{5+3}}{y^8} = \dfrac{-y^8}{y^8} = -y^{8-8} = -y^0 = -1y^0 = -1$

e) $(2a^2b)(-a^3b^2) = 2(-1)a^2a^3bb^2 = -2a^{2+3}b^{1+2} = -2a^5b^3$

f) $\dfrac{(a+b)^7}{2(a+b)^3} = \dfrac{1}{2}(a+b)^{7-3} = \dfrac{1}{2}(a+b)^4$

Example 4.6 Rewrite in its equivalent form as a single exponential expression without parentheses. Identify the numerical coefficient of the final expression.

a) $(-y)^3$ b) $(-2y)^3 2y^2$

Solution:

a) $(-y)^3 = (-1y)^3 = (-1)^3 y^3 = -1y^3 = -y^3$; the coefficient: -1

b) $(-2y)^3 2y^2 = (-2)^3 y^3 2y^2 = -8 \times 2y^3y^2 = -16y^{3+2} = -16y^5$;

the coefficient: -16.

Example 4.7 Write as an algebraic expression using parentheses where appropriate, then remove the parentheses and simplify.

a) The product of $-x$ and y^2, then raised to the sixth power

b) The quotient of $4a^2b^3$ and ab^2 , then raised to the second power

Solution:

a) $\left((-x)(y^2)\right)^6 = (-1x)^6(y^2)^6 = (-1)^6 x^6 y^{2\times 6} = 1x^6 y^{12} = x^6 y^{12}$

b) $\left(\dfrac{4a^2b^3}{ab^2}\right)^2 = (4a^{2-1}b^{3-2})^2 = (4a^1b^1)^2 = (4ab)^2 = 4^2a^2b^2 = 16a^2b^2$

Example 4.8 Simplify the expression $\dfrac{(3x^5)^2}{x^3x^4}$ and then evaluate, when $x = -\dfrac{1}{2}$.

Solution:

$$\frac{(3x^5)^2}{x^3x^4} = \frac{3^2(x^5)^2}{x^{3+4}} = \frac{9x^{5\times 2}}{x^7} = \frac{9x^{10}}{x^7} = 9x^{10-7} = 9x^3$$

To evaluate, we substitute $x = -\dfrac{1}{2}$.

$$9x^3 = 9\left(-\frac{1}{2}\right)^3 = 9\left(\frac{-1}{2}\right)^3 = 9\cdot\frac{(-1)^3}{2^3} = 9\left(\frac{-1}{8}\right) = -\frac{9}{8}.$$

Example 4.9 Which of the following expressions are equivalent to $4x^2y^7$?

$4y^7x^2$, $(2x)^2y^7$, $4x^2(y^5)^2$, $4xy^7x$, $4x^2y^3y^4$

Solution:

$4x^2(y^5)^2 = 4x^2y^{5\times 2} = 4x^2y^{10}$, and hence $4x^2(y^5)^2$ is not equivalent. All others are equivalent. $4y^7x^2$ is equivalent because only the order of factors y^7 and x^2 has been changed.

$(2x)^2y^7 = 2^2x^2y^7 = 4x^2y^7$,

$4xy^7x = 4xxy^7 = 4x^2y^7$,

$4x^2y^3y^4 = 4x^2y^{3+4} = 4x^2y^7$.

Example 4.10 Find the numerical value of Ω such that the following statements are true.

a) $4^6 = 2^\Omega$ b) $36^3 \times 6^7 = 6^\Omega$

Solution:

a) We must express 4^6 as an exponential expression with the base 2 (to match it to 2^Ω). Since $4 = 2^2$, we get $4^6 = (2^2)^6 = 2^{12}$. As a result $2^{12} = 2^\Omega$, hence $\Omega = 12$.

b) We must express $36^3 \times 6^7$ as an exponential expression with the base 6. Notice that $36 = 6^2$, and thus $36^3 \times 6^7 = (6^2)^3 6^7 = 6^6 6^7 = 6^{13}$; $\Omega = 13$.

Example 4.11 Evaluate. a) $\dfrac{9^{81}}{9^{80}}$ b) $\dfrac{4^7 \times 4^6}{4^{11}}$

Solution:

a) $\dfrac{9^{81}}{9^{80}} = 9^{81-80} = 9$

b) $\dfrac{4^7 \times 4^6}{4^{11}} = \dfrac{4^{7+6}}{4^{11}} = \dfrac{4^{13}}{4^{11}} = 4^{13-11} = 4^2 = 16$.

Common mistakes and misconceptions

Mistake 4.1

There is a difference between $-x^2$ and $(-x)^2$. In $-x^2$ only x is squared, in $(-x)^2$, $-x$ is squared, $(-x)^2 = (-x)(-x) = x^2$. Just like $-3^2 = -9$, while $(-3)^2 = 9$.

Mistake 4.2

In the expression $x^7 y^3$, since the bases are not the same, DO NOT add exponents.

Mistake 4.3

Although it is true that $(ab)^2 = a^2b^2$**,** $(a+b)^2 \neq a^2 + b^2$.

We recognize that $(7x)^2 = 49x^2$, however, $(7+x)^2 \neq 49 + x^2$.

Mistake 4.4

Please, remember $x^2x^5x = x^2x^5x^1 = x^{2+5+1} = x^8$ (not $x^{2+5} = x^7$). In other words, if one of the factors does not have an explicit exponent, it means it is raised to the first power, and thus one has to be added.

Exercises with Answers (For answers see Appendix A)

Ex.1 In the expression $3x^m$, 3 is called the____________________ , m is called the____________ or ____________ and x is called the______________ .

Ex.2 Fill in the blanks.

a) An expression x raised to the________ power is equal to itself.

b) An expression x raised to the________ power is equal to 1.

Ex.3 a) In the expression ab^m the exponent pertains to ________________.

b) In the expression $(ab)^n$ the exponent pertains to ________________.

c) In the expression $c(de)^n$ the exponent pertains to ________________.

d) In the expression $(-a)^n$ the exponent pertains to ________________.

e) In the expression $-a^n$ the exponent pertains to ________________.

f) In the expression $\left(\dfrac{2x}{y}\right)^m$ the exponent pertains to ________________.

Ex.4 Write the following statements as algebraic expressions using parentheses where appropriate.

a) The quotient of a and 2, then raised to the fourth power
b) Two thirds of a, then raised to the third power
c) c cubed, and then divided by 7
d) The product of a and 5, then raised to the second power
e) Raise a to the second power, and then multiply the result by 8.
f) The opposite of x, then raised to the fifth power.
g) Raise x to the tenth power, and then take the opposite of the result.

Ex.5 In each of the following expressions, identify the base, exponent and numerical coefficient.

a) $3x^4$
b) $-x^m$
c) $\dfrac{2x^3}{3}$
d) $-(a-bc)^2$
e) $\left(\dfrac{x}{y}\right)^m$
f) $\dfrac{(x+y)^7}{4}$
g) $\dfrac{3}{4}\left(\dfrac{3x+z}{w}\right)^7$
h) $-\dfrac{(ab)^5}{2}$

Ex.6 Write using exponential notation whenever it is possible.

a) $6\times 6\times 6\times 6\times 6$
b) $zzzz$
c) $3a\cdot 3a\cdot 3a\cdot a$
d) $-xyxyxxx$
e) $-a-aaaa$
f) $xxy-yyx$
g) $(a+b)(a+b)(a+b)$
h) $(2t^3)(2t^3)(2t^3)(2t^3)$
i) $m+nn+m$
j) $\dfrac{kkk}{n}\cdot k$
k) $\dfrac{-z-z-z}{zzzz}$
l) $\dfrac{(-z)(-z)(-z)}{z+z+z}$
m) $x\cdot\dfrac{x}{2}\cdot x\cdot x$
n) $(3-x)(-x+3)(3-x)$
o) $\left(\dfrac{2a}{b}\right)\left(2\dfrac{a}{b}\right)\left(\dfrac{2a}{b}\right)$
p) $\left(-\dfrac{3}{x}\right)\left(\dfrac{-3}{x}\right)\left(\dfrac{3}{-x}\right)$
q) $\dfrac{1}{(w+2v)(2v+w)(2v+w)}$
r) $\left(\dfrac{x}{m}-\dfrac{y}{m}\right)\left(\dfrac{x-y}{m}\right)\left(\dfrac{x-y}{m}\right)$
s) $(m+n)(m+n)m+n$
t) $(m+p-n)(p+m-n)(n+m-p)$

Ex.7 Write the following expressions without using exponential notation.

a) $(-4)^5$
b) -4^5
c) $(-m)^3$
d) $-m^3$
e) $(2a)^3$
f) $2a^3$
g) $(a+b)^2$
h) $a+b^2$

Ex.8 Simplify by raising to the indicated power.

a) $(3x)^0$ b) $3x^0$

c) $3^0 x$ d) $a(b+c)^0$

e) abc^0 f) $(abc)^0$

g) $ab+c^0$ h) $a^0b^0c^0$

Ex. 9 While copying expressions from a blackboard, John kept forgetting to copy parentheses.

a) $(x)^3$ John copied as x^3

b) $(-x)^4$ John copied as $-x^4$

c) $-(x)^7$ John copied as $-x^7$

d) $a+(2b)^3$ John copied as $a+2b^3$

e) $(a+2b)^3$ John copied as $a+2b^3$

f) $a+2(b)^3$ John copied as $a+2b^3$

g) $a(bc)^m$ John copied as abc^m

h) $\left(\frac{2x}{y}\right)^4$ John copied as $\frac{2x^4}{y}$

Determine if what was on the board and what John copied has the same meaning, i.e. are the parentheses necessary. Write “same” or “different” for each.

Ex.10 Evaluate a) $2x^3$ b) $(2x)^3$

when $x=100$. Explain why you did not get the same result.

Ex.11 Fill in the blanks, each time using one of the following words: add, subtract, multiply, divide.

a) To multiply exponential expressions with the same bases one needs to ___________ the exponents.

b) To ______________ exponential expressions with the same bases one needs to subtract their exponents.

c) To raise an exponential expression to another power one needs to __________ the exponents.

Ex.12 Write as a single exponential expression.

a) $n^3 \cdot n^{20}$ b) $(s^7)^2$

c) $\frac{x^8}{x^2}$ d) b^7b

e) $(-4x)^2$ f) $2m^4m^5$

g) $\frac{1}{b}\cdot b^2$ h) $\frac{x}{x}$

i) $\frac{9a^{20}}{a^4}$ j) $x^3x^4x^5$

k) $\frac{a^4}{2a^2}$ l) $x^4 \cdot \frac{3}{x}$

m) $6a^{11} \cdot a^3 \cdot a^{18}$

n) $\dfrac{0.5s^{12}}{0.1s}$

o) $-5(t^3)^4$

p) $(2x^7)^3$

Ex.13 Write as an algebraic expression using parentheses where appropriate, then remove the parentheses and simplify.

a) The product of m^2 and $-2m$

b) The quotient of $3x^5$ and $18x^3$

c) The expression $2a^5$ raised to the third power

Ex.14 Simplify the following expressions first, and then evaluate when $a = 2$.

a) $\dfrac{a^7}{a^6}$

b) $2a^2a^2$

c) $(2a)(-\dfrac{1}{2}a^2)$

d) $\dfrac{1}{(a^2)^3} \cdot a^8$

Ex.15 Simplify the following expressions first, and then evaluate when $m = -1$.

a) m^3m^5

b) $\dfrac{m^3}{-2m^2}$

c) $\dfrac{(m^2)^{14}}{3}$

d) $2m^7m^{50}$

Ex.16 Simplify the following expression $\dfrac{-x^{10}}{x^8}$, and then evaluate when

a) $x = 7$

b) $x = -7$

c) $x = -\dfrac{2}{3}$

d) $x = -0.07$.

Ex.17 Rewrite in its equivalent form as a single exponential expression without parentheses. Identify the numerical coefficient of the final expression.

a) $(-B)^5$

b) $(-B)^8$

c) $(-B)^5B^3$

d) $-B^2(-B)^2$

Ex.18 Perform the indicated operations and simplify.

a) $-4x^2(-2x)$

b) $(-4x)^2(-2x)$

c) $(3x)(-2)(x^3)$

d) $(3x^2)(4x^5x)$

e) $\dfrac{a^4}{(2a)^2}$

f) $\dfrac{3x^5}{(3x)^2}$

g) $\dfrac{(4x)^3}{-x}$

h) $\dfrac{(-a)^3}{2a^2}$

i) $-2a \cdot \dfrac{a^7}{(-2a)^2}$

j) $\dfrac{-2x^3x^5}{x^4}$

k) $\dfrac{(3x^2)^3}{x^4}$

l) $\dfrac{a^2}{a^4} \cdot a^2$

m) $\left(\dfrac{1}{v}\right)^2 v^7$

n) $(3aa^2)^2$

o) $\left(\dfrac{2}{3}x^4\right)\left(-\dfrac{3}{5}x^2\right)$

p) $-\left(\dfrac{x^3}{0.4}\right)^2$

Ex.19 Write as an algebraic expression using parentheses where appropriate, then remove the parentheses and simplify.

a) The product of $-3x$ and x^2, then raised to the third power

b) The quotient of $4a^{12}$ and a^2, then raised to the second power

c) $-a^3$ raised to seventh power, then multiplied by a

Ex.20 Perform the indicated operations and simplify.

a) $(x^5y^3)^2$

b) $\left(\dfrac{x^2y}{x}\right)^3$

c) $3a(b^5)^2$

d) $3(ab^2)^2$

e) $-\left(\dfrac{a^3}{4b^2}\right)^2$

f) $x^2y(-x^3)y^4$

g) $\dfrac{-ab^7a}{ba^2b^6}$

h) $x^2 \cdot \dfrac{s^2x}{(4s)^2}$

i) $\dfrac{2(x^2)^3y^2}{x} \cdot y$

j) $4(m-n)^2(m-n)^3$

k) $\dfrac{(a+b)^8}{2(a+b)^4}$

l) $\dfrac{a^{13}b^2c^4}{a^5c^2b}$

Ex.21 Write as an algebraic expression using parentheses where appropriate, then remove the parentheses and simplify.

a) xy^7 raised to the fifth power , then divided by y^3

b) $3ab^3$ squared, and then multiplied by b

Ex.22 Simplify the following expressions, and then evaluate when $m=-2$ and $n=1$.

a) $\dfrac{m^2n^5}{mn}$

b) $\dfrac{(mn^2)^3}{m(n^2)^3}$

c) $\dfrac{(m+n)^4}{(m+n)^2}$

d) $(2m-n)^0$

Ex.23 Circle all expressions that are equivalent to $\frac{2}{5}y^2$.

$\frac{4}{25}y^2$, $\frac{2y^2}{5}$, $y\cdot\frac{2}{5}\cdot y$, $\left(\frac{2}{5}y\right)^2$, $\frac{2}{5}yy$

Ex.24 Circle all expressions that are equivalent to $\left(\frac{a}{3}\right)^{20}$.

$\frac{a^{20}}{3}$, $\frac{a^{20}}{3^{20}}$, $\frac{(a^{12})^8}{3^{20}}$, $\frac{a^{12}a^8}{3^{20}}$, $\frac{(a^4)^5}{3^{20}}$

Ex.25 Circle all expressions that are equivalent to $4x^3y^2$.

$4y^2x^3$, $(2y)^2x^3$, $(-2)y^2(-2)x^3$, $(4xy)^2x$, $4(xy)^2x$

Ex.26 Circle all expressions that are equivalent to $2a^6b^3c^2$.

$2a^6c^2b^3$, $2a^2b^3c^2a^3$, $2(abc)^{11}$, $\frac{7c^2b^3a^6}{14}$, $2(a^3c)^2b^3$

Ex.27 Replace Ψ with a number so the following are equal.

a) $81^4 = 9^\Psi$

b) $7^\Psi = 49^5$

c) $16^{100} = 2^\Psi$

d) $27^4 = 3^\Psi$

e) $\left(\frac{1}{4}\right)^7 = \left(\frac{1}{2}\right)^\Psi$

f) $0.2^\Psi = (0.04)^5$

Ex.28 Evaluate the following expressions.

a) $\frac{15^{248}}{15^{247}}$

b) $\frac{3^{21}\cdot 3^{10}}{3^{29}}$

c) $\frac{(12^3)^6}{-12^{17}}$

d) $\frac{0.5^{10}\times 0.5^6}{(0.5^7)^2}$

e) $\frac{64^5}{4^{13}}$

f) $-\frac{4^{15}}{2^{27}}$

Lesson 5

Topics: Multiplication of algebraic expressions; The Distributive Law; Factorization of a common factor; Factorization of -1; Simplification of algebraic fractions with the use of factorization.

This section is about the Distributive Law, the law that relates addition to multiplication. In many cases, it allows us to rewrite the product of expressions as a sum or the sum of expression as a product.

The Distributive Law

If 5 girls and 7 boys get 2 cookies each, then the girls get $2 \times 5 = 10$ cookies, the boys get $2 \times 7 = 14$ cookies, so the total number of cookies they all receive is

$$2 \times 5 + 2 \times 7 = 10 + 14 = 24$$

Alternatively, one can find the number of cookies by adding the number of girls and boys first, $5 + 7 = 12$ and then since each of the children gets 2 cookies, together they get

$$2 \times (5+7) = 2 \times 12 = 24$$

This property is called The Distributive Law. In general,

The Distributive Law

$$a(b+c) = ab + ac$$
$$a(b-c) = ab - ac$$

To remove parentheses from the expression of the form $a(b + c)$ or $a(b - c)$, multiply the factor outside parentheses by each term inside parentheses.

Example 5.1 Use the Distributive Law to remove parentheses in the following expressions.

a) $x(2-x^2)$ b) $-(x+2y)$ c) $a(b+c+d)$

Solution:

a) $x(2-x^2) = x \cdot 2 - x \cdot x^2 = 2x - x^{1+2} = 2x - x^3$

b) $-(x+2y) = -1 \cdot (x+2y) = -1 \cdot x - 1 \cdot 2y = -x - 2y$

c) The expression inside the parentheses should be treated not as a sum of three terms but as a sum of two $(b+c)$ and d, that is $b+c+d = (b+c)+d$. We can then apply the Distributive Law $a(b+c+d) = a((b+c)+d) = a(b+c)+ad$

If we apply the Distributive Law once again, this time to $a(b+c)$, we get

$$a(b+c+d) = a((b+c)+d) = a(b+c)+ad = ab+ac+ad.$$

Thus $a(b+c+d) = ab+ac+ad$.

We can apply the Distributive Law to remove parentheses in the product of two sums $(x+y)(b+c)$. We need to look at this expression as a sum of two terms in parentheses $(b+c)$ multiplied by the expression $(x+y)$. In other words, $(x+y)$ 'plays the role of a' in the Distributive Law. We may replace a in the Distributive Law with $(x+y)$.

$$(x+y)(b+c)=(x+y)b+(x+y)c$$

If we now apply The Distributive Law to $(x+y)\cdot b$ and $(x+y)\cdot c$, we obtain

$$(x+y)(b+c)=(x+y)b+(x+y)c=xb+yb+xc+yc$$

Thus the following is true.

The Distributive Law For The Product of Two Sums

$$\boxed{(x+y)(b+c)=xb+xc+yb+yc}$$

To multiply two sums, you need to multiply each term of the first sum by each term of the second one and then add all the products.

Please recall, that terms can be rearranged so, equivalently, we can write

$$(x+y)(b+c)=xb+yb+xc+yc.$$

Any other order of terms would be also correct.

Example 5.2 Write an equivalent expression without parentheses.

a) $(a-b)(2a+3)$
b) $(3x-1)2x$
c) $-(x+3y-z)$
d) $(b+c-d+e)a$
e) $(a+b)(c+d)(e-f)$

Solution:

a) Use the Distributive Law for product of two sums.

$$(a-b)(2a+3)=a(2a)+a(3)-b(2a)-b(3)=2a^2+3a-2ab-3b$$

b) Apply the Distributive Law. The order of factors can be changed, and thus

$$(3x-1)2x=2x(3x-1)=2x(3x)-(2x)\cdot 1=6x^2-2x$$

c) A minus sign in front of parentheses is equivalent to multiplication by -1

$$-(x+3y-z)=(-1)(x+3y-z)=-1x-1(3y)-1(-z)=-x-3y+z$$

d) Each term inside parentheses must be multiplied by a.

$$(b+c-d+e)a=ba+ca-da+ea$$

e) Multiply $(a+b)(c+d)$, then the result must be multiplied by $(e-f)$.

$$(a+b)(c+d)(e-f)=(ac+ad+bc+bd)(e-f)=$$
$$ace+ade+bce+bde-acf-adf-bcf-bdf$$

The Distributive Law lets us change an expression from a product to a sum. When we do that 'in reverse', the process is called factorization.

Factorization **Changing the sum of two or more terms to a product is called factorization.**

When factoring algebraic expressions, we merely rewrite them in a different form. The expression obtained in the process is equivalent to the original one.

Factorization of a common factor

Consider $3A + Ax$. The expression has two terms: $3A$ and Ax. Each has A as its factor. A is **a common factor** of both terms. We will factor A from $3A + Ax$.

$$3A + Ax = A\left(\frac{3A}{A} + \frac{Ax}{A}\right) = A(3 + x)$$

The first step follows from the Distributive Law. To convince yourself, multiply each term inside parentheses by A: $A \cdot \left(\frac{3A}{A} + \frac{Ax}{A}\right) = A \cdot \frac{3A}{A} + A \cdot \frac{Ax}{A} = 3A + Ax$. The second step is obtained by canceling A's . $3A + Ax$ was originally written as a sum of two terms. By factoring A from $3A + Ax$, we are now able to express it in a factored form (that is, as a product of two expressions, rather than a sum) $3A + Ax = A(3 + x)$. Notice that, after factorization, you can always check your answer by multiplying factors.

Example 5.3 Factor x^2 from the expression $3x^5 - 2x^3 + x^2$.

Solution:

$$3x^5 - 2x^3 + x^2 = x^2\left(\frac{3x^5}{x^2} - \frac{2x^3}{x^2} + \frac{x^2}{x^2}\right) = x^2(3x^3 - 2x + 1)$$

Example 5.4 Factor $2a$ from the expression $2a - 4a^2 + 6a^3$.

Solution:

$$2a - 4a^2 + 6a^3 = 2a\left(\frac{2a}{2a} - \frac{4a^2}{2a} + \frac{6a^3}{2a}\right) = 2a(1 - 2a + 3a^2)$$

Example 5.5 Factor xy^2 from the expression $4x^2y^2 + xy^4 - 3x^3y^3$.

Solution:
$$4x^2y^2 + xy^4 - 3x^3y^3 = xy^2\left(\frac{4x^2y^2}{xy^2} + \frac{xy^4}{xy^2} - \frac{3x^3y^3}{xy^2}\right) = xy^2(4x + y^2 - 3x^2y)$$

Example 5.6 Factor $(a+2b)$ from the expression $4x(a+2b)-(a+2b)$.

Solution:
$$4x(a+2b)-(a+2b) = (a+2b)\left(\frac{4x(a+2b)}{(a+2b)} - \frac{a+2b}{a+2b}\right) = (a+2b)(4x-1)$$

Please, notice that $\dfrac{a+2b}{a+2b} = 1$ and thus 1 appears as one of the terms inside parentheses.

Example 5.7 Factor $\frac{1}{4}$ from the expression $\frac{1}{4}x - \frac{3}{4}y$.

Solution:
$$\frac{1}{4}x - \frac{3}{4}y = \frac{1}{4}\left(\frac{\frac{1}{4}x}{\frac{1}{4}} - \frac{\frac{3}{4}y}{\frac{1}{4}}\right) = \frac{1}{4}\left(\frac{1}{4}\cdot\frac{4}{1}\cdot x - \frac{3}{4}\cdot\frac{4}{1}\cdot y\right) = \frac{1}{4}(x-3y)$$

Example 5.8 Factor 5 from the expression $x+5$.

Solution:
$$x+5 = 5\left(\frac{x}{5}+\frac{5}{5}\right) = 5\left(\frac{x}{5}+1\right)$$

Factorization of -1

Consider $a-b$. We can factor -1
$$a-b = (-1)\cdot\left(\frac{a}{-1} - \frac{b}{-1}\right) = (-1)(-a+b) = -(-a+b) = -(b-a)$$
Factorization of -1 causes the signs in front of each term in the original expression to change (in our example a changes to $-a$, $-b$ changes to b). Thus, $a-b = -(b-a)$ as we see above.

Example 5.9 Factor -1 from the expression $2x - y + z$.

Solution:
$$2x - y + z = -1(-2x + y - z) = -(-2x + y - z)$$

Recall the operation of simplifying fractions.

$$\frac{15}{25}=\frac{5\times3}{5\times5}=\frac{\cancel{5}\times3}{\cancel{5}\times5}=\frac{3}{5}$$

To simplify this fraction we divide the numerator and denominator by their common factor 5. We 'cancel 5'. **We can always divide the numerator and denominator by the same non-zero expression.** This rule, true for numerical fractions, is also true for algebraic fractions, that is fractions whose numerator and denominator are not necessarily numbers but any algebraic expressions, like for example $\frac{3x}{x-1}$, $\frac{a^2}{a(1-a)}$, or $\frac{xy+x^4}{2x}$. **To simplify an algebraic fraction divide the numerator and denominator by all of their common factors.** For example, consider

$$\frac{3a(b+c)}{ad}, \qquad ad\neq 0$$

Since a is the common factor of the numerator and the denominator, we divide both, the numerator and the denominator, by a. We "cancel a".

$$\frac{3a(b+c)}{ad}=\frac{3\cancel{a}(b+c)}{\cancel{a}d}=\frac{3(b+c)}{d}$$

The following rule must always be followed.

> In algebraic fractions, **only factors can be cancelled, but not terms.**

For example, in the expression $\frac{a+x}{a}$, $a\neq 0$, a cannot be cancelled. Although a in the denominator can be viewed as a factor ($1\cdot a=a$), in the numerator a is used as a term and thus we cannot 'cancel it'. (Just like we cannot cancel 3 in $\frac{3+1}{3}$, $\frac{3+1}{3}\neq 1$).

Application of factorization of a common factor to simplification of algebraic fractions

Factorization can often be used to simplify algebraic fractions. In the expression

$$\frac{3a+ay}{ax}, \qquad ax\neq 0$$

a is a factor of the denominator but not of the numerator. Notice, however, that a is a factor of each term in the numerator. Thus, we can factor a from the numerator, and then cancel it

$$\frac{3a+ay}{ax}=\frac{a(3+y)}{ax}=\frac{\cancel{a}(3+y)}{\cancel{a}x}=\frac{3+y}{x}$$

Example 5.10 Simplify the following expressions, if possible. If not possible, explain why it is not possible.

a) $\frac{m+2mn}{m}$ b) $\frac{xy}{x^2y-3xy^2}$ c) $\frac{v-4z}{4z}$

Solution:

a) One needs to find all common factors of both the denominator and the numerator. In this case the common factor is m. We will factor m in the numerator and then cancel it with m in the denominator and arrive at our result..

$$\frac{m+2mn}{m}=\frac{m(1+2n)}{m}=1+2n$$

b) The common factor is xy. Factor xy in the denominator and cancel it.

$$\frac{xy}{x^2y-3xy^2}=\frac{xy}{xy(x-3y)}=\frac{1}{x-3y}$$

c) It cannot be simplified. In the numerator $4z$ is used only as a term, not as a factor.

Application of the factorization of -1 to the simplification of algebraic fractions

Consider $\frac{-x+y}{x-y}$, $x-y\neq 0$.

There are no common factors that could be canceled, but we should notice that signs in the numerator are exactly the opposite of signs in the denominator (x follows a minus sign in the numerator but a plus sign in the denominator; y is preceded by a plus sign in the numerator but by a minus sign in the denominator). Factoring -1 either in the numerator or in the denominator (the choice is arbitrary) will reverse the signs and allow the simplification.

$$\frac{-x+y}{x-y}=\frac{-1(x-y)}{x-y}=\frac{-1(x-y)}{x-y}=-1$$

Example 5.11 Simplify the following expression $\frac{a+4d}{-4d-a}$.

Solution:

One needs to notice 'the reversed signs' (all terms in the numerators follow a plus sign, while all terms in the denominator follow a minus sign). This requires factorization of -1 (either in the numerator or denominator). We will factor -1 in the denominator.

$$\frac{a+4d}{-4d-a}=\frac{a+4d}{-1(4d+a)}=\frac{a+4d}{-1(a+4d)}=-1$$

Common mistakes and misconceptions

Mistake 5.1

When factoring 3 from $3x+3y+3$,

$$3x+3y+3 \neq 3(x+y)$$

Instead, $3x+3y+3=3(x+y+1)$.

Mistake 5.2

$\dfrac{x+1}{x} \neq 1$. Remember that only factors can be cancelled, not terms.

Exercises with Answers (For answers see Appendix A)

In all exercises of this lesson, we assume that denominators are different from zero.

Ex.1 The Distributive Law states that $c(a+b)=ca+cb$. Explain how we can use it to write an equivalent expression without parentheses. from the expression $(a+b)c$.

Ex.2 Write an equivalent expression without parentheses.

a) $2(L+W)$ b) $R(1-x)$

c) $P(1+rt)$ d) $(R^2-r^2)s$

e) $-(3+xy)$ f) $(x^2-7z)x$

g) $c(2a+c-c^5)$ h) $-(-a+a^2-2)$

Ex.3 The Distributive Law applied to the product of two sums states:

$(x+y)(b+c)=xb+yb+xc+yc$

Can we instead state it as $(x+y)(b+c)=xb+xc+yb+yc$? Why? Reformulate the Distributive Law in four different ways.

Ex.4 Write an equivalent expression without parentheses.

a) $\frac{5}{9}(F-18)$ b) $3y(6y^4+8y^3)$

c) $(x^2-2x)x^4$ d) $-\frac{2}{3}(3c-33d)$

e) $a(a^2+ab+ab^2)$ f) $(2x^3y+\frac{3}{7}-xy^4)xy$

g) $(x-y)(z+w)$ h) $(a-\frac{2}{5}b)(10-a^3)$

i) $(a+b-g)(c-d)$ j) $(x^3+x^2+x+1)(y-1)$

k) $\frac{1}{4}(4x-8)(2y+3)$ l) $(2a-1)(1-b)(c-2)$

Ex.5 First write each of the following statements as an algebraic expression using parentheses where appropriate, and then rewrite it again in its equivalent form without parentheses.

a) The product of $x^2 - y$ and 5

b) The opposite of $-4x+1$

c) The product a and $-c+1$

d) The product of $2y$ and $-a+2b+d$

e) The product of $x-1$ and y^3+2

f) The opposite of $x - x^2 + 2x^4$

Ex.6 Write the following expressions in five different equivalent ways.

a) $3(a+b)$

b) $(2-y)z$

Ex.7 Circle all expressions that are equivalent to $m(n+p)$.

$mn+mp$ $\quad$ $(n+p)m$ $\quad$ $pm+nm$ $\quad$ $mn+p$ $\quad$ $mp+nm$

Ex.8 Circle all expressions that are equivalent to $-(x-y+z)$.

$(-1)(x-y+z)$ $\quad$ $(x-y+z)(-1)$ $\quad$ $-1(x-y+z)$ $\quad$ $(x-y+z)-1$

$-x+y-z$ $\quad$ $-x-y+z$ $\quad$ $-x+y+z$ $\quad$ $-(x+z-y)$

Ex.9 a) After factoring a common factor from a two term expression, how many terms should you have inside parentheses?

b) After factoring a common factor from a three term expression, how many terms should you have inside parentheses?

c) After factoring a common factor from an m-term expression, how many terms should you have inside parentheses?

Ex.10 Factor

a) 5 from the expression $5x+5y$

b) 7 from the expression $7-49a$

c) c from the expression $3c-2c^2$

d) y from the expression $-8xy^6+y$

e) $11t$ from the expression $-11t+44t^3$

f) 2 from the $2lw+2lh+2wh$

g) x from the expression $4x^3-5x^2+5x$

Ex.11 Factor xy from the following expressions

a) $2xy - a^2xy$

b) $-x^2y + xy^2$

c) $axy + xby - yx$

Ex.12 Factor -1 from $\quad$ a) $3+x$ $\quad$ b) $-a+b+1$ $\quad$ c) $a - \frac{x+y-z}{2}$

Ex.13 Factor

a) $5a$ from the following expression $10a-15a^2$

b) $11t$ from the following expression $-\frac{11}{2}t^2+44t$

c) $5x^3$ from the following expression $15x^5+5x^3$

d) $-4y^5$ from the following expression $-8xy^6+4y^5$

e) c^2d^2 from the following expression $8c^2d^2-ac^2d^3$

f) $3x^2y$ from the following expression $-3x^2y+9x^3y^2$

g) $\frac{x}{y}$ from the following expression $\frac{2x}{y}+a\frac{x}{y}$

h) $\frac{1}{a+b}$ from the following expression $\frac{1}{a+b}+s\frac{1}{a+b}$

Ex.14 Factor

a) $\frac{2}{3}$ from the following expression $\frac{2}{3}x^2y-\frac{4}{3}z$

b) $\frac{1}{5}$ from the following expression $\frac{1}{5}x-\frac{1}{25}$

c) 0.6 from the following expression $3.6x-6y+0.6$

d) $\frac{1}{6}$ from the following expression $\frac{5}{6}a+b$

e) $\frac{2}{7}$ from the following expression $\frac{4}{7}x^2-\frac{2}{7}$

f) 0.1 from the following expression $2a+0.3b-0.1$

g) $\frac{3}{11}$ from the following expression $3xy+\frac{6}{11}x-1$

h) 0.02 from the following expression $0.4m-n$

Ex.15 Factor $a+b$ from the following expressions

a) $6(a+b)-x(a+b)$

b) $4(a+b)-3(a+b)^2$

Ex.16 Factor

a) $2xy$ from the following expression $2xy-2xy^2+4x^2y^2$

b) a^3b^3 from the following expression $a^3b^4+5b^3a^7-a^3b^3$

c) $17xy$ from the following expression $17x^5y^3+34x^3y^2+51xy$

d) m^4n from the following expression $m^4n-5n^3m^4+m^8n$

e) $4ac^3$ from the following expression $-16ac^3+8ac^7-12ac^8d$

f) $7ab^2$ from the following expression $35ab^3-14a^2b^4+21ab^2$

g) $x-2y$ from the following expression $-2(x-2y)+(x-2y)z^2$

h) $(c+d)^2$ from the following expression $(c+d)^2-4a(c+d)^3$

Ex.17 Factor a from the following expression $a - a^2 + 3$

Ex.18 From the expression $4 - x$, factor the following.

a) 2 b) x

c) $2x$ d) $4x$

Ex.19 List all terms of the denominator and numerator of the following algebraic fraction. For each such term list all its explicit factors. Find all factors that are common to all terms. If you were asked to simplify the fraction, what would be the expression by which you would divide the numerator and denominator to simplify it?

a) $\dfrac{t}{2t - ty}$

b) $\dfrac{x + xy}{2ax}$

c) $\dfrac{3ab}{ab - a^2}$

Ex.20 In the expression $\dfrac{7x}{x-5}$, can x be viewed as a factor of the denominator? Can x be viewed as a factor of the numerator? Can we "cancel x". If not, why? If yes, what is the resulting expression?

Ex.21 In the expression $\dfrac{7x}{x^2 y}$, can x be viewed as a factor of the denominator? Can x be viewed as a factor of the numerator? Can we "cancel x". If not, why? If yes, what is the resulting expression?

Ex.22 In the expression $\dfrac{a(a+x)}{a}$, can a be viewed as a factor of the denominator? Can a be viewed as a factor of the numerator? Can we "cancel a". If not, why? If yes, what is the resulting expression?

Ex.23 In the expression $\dfrac{-ab}{ab^2 + c}$, can ab be viewed as a factor of the denominator? Can ab be viewed as a factor of the numerator? Can we "cancel ab". If not, why? If yes, what is the resulting expression?

Ex.24 Simplify, if possible. Otherwise write "not possible". Also, name the expression by which you divide the numerator and denominator.

a) $\dfrac{3xy}{9yx}$ b) $\dfrac{2abc}{8ab}$

c) $\dfrac{-a^2}{b^2 a^2}$ d) $\dfrac{5xy^4}{20y^3}$

e) $\dfrac{15x(a-b)}{25x}$ f) $\dfrac{a^2(b-c)}{2a}$

g) $\dfrac{4x(-5x^2)}{x}$

h) $\dfrac{bc(b+e)}{b}$

i) $\dfrac{a+b}{7(a+b)}$

j) $\dfrac{2x+3}{2x}$

Ex.25 Simplify, if possible. Otherwise write "not possible".

a) $\dfrac{2}{2x+2y}$

b) $\dfrac{xy+xz}{3x}$

c) $\dfrac{4x-5x^2}{x}$

d) $\dfrac{3x+3x^2}{6x}$

e) $\dfrac{3x}{3x-9x^2}$

f) $\dfrac{a^3b-4b^4a^4}{a}$

g) $\dfrac{a-3b}{-3b+a}$

h) $\dfrac{x^2}{2x^2-x^5}$

i) $\dfrac{a+b+c}{a+b}$

j) $\dfrac{xy}{xy+xy^2}$

k) $\dfrac{12x-4}{3x}$

l) $\dfrac{-4x+12y+8z}{4}$

m) $\dfrac{xy-xy^2+x^2y}{3yx}$

n) $\dfrac{u+2v-s}{s-2v-u}$

o) $\dfrac{4uv^2-6vu^2+2vu}{2uv}$

p) $\dfrac{xy}{3x^4y-6x^2y^2z}$

Lesson 6

Topics: Addition and subtraction of algebraic expressions.

In this lesson we will learn how and when to perform the operations of addition and subtraction of algebraic expressions.

Like terms

Consider $3x - 4y^2 + z$. The expression consists of three terms: $3x$, $-4y^2$, z. Each term can be viewed as a product of a numerical and non-numerical factor (recall that numerical factors are also called numerical coefficients). And so,

$3x$ has a numerical factor 3, non-numerical factor x
$-4y^2$ has a numerical factor -4, non-numerical factor y^2
z has a numerical factor 1, non-numerical factor z

Like Terms **Like terms are terms that have equal non-numerical factors.**

What is meant by non-numerical factors being equal is that they are equal (equivalent) as algebraic expressions. That does not mean that they 'look identical'. You will notice that in the second example below, xy and yx are equivalent even though they do not 'look identical'.

Examples:

$3a$ and $7a$ are like terms because their non-numerical factors, both a's, are equivalent.
$4xy$ and $-3yx$ are like terms because xy and yx are equivalent.
x^2 and x^3 are not like terms since x^2 is not equivalent to x^3.

Another way of looking at like terms is that two terms are alike if they can be written as expressions that have the same variables with the same exponents. For example, $5x^3y^2$ and $4yx^3y$ are like terms, because $4yx^3y$ can be written as $4x^3y^2$, and thus both expressions consist of the variable x raised to the third power and variable y raised to the second power. It should be stressed once again, that **being like terms does not mean 'looking identical'.**

Example 6.1 Circle all expressions that are like $3x^3y^2$.

$-4y^2x^3$ $\qquad \dfrac{yx^3y}{2}$ $\qquad 0.2(xy)^2y$ $\qquad -2x^2y^3$

Solution:

We should circle $-4y^2x^3$ (because $-4y^2x^3 = -4x^3y^2$) and $\dfrac{yx^3y}{2}$ (because $\dfrac{yx^3y}{2} = \dfrac{x^3y^2}{2} = \dfrac{1}{2}x^3y^2$).

If we are to add or subtract objects they must have the same units. For example, we can add dollars to dollars and pounds to pounds, but we cannot add dollars to pounds. If we have 3 oranges and 5 oranges, together we have 8 oranges, but if we have 3 oranges and 5 plums, we cannot add them together. The same idea applies to addition and subtraction of algebraic expressions. **Like, and only like, terms can be added or subtracted.**

We apply the Distributive Law to add like terms.

$$3a+5a=(3+5)a=8a$$

(or, 3 oranges and 5 oranges equal 8 oranges)

To add (or subtract) like terms, add (or subtract) their numerical coefficients and keep non-numerical factors the same.

For example,

$$3x-8x=(3-8)x=-5x$$

Similarly,

$$\frac{1}{7}xy+\frac{3}{7}yx=\left(\frac{1}{7}+\frac{3}{7}\right)xy=\frac{4}{7}xy$$

$$0.3y^2-0.1y^2=(0.3-0.1)y^2=0.2y^2$$

The process of adding and subtracting like terms is called **combining like terms** or **collecting like terms.**

<u>Example 6.2</u> If possible, collect like terms. Otherwise, write "not possible".

a) $-3x+4x$

b) $\frac{1}{2}a^2-\frac{3}{7}a^2$

c) $0.5mn-0.7m$

d) $\frac{ab}{3}-\frac{1}{6}ba$

Solution:

a) $-3x+4x=(-3+4)x=x$

b) $\frac{1}{2}a^2-\frac{3}{7}a^2=\left(\frac{1}{2}-\frac{3}{7}\right)a^2=\left(\frac{7}{14}-\frac{6}{14}\right)a^2=\frac{1}{14}a^2$

c) Since terms mn and m are not like terms, terms cannot be combined. It is "not possible".

d) $\frac{ab}{3}-\frac{1}{6}ba=\frac{1}{3}ab-\frac{1}{6}ab=\left(\frac{1}{3}-\frac{1}{6}\right)ab=\left(\frac{2}{6}-\frac{1}{6}\right)ab=\frac{1}{6}ab$

Simplification of expressions with two or more terms by collecting like terms

Let us examine the following expression $3x+4y+2x-6y$. There are two "groups" of like terms in this expression, $x's$ and y's. By changing the order of terms, we can group all like terms together, collect them, and as a result obtain a simplified expression.

$$3x+4y+2x-6y=3x+2x+4y-6y=(3+2)x+(4-6)y=5x-2y$$

Example 6.3 Simplify by collecting like terms.

a) $3x+a-4x-a$ b) $7y+5-12y+y-6$

Solution:

a) $3x+a-4x-a=3x-4x+a-a=(3-4)x+a-a=-x$

b) $7y+5-12y+y-6=7y-12y+y+5-6=(7-12+1)y-1=-4y-1$

Simplification of expressions requiring the removal of parentheses

Collecting like terms often has to be preceded by removing parentheses. This is always done according to the Distributive Law.

$4m+(-2+m)=4m-2+m=4m+m-2=5m-2$ Parentheses following a plus sign are redundant, and thus can be dropped. Each term inside parentheses stays the same.

$3x-(5x-1)=3x-5x+1=-2x+1$ Parentheses following a minus sign are removed by reversing the sign of each term inside parentheses (operation equivalent to multiplication by -1).

$a+3(a-2)=a+3a+3(-2)=a+3a-6=4a-6$ We multiply 3 by each term in parentheses, and then collect all like terms.

Example 6.4 Write each of the following expressions using algebraic symbols, then rewrite it in its equivalent form without parentheses and, if possible, collect like terms.

a) add $3x-2$ and $-4x+2$

b) subtract $3x-2$ from $-4x+2$

c) multiply $3x-2$ and $-4x+2$

Solution:

a) This should be written as $3x-2+(-4x+2)$. We remove parentheses (using the Distributive Law) and collect like terms (x's separately, numbers separately) to get

$3x-2+(-4x+2)=3x-2-4x+2=-x$

b) This should be written as $-4x+2-(3x-2)$ Notice that "subtract from" causes us to reverse the order of the expressions. Removing parentheses following a minus sign reverses signs of *all* terms inside parentheses ($3x$ 'becomes' $-3x$, 2 'becomes' -2). After removing parentheses we collect like terms (x's separately, numbers separately) to get $-4x+2-(3x-2)=-4x+2-3x+2=-7x+4$

c) This should be written as $(3x-2)(-4x+2)$. To remove parentheses we apply the Distributive Law $(3x-2)(-4x+2)=3x(-4x)+3x(2)-2(-4x)-2(2)=$

$$-12x^2+6x+8x-4=-12x^2+14x-4$$

Example 6.5 Rewrite the expression $3(x^5-2)-(4-x^5)$ in its equivalent form without parentheses and simplify by collecting like terms.

Solution:

$$3(x^5-2)-(4-x^5)=3x^5+(3)(-2)-4+x^5=3x^5+x^5-6-4=4x^5-10$$

The square of the sum or the difference of two expressions

The Distributive Law, together with the ability to collect like terms, allows us to derive two important formulas.

Consider $(a+b)^2$. Our goal is to write the expression in its equivalent form without parentheses.

$(a+b)^2=$	use the definition of the square of the expression
$(a+b)(a+b)=$	apply the Distributive Law for the product of two sums
$a^2+ab+ba+b^2=$	collect like terms
$a^2+2ab+b^2$	

Thus the following is true.

The Square of the Sum of Two Expressions.

$$\boxed{(a+b)^2=a^2+2ab+b^2}$$

Notice that this means that the square of **the sum of two terms is not equal to the sum of their squares.** In other words $(a+b)^2\neq a^2+b^2$.

Using exactly the same technique, one can prove that

The Square of the Difference of Two Expressions.

$$\boxed{(a-b)^2=a^2-2ab+b^2}$$

Again, this means that the square of **the difference of two terms is not equal to the difference of their squares.** In other words $(a-b)^2\neq a^2-b^2$.

Example 6.6 Rewrite $(2-x)^2$ in its equivalent form without parentheses. Simplify by collecting like terms.

Solution:
$$(2-x)^2 = (2-x)(2-x) = 2\times 2 + 2(-x) - x(2) - x(-x) = 4 - 2x - 2x + x^2 = 4 - 4x + x^2$$

Common mistakes and misconceptions

Mistake 6.1

$2\,\text{apples} + 3\,\text{apples} \neq 5(\text{apples})^2$**.** Similarly, $2x + 3x \neq 5x^2$.

Instead, $2x + 3x = 5x$ **(**although $2x(3x) = 6x^2$).

Mistake 6.2

$3x-(a+b) \neq 3x - a + b$

The minus sign pertains to the whole expression $(a+b)$, and all signs must be 'changed to the opposite', not only the first one, $3x-(a+b) = 3x - a - b$**.**

Mistake 6.3

Although it is true that $(ab)^2 = a^2b^2$**,** $(a+b)^2 \neq a^2 + b^2$

Mistake 6.4

When collecting like terms in $\frac{x}{3}+\frac{2x}{3}$ you CANNOT multiply the expression by 3 to get rid of fractions. $\frac{x}{3}+\frac{2x}{3} \neq x + 2x$ (it would be like saying that $\frac{1}{3}+\frac{2}{3} = 1+2$).

Exercises with Answers (For answers see Appendix A)

Ex.1 Which of the following rows consists of all like terms?

a) $-\frac{1}{2}x,\quad 4x^2,\quad 0.5x^2,\quad -\frac{1}{6}x^2,\quad 3x$

b) $-3xy,\quad 8yx,\quad -0.6xy,\quad 2xy,\quad -\frac{3}{7}xy$

c) $5x,\quad 5x^2,\quad 5x^3,\quad 5x^4,\quad 5$

d) $-2a,\quad -5a,\quad 7a,\quad 29a,\quad a,\quad -11a$

Ex.2 Are x and $-x$ like terms?

Ex.3 Are any of the following like terms: $7x^2y$, $7xy^2$, and $7(xy)^2$?

Ex.4 Circle all terms that are like $6x^2y$.

$6x^2y^2$ $\quad -2x^2y$ $\quad 5xy$ $\quad 0.3yx^2$

Ex.5 Circle all terms that are like $-\frac{3}{7}a^2b$.

$5(ab)^2$ $\quad 2a^2b$ $\quad -\frac{3}{7}ab^2$ $\quad \frac{ba^2}{7}$

Ex.6 Circle all expressions that are like x^2y^3.

y^3x^2 $\quad xy^3x$ $\quad 2x^3y^2$ $\quad y^2xx$ $\quad yyxxy$

Ex.7 Circle all expressions that are like a^5b^3.

$a^2b^5a^3$ $\quad -b^2a^5b$ $\quad (ab)^3b^3$ $\quad 2(ab)^3a^2$ $\quad a^3b^5$

Ex.8 If possible, add (or subtract) the following expressions. Otherwise, write “not possible”.

a) $4x-2x$
b) a^2-a
c) $y-y$
d) $a+ab$
e) $st+ts$
f) ac^2+cac
g) ac^2+6ca^2
h) $7mhv-hmv$
i) $2xy^3z^5+y^3z^5x$
j) $3m^3n+6m^2nm$
k) $7xy^3z^5-xy^4z^5$
l) $9a^2b^2-7(ab)^2$

Ex.9 Collect like terms by first factoring x.

a) $3x-4x$
b) $\frac{1}{3}x-\frac{2}{7}x$
c) $\frac{2}{11}x-\frac{3x}{22}$
d) $0.3x-0.5x$
e) $-\frac{7}{9}x-\frac{2x}{5}$
f) $\frac{x}{5}-\frac{2}{3}x+\frac{3x}{10}$

Ex.10 If possible, collect like terms. Otherwise, write “not possible”.

a) $7x-8x$
b) $2a-7a$
c) $\frac{1}{2}x+\frac{1}{2}x$
d) $\frac{1}{2}x-\frac{3}{2}x$
e) $ab-8ba$
f) $4-7x$
g) $5c^3-10c^4$
h) $-0.4x^2y-0.8yx^2$

Ex.11 Rewrite by grouping all like terms together. Then collect like terms. For example, $3x-7y+2x-2y=3x+2x-7y-2y=5x-9y$.

a) $-3x+8y-8x-2x$
b) $-5a-3b-2b-7a$
c) $-2ab-4ba+2+3ab-1$

Ex.12 Simplify by collecting all like terms.

a) $3j-4j+2j$

b) $2+a-\frac{2}{3}a$

c) $0.2z-0.5z+z$

d) $2-7m-4$

e) $-x^3+x^4-x^3$

f) $2x+y+x-2y$

g) $-\frac{3x}{2}+\frac{1}{2}x-1$

h) $-3y+\frac{1}{2}x-x+4y$

i) $\frac{cd}{3}+\frac{1}{3}dc-d+c$

j) $a^2b^3-4b^2a^3-6a^2b^3$

k) $x^4y-2x+3yx^4-5x$

l) $\frac{3}{38}ab-\frac{7}{19}ba+2\frac{3}{4}+3\frac{1}{8}$

Ex.13 Students were asked to simplify the expression $-2(a-b)+3a-3b$. The following answers were given.

Student A: $a-b$
Student B: $b-a$
Student C: $-b+a$
Student D: $-a+b$

List all students who gave the right answer.

Ex.14 Collect like terms, and only then evaluate $-3x+2y-\frac{2}{3}y+4x-\frac{y}{3}$ when

a) $x=-1, \quad y=-3$

b) $x=-2, \quad y=\frac{1}{5}$

c) $x=0.5, \quad y=-3.2$

Ex.15 Evaluate $\frac{2}{5}x+\frac{1}{2}x+\frac{x}{10}$ when

a) $x=4$

b) $x=-10$

c) $x=\frac{2}{3}$.

If you have not done this yet, simplify the expression by collecting like terms. Compare the obtained expression with the results of the evaluations.

Ex.16 Write an equivalent expression without parentheses and then collect like terms

a) $(6a-2)+12a$

b) $4a+7a+2(8a+1)$

c) $y-3(-y+2)$

d) $-100(t-0.1)+102t$

e) $\frac{2x}{5}-(2x-1)$

f) $a-0.1(2-3a)$

g) $-(x-1)-x$

h) $(q+6)(-8)-21q$

i) $3a-9b-(4a-\frac{4}{5}b)$

j) $2x-5y-4(3x+y)$

k) $-6(\frac{2}{3}d-\frac{1}{2}a)-a-d$

l) $3a^2-1-(4a^2+2)$

m) $-3xy+7yx-(xy+3)$

n) $-3cb+abc-2(a+bc)$

o) $\frac{1}{3}a-b-\frac{1}{3}(a-2b)$

p) $(6x^4+3x^3-1)-(3x^3-3x^2-3)$

q) $(1+4x+6x^2+7x^3)+(5-4x+6x^2-7x^3)$

r) $-(0.2m+0.03)-(2.3m-4)$

Ex.17 a) Students were asked to write $(A+B)^2$ in its equivalent form without parentheses and collect like terms of the resulting expression. One student claimed that the answer was A^2+B^2, the other one that it was $A^2+2AB+B^2$. Was either of them right?

b) If you were asked to write $(A-B)^2$ in its equivalent form without parentheses and then collect like terms, what would be your answer?

Ex.18 Write an equivalent expression without parentheses, and then collect like terms.

a) $(x^4+2)^2$

b) $(3x-1)^2$

c) $(a-2)(a+4)$

d) $(3b-c)\left(b+\frac{1}{2}c\right)$

e) $\frac{1}{3}(3c-x)+2(5x-c)$

f) $(1+x-2x^2)(2-x)$

g) $(a^3+b)(a^2-2b)$

h) $\left(\frac{2}{3}-2x\right)^2$

i) $-(a-3b+c)-(c-a)$

j) $-2(-x+2y)-3(5x-3y)$

k) $3x+(4-x)(x+2)$

l) $5-a^2-(3+2a)^2$

m) $6x^2-(x+1)(3x-2)-2$

n) $4-m-2(m-1)^2$

o) $3k^2-3k-(k-2)(2k+3)$

p) $x^4+9-(x^2-3)^2-6x^2$

Ex.19 Write each of the following expressions using algebraic symbols, then rewrite it in its equivalent form without parentheses and, if necessary, collect like terms.

a) Subtract $-2xy$ from $3yx$

b) The sum of $3x-1$ and $-4x+2$

c) The difference of $-4a^3+2a$ and $4a^3-2$

d) The difference of $-a+2+3b$ and $-2b+a$

e) The product of $a-\frac{1}{3}$ and $\frac{2}{5}-2a$

f) The product of $2x^2-y$ and $3y-x^2$

g) The sum of $-mnk$, $4nmk$, and $-3mn$

h) The sum of $3x$ and 2, then raised to the second power

i) The difference of $2a$ and b, then raised to the second power

j) The product of 2 and $-4x+1$, then added to $5x+2$

Ex.20 Rewrite the expression $-a-2(a-b)-5b$ in its equivalent form without parentheses, collect like terms, and only then evaluate if

a) $a = -1, \quad b = -2$

b) $a = \dfrac{3}{14}, \quad b = \dfrac{2}{7}$

c) $a = -2.4, \quad b = 0.6$

Ex.21 Rewrite the expression $4x^2 - xy - 2(yx - 1 + 2x^2) - 2$ in its equivalent form without parentheses, collect like terms, and only then evaluate when

a) $x = -1, \quad y = -3$

b) $x = -2, \quad y = \dfrac{1}{6}$

c) $x = 0.5, \quad y = -0.2$

Lesson 7

Topics: Evaluation of more complicated algebraic expressions; Substitution of not only numbers but also algebraic expressions.

Recall,

If two quantities are equal, you can always substitute one for the other. **"equals can be substituted for equals"**

Substitution of numbers for entire parts of expressions

According to the principle **"equals can be substituted for equals"** not only can we substitute numbers for variables, but also for entire 'parts of expressions'.

If, for example, we know that $a+b=2$, we can evaluate $(a+b)^2-3(a+b)$ by substituting the value 2 for $a+b$.

$$(a+b)^2-3(a+b)=2^2-3\times 2=4-6=-2$$

Example 7.1 Evaluate the expression $\frac{x-y}{z^2}$, if $x-y=-0.2$ and $z^2=0.4$

Solution:

$$\frac{x-y}{z^2}=\frac{-0.2}{0.4}=-\frac{2}{4}=-\frac{1}{2}$$

Using equivalent forms of an algebraic expression for its evaluation

If we are asked to evaluate a 'complicated' algebraic expression, we may be able to simplify the expression first, and then use its simplified form to perform the evaluation. Notice, that what we do is replace ('substitute for') the original algebraic expression with its equivalent ('equal') form. The principle **"equals can be substituted for equals"** is used.

Example 7.2 Simplify the expression $3x+3(y-x)$, and then evaluate it when $x=3$ and $y=-2$.

Solution:

$$3x+3(y-x)=3x+3y-3x=3x-3x+3y=3y$$

If $y=-2$, then $3y=3(-2)=-6$.

Sometimes, the *only* way to perform the evaluation is to first replace the expression with a certain equivalent form of it. Suppose, for example, that we know that $x=5$, and $y+z=4$. Is it possible to evaluate $xy + xz$? We will be able to do that if we can find an equivalent form of $xy + xz$, a form that would 'match' the information we have. If we factor x, we obtain

$$xy + xz = x(y + z)$$

Now, we can replace x with 5, and $y + z$ with 4,

$$xy + xz = x(y + z) = 5(4) = 20$$

Example 7.3 Evaluate the following expressions, if $a - b = -1$ and $c = 2$

a) $a - c - b$ b) $2c - (b - a)$

c) $\frac{a}{c} - \frac{b}{c}$ d) $\frac{3a - 3b}{c}$

Solution:

a) We change the order of terms $a - c - b = a - b - c = -1 - 2 = -3$

b) We remove parentheses, and then change the order of terms.

$$2c - (b - a) = 2c - b + a = a - b + 2c = -1 + 2(2) = -1 + 4 = 3$$

c) We write the expression as one fraction $\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c} = \frac{-1}{2}$

d) We factor 3 in the numerator $\frac{3a - 3b}{c} = \frac{3(a - b)}{c} = \frac{3(-1)}{2} = -\frac{3}{2}$

Example 7.4 Evaluate the following expressions, if $\frac{m}{n} = 4$.

a) $\frac{m^2}{n^2}$ b) $\frac{n}{m}$

Solution:

a) $\frac{m^2}{n^2} = \left(\frac{m}{n}\right)^2 = 4^2 = 16$

b) $\frac{n}{m} = \frac{1}{\frac{m}{n}} = \frac{1}{4}$ Notice that $\frac{n}{m} = \frac{1}{\frac{m}{n}}$ because $\frac{1}{\frac{m}{n}} = 1 \div \frac{m}{n} = 1 \times \frac{n}{m} = \frac{n}{m}$

Replacing parts of algebraic expressions with other equivalent expressions

If we know that $x = -3m$ and $y = 5m$, then we can **express $x + y$ in terms of m** i.e. write it as an expression that depends *only* on the variable m. "**Equals can be substituted for equals",** and so we replace x with $-3a$, y with $5a$.

$$x + y = -3a + 5a = 2a$$

This exhibits the direct relationship between $x + y$ and a, and, in turn, allows us to find the value of $x + y$ any time we know the value of a.

For instance, the area of a rectangle is equal to the product LW where L represents its length and W its width. Let us consider a rectangle with the length and width equal to each other. Recall that this type of a rectangle is called a square. Let a denote the side of the square. We have $W = a, L = a$. We can express the area of a square in terms of a. To this end, we replace both W and L with a:

$$LW = aa = a^2$$

And thus the area of a square is equal to the square of its side.

Now, if for example, we would like to fine the area of a square, whose side is 2 inches long, we determine that $a = 2$ (side is 2 inches long), and the area can be found by evaluating

$$a^2 = 2^2 = 4$$

The area of the square is 4 (actually 4 in^2).

Example 7.5 If $x = -z$, express the following expressions in terms of z.

a) x^2 b) $-x$

Solution:

a) $x^2 = (-z)^2 = z^2$

b) $-x = -(-z) = z$

Example 7.6 Express $(m+n)(m^2 - mn + n^2)$ in terms of

a) x, if $m = 2$, and $n = 3x$.

b) m, if $n = m$.

Solution:

a) We replace m with 2, and n with $3x$, and simplify.

$$(m+n)(m^2 - mn + n^2) = (2+3x)(2^2 - 2(3x) + (3x)^2) = (2+3x)(4 - 6x + 3^2 x^2)$$
$$= (2+3x)(4 - 6x + 9x^2)$$

b) We replace n with m, and simplify.

$$(m+n)(m^2 - mn + n^2) = (m+m)(m^2 - mm + m^2) = 2m(m^2 - m^2 + m^2) = 2mm^2 = 2m^3$$

Example 7.7 If $A^2 = v$ and $B^2 = 2v$, express $(A+B)(A-B)$ in terms of v.

Solution:

We must find an equivalent form of the above expression expressed only in terms of A^2 and B^2. We will remove parentheses.

$$(A+B)(A-B) = A^2 - AB + BA - B^2 = A^2 - B^2.$$

Now, we can replace A^2 with v, B^2 with $2v$ to get

$$(A+B)(A-B) = A^2 - B^2 = v - 2v = -v.$$

Exercises with Answers (For answers see Appendix A)

Ex.1 If $x^2 = 10$, evaluate the following.

a) $2x^2$ b) $\left(\frac{1}{30}x^2\right)^3$ c) $-3x^2-2$

Ex.2 If $A^5=-3$, evaluate the following.

a) $-2A^5$ b) $-(2A^5)^2$

c) $-2-2A^5$ d) $3A^3A^2$

Ex.3 Evaluate the following expressions, if $\frac{a+b}{c}=-2$.

a) $\frac{a+b}{c}-2$

b) $-\left(\frac{a+b}{c}\right)^2$

c) $\left(-\frac{a+b}{c}\right)^2$

Ex.4 If $x+y=-2$, evaluate.

a) $7x+7y$ b) $\frac{x}{7}+\frac{y}{7}$

c) $-x-y$ d) $(-x-y)^2$

Ex.5 If $\frac{1}{A}=-\frac{2}{7}$, evaluate.

a) $1\div A$ b) $\frac{-1}{A}$

c) $\frac{1}{-A}$ d) $\frac{1}{A^2}$

Ex.6 Evaluate the following expressions, if $2a+b-c=-7$.

a) $-c+b+2a$

b) $-2a-b+c$

c) $-(-2a-b+c)$

d) $2a-(c-b)$

e) $a+b-c+a$

Ex.7 Evaluate the following expressions, if $GHJ=-1$.

a) $-JHG$ b) $\frac{1}{GHJ}$

c) $G(-2)H(-1)J$ d) $G^3(HJ)^3$

Ex.8 Evaluate the following expressions, if $\frac{a}{b}=3$.

a) $\left(\dfrac{a}{b}\right)^2$

b) $-\dfrac{a}{b}$

c) $a \div b$

d) $\dfrac{b}{a}$

e) $a \cdot \dfrac{2}{b}$

f) $\dfrac{a^2 a}{b^3}$

Ex.9 Evaluate the following expressions, if $\dfrac{xy}{z} = -\dfrac{1}{3}$.

a) $\dfrac{yx}{z}$

b) $\dfrac{xy}{-z}$

c) $-\dfrac{x^2 y^2}{z^2}$

d) $\dfrac{1}{z} \cdot xy$

e) $\dfrac{z}{xy}$

f) $xy \div z$

Ex.10 Evaluate the following expressions, if $a + b = -2 \quad c - d = 3$.

a) $a + b + c - d$

b) $2(a - d + c + b)$

c) $9(a + b)^2 - d + c$

Ex.11 Evaluate the following expressions, if $x^2 y = -0.2, \quad x - z = 0.6$

a) $\dfrac{x - z}{x^2 y}$

b) $-(x - z) - x^2 y$

c) $y(x - z)x^2$

Ex.12 Evaluate the following expressions, if $xy = -1, \quad zt = -3$.

a) $-4xyzt$

b) $-zxty$

c) $-xy + zt$

d) $\dfrac{yx - tz}{xy}$

Ex.13 Evaluate the following expressions, if $x^3 y^4 z^6 = -\dfrac{2}{3}$.

a) $-z^6 y^4 x^3$

b) $x^3 (y^2 z^3)^2$

c) $-\dfrac{x^3 y^4}{3} \cdot z^6$

d) $z^6 (-2) y^4 (-3) x^3$

Ex.14 Evaluate the following expressions, if $x^2 + y^2 = 0.1$

a) $y^2 - 0.3 + x^2$

b) $\dfrac{x^2}{0.2} + \dfrac{y^2}{0.2}$

c) $\dfrac{1}{2} x^2 + \dfrac{1}{2} y^2$

d) $\dfrac{1}{x^2 + y^2}$

Ex.15 Simplify the following expressions, and then evaluate when $x = 2,\ z = -3$.

a) $\dfrac{3x + 3z}{3}$ b) $4x - (z + 4x)$

Ex.16 Rewrite the expression $\dfrac{3a}{2}$ in terms of x, if it is given that $a = 2x$. Simplify.

Ex.17 Let $C = -3x, \quad D = \dfrac{x}{2}$. Express the following expression in terms of x. Simplify.

a) CD b) $C - D$

c) $3C^2D$ d) $\dfrac{C - 2D}{4}$

Ex.18 Rewrite the expression a^2 in terms of x for each of the following. Write your answer without parentheses.

a) $a = x$ b) $a = 5x$

c) $a = -x$ d) $a = -5x$

e) $a = x + 1$ f) $a = x^3$

Ex.19 Rewrite the expression $a^2 - 2ab + b^2$ in terms of x, if

a) $a = 1, b = x$ b) $a = 3x, b = 2x$

c) $a = \dfrac{x}{2}, b = -x$ d) $a = b = x$

Ex.20 Let $P = 3x^2 - 2x + 1,\ Q = x - 2$, and $R = 3x - 1$. Express the following in terms of x. Remove parentheses and simplify.

a) $Q + R$ b) $-P$

c) $2P$ d) $P - R$

e) R^2 f) QR

g) $3R - Q$ h) $P - Q^2$

Ex.21 Express the following expression in terms of s, if $x = s^3,\ y = 2s$. Simplify, if possible.

a) $2x - y$ b) x^2y

c) $2x + y^3$ d) $\dfrac{xy^2}{4}$

Ex.22 Express ab in terms of x, if $a = \dfrac{5x}{126}, \quad b = \dfrac{126}{5}$. Simplify.

Ex.23 Express the expression $(mn)^2 - mn^2$ in terms of

a) m, if $n = m$. Simplify.

b) s, if $m = 2s, n = s$. Simplify

Ex.24 Express the expression $m^3n - 2m^2n^2$ in terms of s, if

a) $m = s, n = -2s$

b) $m = 2s^2, n = s^4$

c) $n = m = s$

Ex.25 Express the expression $(m-n)^2 + (m+n)^2$ in terms of

a) s, if $m = 2s, n = 3s$

b) m, if $n = m$

c) m, if $n = -m$

Ex.26 The following is true: $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

a) Express $(a+b)^4$ in terms of b, if $a = b$.

b) Express $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ in terms of b, if $a = b$. Simplify.

c) Compare the results of (a) and (b).

d) Express $(a+b)^4$ in terms of b, if $a = -b$.

e) Express $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ in terms of b, if $a = -b$. Simplify.

f) Compare the results of (d) and (e).

Ex.27 Rewrite the expression $a - 2b + 3c + 4d$ in terms of x, if it is given that $a + 3c = 5x$ and $4d - 2b = -x$. Simplify.

Ex.28 Express the following expression in terms of m, if $ab = m$

a) $-3ba$

b) $\frac{a}{2}b$

c) $\left(6^a\right)^b$

d) a^3b^3

e) $(-2b)(-\frac{1}{2}a)$

f) $a \div \frac{1}{b}$

Ex.29 Express the following expression in terms of y, if $\frac{x}{z} = -y$, $\frac{t}{z} = 2y$. Simplify.

a) $\frac{x}{z} - \frac{t}{z}$

b) $\left(\frac{x}{z}\right)^2 - \left(\frac{t}{z}\right)^2$

c) $\frac{x+t}{z}$

d) $\frac{1}{z}x - \frac{2}{z}t$

Ex.30 Rewrite the expression $(a-b)(a+b)$ in terms of x, if it is given that $a = 5x$ and $b = 2 - x$. Simplify.

Lesson 8

Topics: Generalities on equations; Solving linear equations in one unknown.

This lesson introduces an important algebraic concept: equations.

Equations

Equation **A mathematical statement consisting of two expressions separated by an equal sign is called an equation.**

The following are examples of equations.

$$2+3=5, \qquad 3x=5, \qquad x+3=y.$$

We will refer to expressions on the left of the equal sign as **the left-hand side of the equation** and to the expressions on the right of the equal sign as **the right-hand side of the equation.** In $x+3=y$, $x+3$ is the left-hand side of the equation, and y is the right-hand side of the equation. We can always reverse sides of the equation. For instance, instead of $x+3=y$, we can write $y=x+3$. Both statements have exactly the same meaning.

The difference between equations and algebraic expressions

Notice the difference between an equation and algebraic expression. Equations are two algebraic expressions separated by the equal sign. There is always the left-hand side and the right-hand side of each equation. Algebraic expressions are different from equations. For example, $x-y=3x+2$ is an equation, but $x-y$, $3x+2$ are simply algebraic expressions.

Example 8.1 Determine whether the following mathematical sentences represent an equation or an algebraic expression. Any time you find an equation, circle its left-hand side.

a) $4x-2$ b) $4x-2=7$ c) $x^2+3y^2=4-x$

Solution:
Only b) and c) are equations. The left hand sides: $4x-2=7$, $x^2+3y^2=4-x$

Equations are often used to state the equality of two expressions containing one or more variables (often called **unknowns**).

The solution set **A value of the variable for which the equation is true (that is, a value for which the left hand side of the equation is equal to the right hand side) is called a solution. The solution set of the equation is the set of *all* solutions of the equation.**

For example, $x = 7$ is **a solution** of the equation $x + 2 = 9$ (when we replace x with 7, the right-hand side is equal to the left-hand side of the equation $7 + 2 = 9$). Later on, we will learn that $x = 7$ is the only solution of this equation, and thus, since there are no other solutions, $x = 7$ is also **the solution set** of $x + 2 = 9$. Consider the equation $x^2 = 36$. One can check that $x = 6$ is a **solution** of $x^2 = 36$ (substitute 6 for *x* to get $6^2 = 36$). It is not **the solution set**, because it is not the *only* solution of this equation. For example, $x = -6$ is another solution, since $(-6)^2 = 36$. **The solution set of an equation consists of all solutions.** Therefore, in the previous example, the solution set would need to include both 6 and -6.

To Solve an Equation **To solve an equation is to find the set of *all* solutions of the equation, or prove that it does not have a solution.**

Example 8.2 Determine which of the following numbers are solutions of the equation $x^3 = 6 + x$.

a) $x = 2$ b) $x = -2$

Solution:

a) Evaluate the left-hand side of the equation when $x = 2$.

$$x^3 = 2^3 = 8.$$

Evaluate the right-hand side of the equation when $x = 2$.

$$6 + x = 6 + 2 = 8.$$

Since both the left-hand and right-hand side of the equations are equal, 2 is a solution.

b) Evaluate the left-hand side of the equation when $x = -2$.

$$x^3 = (-2)^3 = -8$$

Evaluate the right-hand side of the equation when $x = -2$.

$$6 + x = 6 + (-2) = 6 - 2 = 4.$$

Since the left-hand side is not equal to the right-hand side ($-8 \neq 4$), -2 is not a solution.

Why do we solve equations? Here is a very basic example. Suppose that you bought some candies and you did not know how many you bought but you know that you spent 80 cents and that each candy cost 2 cents. To find out how many candies you actually bought, you set up an equation $2x = 80$, where *x* represents number of candies you bought. If you solve this equation, you would find that $x = 40$ and this means that you bought 40 candies. Now, let us learn how to solve certain types of equations.

Operations that can be performed on any equation without changing the set of solutions

The following operations can be done to any equation without changing its solution.

Any quantity can be added to, or subtracted from both sides of an equation.

For example, If $x = y$ then $x + 2 = y + 2$

If $x = y$ then $x - 2 = y - 2$

Both sides of an equation can be multiplied, or divided by any **nonzero** quantity.

For example, If $x = y$ then $2x = 2y$

If $x = y$ then $\frac{x}{2} = \frac{y}{2}$

It is important to remember that, any time we perform any operation on one side of an equation, exactly the same operation must be performed on the other side. We always perform **the same operation on both sides.**

Solving linear equations in one variable

There are many types of equations, and depending on the type of equation, different solving techniques are involved. We will learn how to solve linear equations in one variable (the formal definition of a linear equation will be provided later).

To solve a linear equation we perform the operations of adding, subtracting, multiplying or dividing both sides of an equation by suitable quantities with the goal of isolating the variable (often *x*) on one side, and 'bringing all numbers to the other side' of the equation. The numerical value obtained on the other side is the solution.

We will learn how to implement the above idea by solving several equations.

■ Solve the following equation $x + 2 = 7$.

We need to isolate x, and to this end, we must somehow 'get rid of 2' on the left-hand side of the equation. 2 is added to x, so if we subtract 2 (opposite operation to addition), we will get zero, and 2 will no longer be on the left-hand side. But if we subtract 2 from the left-hand side of the equation, the same operation must be performed on the right-side. Thus we need to subtract 2 from **both sides**. We could say that we are 'bringing 2 to the other side'

$x + 2 = 7$ Subtract 2 from **both sides**.

$x + 2 - 2 = 7 - 2$ Simplify.

$x = 5$

The solution of $x + 2 = 7$ is $x = 5$. ■

■ Solve the following equation $3x = 6$.

$$3x = 6$$

To isolate x, 'bring 3 to the other side'. 3 is multiplied by x, thus we need to divide (opposite operation to multiplication) **both sides** by 3.

$$\frac{3x}{3} = \frac{6}{3}$$

Simplify.

$$x = 2$$

The solution of $3x = 6$ is $x = 2$ ■

■ Solve the following equation $\frac{x}{4} = 9$.

$$\frac{x}{4} = 9$$

To isolate x multiply **both sides** by 4 (since x is divided by 4 and the opposite operation to multiplication is division).

$$4 \cdot \frac{x}{4} = 9 \cdot 4$$

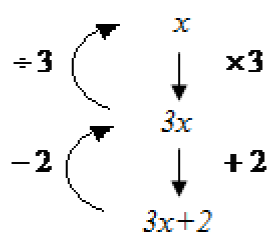
Simplify.

$$x = 36$$

The solution of $\frac{x}{4} = 9$ is $x = 36$ ■.

■ Solve the following equation $3x + 2 = 5$.

The goal is to isolate x on one side by 'undoing' the operations that were performed on x.



The unknown x was multiplied by 3, and then 2 was added. To solve for x we reverse the order. Since the last operation was adding 2, we "undo" that first by subtracting 2. Next we have to "undo" multiplication of x by 3, so we divide by 3.

$$3x + 2 = 5$$

Subtract 2 from **both sides** of the equation.

$$3x + 2 - 2 = 5 - 2$$

Simplify.

$$3x = 3$$

Divide **both sides** by 3.

$$\frac{3x}{3} = \frac{3}{3}$$

Simplify.

$$x = 1$$

The solution of $3x + 2 = 5$ is $x = 1$■

■ Solve the following equation $2x-9=3-4x$.

$2x-9=3-4x$	If $x's$ appear on both sides, 'bring all $x's$ on one side'. $4x$ is subtracted, thus we need to add (opposite operation) $4x$ to **both sides**.
$2x-9+4x=3-4x+4x$	Collect like terms (on both sides separately!)
$6x-9=3$	Now, the equation is in the form we already know how to solve. We will continue solving it using methods introduced earlier. To isolate x, we first 'bring 9 to the other side' by adding it to **both sides**.
$6x-9+9=3+9$	Simplify.
$6x=12$	Divide **both sides** by 6.
$\frac{6x}{6}=\frac{12}{6}$	Simplify.
$x=2$	

The solution of $2x-9=3-4x$ is $x=2$ ■

■ Solve the following equation $3(x-2)=9$.

$3(x-2)=9$	If parentheses are involved on one or both sides of the equation, we **first remove parentheses** by applying the Distributive Law.
$3x-6=9$	The equation is in the form we already know how to solve: add 6 to **both sides**. The operation of adding 6 to -6 can easily be performed mentally, without recording it and this is how we will be doing it from now on. You are encouraged to do the same.
$3x=15$	Divide each side by 3.
$x=\frac{15}{3}$	Simplify.
$x=5$	

The solution of $3(x-2)=9$ is $x=5$. ■

<u>Example 8.3</u> Solve the following equations.

a) $98=-x$ b) $4x-5=15+x$

Solution:

a) $-x=98$	To eliminate the minus sign, multiply (or equivalently divide) both sides by -1 (recall that $-x=-1\cdot x$).
$-1(-x)=-1(98)$	Perform the indicated operations.
$x=-98$	

The solution of $-x=98$ is $x=-98$.

b) $4x-5=15+x$ 'Bring all x's on one side' by subtracting x from both

	sides (subtracting $4x$ from both sides would also be correct).
$4x - x - 5 = 15$	Collect like terms.
$3x - 5 = 15$	Add 5 to both sides
$3x = 15 + 5$	Perform the operation of addition.
$3x = 20$	Divide both sides by 3.
$x = \frac{20}{3}$	

The solution of $4x - 5 = 15 + x$ is $x = \frac{20}{3}$.

<u>Checking solution of a linear equations</u>

It is always possible **to check your solution.** To this end, replace the variable in the original equation with the value of the solution and verify that the left-hand side of the equation is equal to the right-hand side. For example, to check that $x = 5$ is indeed a solution of the equation $3(x - 2) = 9$, substitute 5 for x in the equation.

$$3(5 - 2) = 9$$
$$3 \times 3 = 9$$
$$9 = 9$$

Since the left hand side is equal to the right hand side of the equation, 5 is indeed a solution.

<u>Example 8.4</u> Solve the equation $3 = -2x - 5$ and check your solution.

Solution:

$3 = -2x - 5$	Add 5 to both sides (x was multiplied by -2, and then 5 was subtracted, so to 'undo' that, we first add 5)
$8 = -2x$	Divide each side by -2 (divide, since x is *multiplied* by -2: there is no subtraction!). Simplify.
$-4 = x$	

The solution of $3 = -2x - 5$ is $x = -4$.

To check the answer we replace x with -4 in the original equation $3 = -2(-4) - 5$, evaluate both sides, $3 = 3$, and conclude that, indeed, -4 is a solution.

<u>Linear equations with no solution or a solution set consisting of all real numbers</u>

All of the above equations had **exactly one solution**. However, there are some linear equations that have no solutions. The solution set of other equations may consist of all real numbers. We will see below an example of an equation that has **no solution**, followed by an example whose solution consists of **all real numbers**.

■ Solve the following equation $2x-1=2x+5$.

$$2x-1=2x+5 \qquad \text{Subtract } 2x \text{ from } \textbf{both sides}.$$
$$-1=5$$

As we know, subtracting $2x$ from both sides produces an equation with exactly the same solution set as the original one. But $-1=5$ is a false statement. No matter what the value of x is, it is never true. In other words, there is no value of x that would satisfy it. We conclude:
The equation $2x-1=2x+5$ does not have a solution. We write 'no solution' as the answer. ■

■ Solve the following equation $2(x-3)=2x-6$.

$$2(x-3)=2x-6 \qquad \text{Remove parentheses.}$$
$$2x-6=2x-6 \qquad \text{Subtract } 2x \text{ from } \textbf{both sides}.$$
$$-6=-6$$

The statement $-6=-6$ is true. It is true regardless of the value of x. It is true if $x=1$, $x=2$, or $x=1000$. It is true for any value of x. All real numbers make it true. Therefore, the solution of $2(x-3)=2x-6$ is all real numbers. We write 'all real numbers' as the answer. ■

In general, if in the process of solving an equation the variable is cancelled (disappears from the equation), then the equation either has no solution or its solution is all real numbers.

If an equation reduces to a false numerical statement, the original equation does not have a solution.

If an equation reduces to a true numerical statement, the original equation has a solution consisting of all real numbers.

<u>Example 8.5</u> Solve the following equation $4x=4(x-1)$

Solution:

$$4x=4(x-1) \qquad \text{Remove parentheses using the Distributive Law.}$$
$$4x=4x-4 \qquad \text{Subtract } 4x \text{ from both sides.}$$
$$0=-4$$

Since the statement $0=-4$ is false, the equation $4x=4(x-1)$ has no solutions.

Common mistakes and misconceptions

Mistake 8.1

Algebraic expressions cannot be 'solved'. To solve means to find all values of variables for which the left-hand side of the equation is equal to the right-hand side. So, for example, it would be meaningless to use the phrase 'Solve $3x+2$'.

Mistake 8.2

While solving an equation, DO NOT display the work as a sequence of equations.

$$3x+2=6=3x=4\ldots$$

Each equation must always have a left-hand side and a right-hand side. Any time you perform an operation rewrite the whole equation.

Mistake 8.3

When solving $6x+9=5$, first subtract 9. DO NOT divide by 6 as a first step.

Mistake 8.4

If in the equation $5x-5=4x$ you decide to subtract $4x$ from both sides, remember that on the right hand-side you get 0, and you must record that zero: $5x-5-4x=0$ (the right-hand side DOES NOT 'disappear').

Mistake 8.5

DO NOT write 'no solution' when you get $x=0$. 0 IS the solution.

Mistake 8.6

When asked to solve an equation and after all operations you get $3=-5$, you need to write the final answer: *there is no solution* (similarly, write *all real numbers are solutions*, when you get $3=3$).

Exercises with Answers (For answers see Appendix A)

Ex.1 Fill in the blanks using the following words: 'equation', 'algebraic expression', and 'solution' as appropriate.

One can solve a(n)____________ but not a(n) __________________ .

If the left hand side of an equation is equal to the right hand side of the equation for $x=7$, then 7 is called a ____________ .

The __________(s) of an equation are all values of variables that make the equation true.

The statement that contains two quantities separated by an equal sign is called a(n) ___________ .

A(n) ______________ always makes the ______________ true.

Ex.2 Determine whether the following mathematical sentences represent an equation or an algebraic expression. In the case of an equation, circle the right-hand side of the equation.

a) $5x$
b) $5x=2$
c) $x^2=36$
d) $\dfrac{x-1}{2}+x$
e) $x=-4+2x$

Ex.3 Tom found the solution of an equation to be $x=3$, but the teacher gave as a correct solution $3=x$. Is Tom's answer right? Mary's answer to the same question is $-x=-3$. Did Mary correctly solve the equation? Tell why or why not for both Tom and Mary.

Ex.4 Does $x = 7$ make the statement $2(x+1) - x = 7$ true or false? Does that mean that 7 is, or is not a solution of $2(x+1) - x = 7$?

Ex.5 Determine if any of the following numbers $-2,\ 16,\ \frac{1}{2},\ 2$ is a solution of the equation. $-x^4 = 16$. How about the equation $x^4 = 16$?

Ex.6 Is $x = -1$ a solution of

a) $(-x)(-x) = 2$

b) $-x - x = 2$

Ex.7 Is $x = 2$ a solution of

a) $6^x = 36$

b) $-6^x = 36$

c) $(-6)^x = 36$

Ex.8 Does $a = \frac{2}{5}$ make the following statements true or false?

a) $-a^2 = \frac{4}{25}$

b) $a - \left(-\frac{3}{5}\right) = 1$

c) $-\frac{a}{2} = -5$

d) $-a = -\frac{12}{5} + 5a$

Ex.9 Determine if $y = 0.3$ is a solution of any of the following equations.

a) $0.027 = y^3$

b) $\frac{y}{1-y} = \frac{3}{7}$

c) $(-y - 0.7)^{26} = -1$

Ex.10 Determine whether the following mathematical sentences represent an equation or an algebraic expression. In the case of an equation, determine whether or not $x = -3$ is a solution of it.

a) $\frac{x}{x+3}$

b) $\frac{x}{x+3} = 0$

c) $-x = x^2 - 12$

d) $x^2 = -2x$

e) $x^2 - 2x$

f) $x^3 = x^2 + x$

Ex.11 Determine whether the following mathematical sentences represent an equation or an algebraic expression. In the case of an equation, determine whether or not $x = 0.6$ is a solution of it.

a) $-10x^2$

b) $-3.6 = -10x^2$

c) $(0.5 - x)^2 = 0.01$

d) $(0.5 - x)^2$

e) $\frac{12}{-x} = -20$

f) $\frac{-x + 0.6}{x} = 0.6$

Ex.12 Determine whether the following mathematical sentences represent an equation or an algebraic expression. In the case of an equation, guess one of its solutions.

a) $2x^3 = -2$
b) $2x + 3x^3$
c) $x^2 + 5 = x + 5$
d) $\frac{4}{x} = 4$
e) $-8x^7$
f) $3(x-7) = 0$

Ex.13 Find a number that makes the equation $\frac{1}{x} = -\frac{1}{2}$
a) true
b) false

Ex.14 Find a number that is
a) a solution of $-x = 5$
b) not a solution of $-x = 5$

Ex.15 Check each of the following to determine whether or not it is a solution of the equation

$$(4-x)(2x+5)(x+3)x = 0$$

a) $x = -3$
b) $x = 0$
c) $x = -\frac{5}{2}$
d) $x = \frac{5}{2}$
e) $x = -4$
f) $x = 4$

Ex.16 Guess four solutions of the equation $-(2-x) = -2 + x$.

Ex.17 Mr. X tried to solve the following equation $x = 2x$ by dividing each side by x. As a result, he obtained the equation $1 = 2$ and concluded that the equation $x = 2x$ has no solution. The correct solution of this equation is $x = 0$. What did Mr. X do wrong?

Ex.18 Solve the following linear equations and check your solution.
a) $x + 4 = 1$
b) $3x = 18$
c) $-7 + x = -2$
d) $-5 = \frac{x}{4}$
e) $-10 = 0.2x$
f) $-5 = -2 + x$
g) $4 = -2x$
h) $-x = 8$

Ex.19 Solve the following equations and check your solution.
a) $2x + 15 = 7$
b) $4x - 7 = 5$
c) $9 = 3 - 2x$
d) $-5x + 4 = 0$
e) $-x - 13 = 5$
f) $10 = -3x - 5$

Ex.20 Solve the following equations.
a) $-4x = x + 12$
b) $3x = 5 - 4x$
c) $6x = 15x$
d) $-4x + 2 = 3x$
e) $3 - 7x = x$
f) $3 = -1 - 4x$

Ex.21 Solve the following equations.

a) $4(x-3)=12$ b) $0=2(1+x)$

c) $-6=2(x+5)$ d) $-(x+3)=2x$

e) $-2(x-3)=-x$ f) $x-4=-(x+2)$

Ex.22 Determine which of the following equations has no solution, exactly one solution, or a solution set consisting of all real numbers.

a) $x-8=x-9$

b) $-x=0$

c) $x-x=0$

d) $4(x+2)=2+4x$

e) $3x+1=3\left(x+\frac{1}{3}\right)$

Ex.23 Solve the following equations.

a) $6x=-12$ b) $\frac{a}{4}=-15$

c) $-x=-7$ d) $-x=0$

e) $\frac{1}{6}a+4=4$ f) $3x-7=8$

g) $0.8y-0.1=1.5$ h) $22-5y=y$

i) $-x-4x=-2x$ j) $x-5=3x+7$

k) $4x+3=1-x$ l) $2.3a-5=1.8a$

m) $3(2B+5)=16$ n) $-3x=12-7x$

o) $-3(x-2)=6-3x$ p) $x-7=-3(x+5)$

q) $3-7x=2-4x$ r) $3(2m-4)=6m+6$

s) $-(x+3)=2x+3(1-x)$ t) $-2(4x-1)+7x=4-x$

u) $4(y-2)=-(1+y)$ v) $-(x+3)+x=-(2x-1)$

Lesson 9

Topics: Solving linear equations involving fractions; Solving equations for a given variable.

We will continue solving equations.

Solving linear equations involving fractions

If you are to solve an equation involving fractions, you can eliminate the fractions by applying the following 'trick': **multiply both sides by a common multiple of all denominators of the equation**. This step or *'trick'* is not a requirement; however, if you chose to omit this step, you would have to continue to work with fractions in order to solve the equation.

■ Solve the following equation $\frac{4x}{5}+1=\frac{2}{3}$.

$\frac{4x}{5}+1=\frac{2}{3}$	In this equation, "15" is a common multiple of both denominators (which are 5 and 3). We multiply both sides by 15.
$15\left(\frac{4x}{5}+1\right)=15\left(\frac{2}{3}\right)$	Remove parentheses.
$15\cdot\frac{4x}{5}+15\cdot1=15\cdot\frac{2}{3}$	Cancel the denominators.
$3\cdot4x+15=5\cdot2$	Simplify.
$12x+15=10$	'Bring 15 to the other side' by subtracting 15 from both sides
$12x=-5$	Divide each side by 12.
$x=-\frac{5}{12}$	

The solution of the equation $\frac{4x}{5}+1=\frac{2}{3}$ is $x=-\frac{5}{12}$. ■

Example 9.1 Solve the following equation $\frac{3x-1}{2}=1-\frac{x}{3}$.

Solution:

$\frac{3x-1}{2}=1-\frac{x}{3}$	To get rid of fractions, multiply both sides by 6 (common multiple of all denominators on both sides, i.e. 2 and 3).

$6 \cdot \frac{3x-1}{2} = 6 \cdot \left(1 - \frac{x}{3}\right)$	Remove parentheses applying the Distributive Law.
$6 \cdot \frac{3x-1}{2} = 6 - 6 \cdot \frac{x}{3}$	Simplify.
$3(3x-1) = 6 - 2x$	Remove parentheses once again.
$9x - 3 = 6 - 2x$	Add $2x$ to both sides to 'group' all x's on one side.
$9x - 3 + 2x = 6 - 2x + 2x$	Collect like terms on each side separately.
$11x - 3 = 6$	Add 3 to both sides.
$11x - 3 + 3 = 6 + 3$	Perform the indicated operations.
$11x = 9$	Divide each side by 11 to isolate x.
$x = \frac{9}{11}$	

The solution of $\frac{3x-1}{2} = 1 - \frac{x}{3}$ is $x = \frac{9}{11}$.

Example 9.2 Let $a = 3x - 2$ and $b = -x + 2$. Find the value of x so that the following is true.

a) $a = b$ b) $a = \frac{b}{2}$

Solution:

a) In the equation $a = b$, use substitution to replace a with $3x - 2$, and b with $-x + 2$. Solve the obtained equation.

$3x - 2 = -x + 2$	'Bring all x's on one side' by adding x to both sides.
$3x + x - 2 = 2$	Collect like terms.
$4x - 2 = 2$	Add 2 to both sides. Simplify.
$4x = 4$	Divide each side by 4. Simplify.
$x = 1$	

If $a = 3x - 2$ and $b = -x + 2$, then $a = b$ when $x = 1$.

b) In the equation $a = \frac{b}{2}$, substitute a with $3x - 2$, and b with $-x + 2$. Solve the obtained equation.

$3x - 2 = \frac{-x+2}{2}$	Multiply each side by 2.
$2(3x - 2) = 2 \cdot \frac{-x+2}{2}$	Remove parentheses on the left-hand side of the equation.
	Simplify the right side of the equation.
$6x - 4 = -x + 2$	Add x to both sides of the equation. Simplify.
$7x - 4 = 2$	Add 4 to both sides. Simplify.
$7x = 6$	Divide each side by 7.
$x = \frac{6}{7}$	

If $a = 3x - 2$ and $b = -x + 2$, then $a = \frac{b}{2}$ when $x = \frac{6}{7}$.

Often, equations express a relationship between more than one variable. For instance,

$$A = L \cdot h, \quad P = 2L + 2W, \quad \text{or } v = \frac{d}{t}$$

Consider the first equation. In this equation A is expressed in terms of L and h. Suppose that we are asked to find h when the value of L and A are given, and then to repeat this calculation for several different values of L and A. Rather then substituting the values of L and A each time in the original equation and then solving it for h, it seems to be easier to first express h in terms of L and A. Then we could more easily calculate the value of h. Expressing h in terms of the other variables is called solving an equation for h.

Solving an equation for a given variable

To solve an equation for a given variable means to isolate that variable on one side of the equation with all other quantities on the other side.

The steps used in the process of solving for a given variable are exactly the same as those used in solving linear equations. Treat the specified variable as if it were the only variable in the equation and treat the other variables as if they were numbers. Isolate the specific variable by adding, subtracting, multiplying or dividing both sides by a suitable expression.

■ Solve $A = L \cdot h$ for h.

$A = L \cdot h$ — Since h is multiplied by L, we divide both sides by L (we assume that $L \neq 0$, otherwise the operation cannot be performed).

$\frac{A}{L} = \frac{L \cdot h}{L}$ — Simplify.

$\frac{A}{L} = h$

The equation $A = L \cdot h$ is solved for h: $h = \frac{A}{L}$. ■

■ Solve $d = \frac{d^2}{t}$ for t (assume, $t \neq 0$)

$d = \frac{d^2}{t}$ — Remove t from the denominator by multiplying both sides by t.

$td = \frac{d^2}{t} \cdot t$ — Simplify.

$td = d^2$ — Divide both sides by d (assume that $d \neq 0$).

$\frac{td}{d} = \frac{d^2}{d}$ — Simplify.

$t = d$

The equation $d = \frac{d^2}{t}$ is solved for t: $t = d$. ■

■ Solve $xa = c + ya$ for a.

$xa = c + ya$	By subtracting ya from both sides, group all terms with a on one side.
$xa - ya = c$	If a variable for which we solve an equation appears in several terms, factor it. In this case, factor a.
$a(x-y) = c$	Divide each side by $x-y$ (assume $x - y \neq 0$).
$a = \dfrac{c}{x-y}$	

The equation $xa = c + ya$ is solved for a: $a = \dfrac{c}{x-y}$. ■

<u>Example 9.3</u> Solve the equation $\dfrac{A-B^2}{B} = B$ for A, and then find the value of A, if $B = 3$.

Solution:

$\dfrac{A-B^2}{B} = B$	Multiply each side by *B*.
$B \cdot \dfrac{A-B^2}{B} = BB$	Simplify.
$A - B^2 = B^2$	Add B^2 to both sides.
$A = 2B^2$	

To find the value of A, if $B = 3$, replace B in $A = 2B^2$ with 3 and evaluate.
$A = 2\times 3^2 = 2\times 9 = 18$.

<u>Example 9.4</u> Solve each equation for the specified variable. Always assume that the denominator (divisor) is different from zero. Simplify your final answer.

a) $\dfrac{a}{b} - b^2 = 3b^2$ for a b) $xy = 1 - y + x$ for x

Solution:

a) $\dfrac{a}{b} - b^2 = 3b^2$	Add b^2 to both sides.
$\dfrac{a}{b} = 3b^2 + b^2$	Collect like terms.
$\dfrac{a}{b} = 4b^2$	Multiply each side by b.
$b \cdot \dfrac{a}{b} = 4b^2 b$	Simplify.
$a = 4b^3$	
b) $xy = 1 - y + x$	Bring all terms with *x*'s on one side.

$xy - x = 1 - y$	Factor x.
$x(y-1) = 1 - y$	Divide each side by $y-1$
$x = \dfrac{1-y}{y-1}$	Factor -1 from the numerator (or you could choose to factor -1 from the denominator).
$x = \dfrac{-1(y-1)}{y-1}$	Cancel $y-1$.
$x = -1$	

Exercises with Answers (For answers see Appendix A)

When needed, assume that all denominators (divisors) are not equal to zero.

Ex.1 Solve the following equations.

a) $\dfrac{x-2}{4} = -5$

b) $-1 = 8x - \dfrac{x}{2}$

c) $-3x - \dfrac{x}{5} - 1 = 0$

d) $\dfrac{x-1}{3} = 1 - \dfrac{x}{2}$

e) $\dfrac{1}{5}y - \dfrac{1}{5} = \dfrac{3}{10}y$

f) $\dfrac{x}{2} + \dfrac{2}{3} = \dfrac{3}{4}$

Ex.2 Solve the following equations.

a) $8x - 2 = 3x + 5(x+2)$

b) $-x + (2-x) = 5x - 12x$

c) $1 - \dfrac{2}{3}a = \dfrac{5}{6}$

d) $\dfrac{3x}{2} - \dfrac{3}{4} = -1$

e) $4(3x-1) = 12x - 2$

f) $4x - 2 = 3x - (1 - 5x)$

g) $2(x-3) + 4 = x - 2$

h) $-\dfrac{3x}{2} = -1 + \dfrac{3x}{5}$

i) $3(4-x) = -2(x-6) - x$

j) $\dfrac{5}{8}x - \dfrac{7}{12} = x - \dfrac{3}{4}$

k) $3(4-7x) = 1 - 5(3+x)$

l) $-(x+2) + 2(x-1) = 3x + 5$

Ex.3 Let $A = 2x,\ B = -x.$ Find x, so the following are true (Hint: Substitute algebraic expressions for A and $B.$ Then solve for x).

a) $A + B = 0$

b) $A - 3B = 1$

c) $\dfrac{A}{3} = \dfrac{B}{8}$

Ex.4 Let $P = 3(x-1)$ and $Q = 4x + 5$. Find x such that the following is true (Hint: Substitute algebraic expressions for P and Q. Then solve for x).

a) $P = Q$

b) $P = -Q$

c) $\dfrac{Q}{2} = P$

Ex.5 Let $x = -3a + 2, \quad y = 2a + 1, \quad z = 2 - a$. Find a, so the following are true (Hint: Substitute algebraic expressions for $x, y,$ and z. Then solve for a).

a) $x + y = z$

b) $2x = y + z$

c) $\dfrac{x}{8} + \dfrac{y}{2} = \dfrac{z}{4}$

Ex.6 Does the following statement make sense: "Solve $\dfrac{x+1}{2} - 5y$ for x"? Why or why not?

Ex.7 Solve for x. a) $\dfrac{x}{2} = 7$ b) $\dfrac{x}{a} = b$

Ex.8 Solve for x. a) $x + 3 = 8$ b) $x + a = b$

Ex.9 Solve for x. a) $3x - 7 = 11$ b) $ax - b = c$

Ex.10 Solve for x. a) $5x - 2x - 6 = 0$ b) $ax - bx - c = 0$

Ex.11 Solve for x. a) $4x = 2(x + 1)$ b) $ax = b(x + c)$

Ex.12. Solve for the indicated variable. Simplify your answer whenever possible.

a) $-x = a$ for x

b) $\dfrac{b}{a} = ac$ for b

c) $\dfrac{a}{b^3} = \dfrac{b}{c}$ for a

d) $ax + b = 4b$ for a

e) $\dfrac{a^2}{u} = a$ for u

f) $abc^2 = (ac)^3$ for b

g) $\dfrac{2m}{n^2} = 4n$ for m

h) $2x + y = t + 3x$ for y

i) $x^3 + xy = 2x^3$ for y

j) $AX - A = 1 - X$ for A

k) $s(x-1) = s^2$ for x

l) $ax - by = 3ax + 4by$ for a

m) $3(v - t) = s$ for v

n) $ax + x(a+2) = 1$ for x

o) $\dfrac{m - 2n}{x} = 2n - m$ for x

p) $mb + 2mb^2 = 3b$ for m

Ex.13 Solve the following equation $\dfrac{x}{y} = d + e$ for

a) d b) x c) y

Ex.14 Solve the following equation $3at + b = 2at + t$ for

a) b b) a c) t

Ex.15. Solve $\frac{x}{a^3} = a^6$ for x, and then evaluate, if $a = -1$.

Ex.16. Solve $3mn = mn - 2m^2$ for n, and then evaluate, if $m = -4$.

Ex.17. If b represents the base of a triangle, h is the height, and A is the area of the triangle, then the following is true: $A = \frac{bh}{2}$.

a) Solve the above formula for h

b) Find the height of a triangle with base $b = 2$ inches and the area $A = 4$ square inches

Ex.18. If L represents the length of a rectangle and W its width, then the perimeter P of the rectangle is given by the formula $P = 2(L + W)$

a) Solve the above formula for W.

b) Find the value of W when $P = 4,$ and $L = 1$.

c) Find the width of a rectangle with perimeter 10 inches and length equal to 3 inches.

Lesson 10

Topics: Linear Inequalities; Graphing sets of the type $x > a$, $x \geq a$ on a number line.

We will now be discussing inequalities.

Meaning and symbolic notation for 'less (greater) than', 'less (greater) than or equal to'

Recall that

$x < 2$ **means that a number x is less than 2**
$x > 2$ **means that a number x is greater than 2.**

Notice, that $x < 2$ has exactly the same meaning as $2 > x$**,** and $x > 2$ the same as $2 < x$ (just as $1 < 2$ has exactly the same meaning as 2 > 1).

The following symbols are also in use

$x \leq 2$ **means that a number x is less than or equal to 2**
$x \geq 2$ **means that a number x is greater than or equal to 2.**

Again, $x \leq 2$ is equivalent to $2 \geq x$, $x \geq 2$ is equivalent to $2 \leq x$.

The only difference between '$x \leq 2$' and '$x < 2$' is that 2 satisfies the first inequality but does not satisfy the second one (2 is not less than 2). Similarly, 2 satisfies **'** $x \geq 2$ **',** but does not satisfy **'** $x > 2$ **'.**

Example 10.1 Describe the following set of numbers using inequality signs.

a) All numbers x that are positive
b) All numbers x that are non-negative
c) All numbers x that are at most equal to 9
d) All numbers x that are at least equal to 9
e) All numbers x that are not less than 9

Solution:

a) $x > 0$ (zero should not be included)
b) $x \geq 0$ (zero should be included)
c) $x \leq 9$ (at most means that amount or less)
d) $x \geq 9$ (at least means that amount or higher)
e) $x \geq 9$ (not less than includes the given number and higher)

Example 10.2 Determine which of the following numbers satisfies the condition $x \leq -2$.

$-4,\quad -2.1,\quad -2,\quad -0.9,\quad 0,\quad 1$

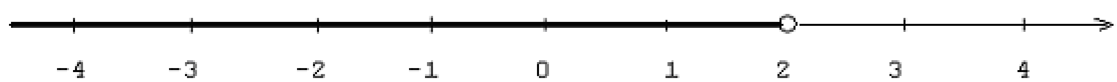
Solution:

For each number you should check if it satisfies the inequality by replacing x with this number. For example, the number -4 satisfies the inequality $-4 \le -2$ (it might help to plot the 6 points on a number line and note which ones are either equal to -2 or to the left of -2). The final answer is $-4, \quad -2.1, \quad -2$

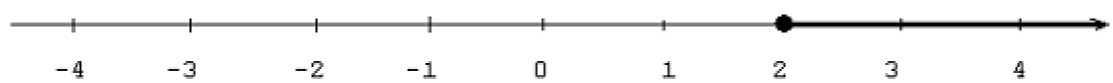
Graphing of inequalities

To Graph an Inequality **To graph an inequality means to plot all numbers satisfying the inequality on a number line.**

Graphing gives a visual representation of a given set of numbers. For example, the graph of $x < 2$ consists of all numbers less than 2. These numbers are represented by points that are to the left of 2 on a number line. The number 2 does not belong to the set. We shade **all points to the left of 2 and place 'an open circle' at 2, to indicate that 2 is excluded.**



The graph of $x \ge 2$ consists of all numbers greater than or equal to 2, represented by points to the right of 2 on a number line, including 2. The number 2 does belong to the set. We will **shade all points to the right of 2 and place 'a closed circle' at 2, to indicate that 2 is included.**



A solution, the solution set, and what it means to solve an inequality

The concept of solution(s) of an inequality is identical to the concept used in equations.

Solution Set of an Inequality **The values of the variables for which the inequality is true are called solutions. The solution set of the inequality is the set of all solutions of the inequality.**

For example, $x = 3$ is **a solution** of $x + 2 < 6$ (when we replace x with 3, we obtain the true statement $3 + 2 < 6$). However, it is not **the solution set,** because there are other solutions of the inequality, for example $x = 1$ $(1 + 2 < 6)$. As expected,

To Solve an Inequality **To solve an inequality is to find the set of solutions of the inequality or prove that it does not have a solution.**

Operations that can be performed on any inequality without changing its solution

The following operations can be performed to any inequality without changing its solution.

Any quantity can be added to, or subtracted from both sides of an inequality.
For example, If $x < y$ then $x+2 < y+2$
If $x < y$ then $x-2 < y-2$

Both sides of an inequality can be multiplied, or divided by any **nonzero** quantity, but **if the quantity is negative the direction of the inequality symbol has to be reversed.**
For example, If $x < y$ then $2x < 2y$ but $-2x > -2y$
If $x < y$ then $\frac{x}{2} < \frac{y}{2}$ but $\frac{x}{-2} > \frac{y}{-2}$

We must always perform the same operation on both sides of an inequality, just as we did in our work with equations.

Example 10.4 Knowing $6 \le -3x$, which of the following inequalities must also be true?
a) $-2 \le x$ b) $0 \le -3x-6$

Solution:
a) Since x appears without -3 on the right-hand side of the inequality $-2 \le x$, we conclude that the original inequality must have been divided by -3 on both sides. But any time you divide an inequality by a negative number, the inequality sign must be reversed. The resulting inequality would have been $-2 \ge x$ (not $-2 \le x$). Thus $-2 \le x$ does not have to be true.
b) Since 6 on the left-hand side of the inequality $6 \le -3x$ is 'replaced' with zero, we conclude that 6 must have been subtracted from both sides of the inequality. The resulting inequality would be $6-6 \le -3x-6$, which after simplification gives us $0 \le -3x-6$. Thus, if $6 \le -3x$ is true, $0 \le -3x-6$ must also be true.

Solving linear inequalities in one variable

There are many types of inequalities, but in this course we will focus on linear inequalities only. To solve a linear inequality, we apply the same strategy as in the case of a linear equation. That is, we add, subtract, multiply or divide both sides of an inequality by suitable quantities with the goal of isolating the variable on one side of an inequality. However, we must remember the following rule.

Any time we multiply or divide both sides of an inequality by a *negative* number, the direction of the inequality sign must be reversed.

Consider the following example.

■ Solve the inequality $-4x+9 \ge 1$.

$-4x+9 \ge 1$	Subtract 9 from both sides.
$-4x+9-9 \ge 1-9$	Perform the indicated operations (on each side separately).
$-4x \ge -8$	Divide both sides by -4 and **reverse the direction of the inequality sign.**
$\dfrac{-4x}{-4} \le \dfrac{-8}{-4}$	Cancel fractions.
$x \le 2$	

The solution of the inequality $-4x+9 \ge 1$ is $x \le 2$ ■

This means that all numbers that are less than or equal to 2 make this inequality true. That is, if in the original inequality x is replaced by any number less than or equal to 2, the resulting numerical statement will always be true. It also means that any other number, i.e. any number that is greater than 2, when substituted for x, would give us a false numerical statement.
Observe that there are infinitely many solutions and that is why, instead of listing them, we must use the appropriate inequality symbol in recording the result.

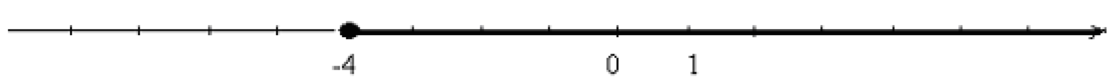
<u>Example 10.5</u> Solve the following inequalities and graph their solution.

a) $-x \le 4$ b) $3x-7 \le -2$

Solution:

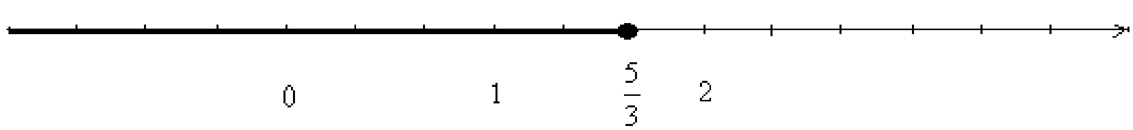
a) $-x \le 4$	Multiply both sides by -1, remember about reversing the inequality sign.
$(-1)(-x) \ge (-1)(4)$	Perform the indicated operations.
$x \ge -4$	

The graph of the solution:



b) $3x-7 \le -2$	Add 7 to both sides.
$3x-7+7 \le -2+7$	Simplify.
$3x \le 5$	Divide each side by 3.
$\dfrac{3x}{3} \le \dfrac{5}{3}$	Simplify.
$x \le \dfrac{5}{3}$	

The graph of the solution:



Example 10.6 Solve the following inequality $\frac{4-x}{2}-\frac{x+2}{4}>5$

Solution:

$\frac{4-x}{2}-\frac{x+2}{4}>5$	To get rid of fractions, multiply both sides by 4 (a common multiple of both denominators).
$4\cdot\left(\frac{4-x}{2}-\frac{x+2}{4}\right)>5\cdot 4$	Apply the Distributive Law to remove parentheses.
$4\cdot\frac{4-x}{2}-4\cdot\frac{x+2}{4}>5\cdot 4$	Simplify (place the parentheses correctly)
$2(4-x)-(x+2)>20$	Again apply the Distributive Law to remove parentheses.
$8-2x-x-2>20$	Collect like terms on each side of the inequality separately
$-3x+6>20$	Subtract 6 from both sides.
$-3x+6-6>20-6$	Simplify.
$-3x>14$	Divide each side by -3 and reverse the inequality sign.
$\frac{-3x}{-3}<\frac{14}{-3}$	Simplify.
$x<-\frac{14}{3}$	

The solution of $\frac{4-x}{2}-\frac{x+2}{4}>5$ is $x<-\frac{14}{3}$.

Linear inequalities with no solution or solution consisting of all real numbers

It might happen that during the process of solving an inequality, the variable gets cancelled and the inequality reduces to a numerical statement (you can remember this occurring when solving some equations).

■ Solve the following inequality $x+3\le x+5$.

$x+3\le x+5$	Subtract x from each side
$x+3-x\le x+5-x$	Simplify
$3\le 5$	

The statement $3\le 5$ is always true, regardless of the value of x. Thus, the solution set of $x+3\le x+5$ consists of all real numbers. ■

■ Solve the following inequality $x+5 \le x+3$.
This time, after subtracting x from each side we obtain

$$5 \le 3$$

This is a false statement, false for all values of x. There is no value of x that would make this statement true. Thus $x+3 \le x+5$ has no solutions. ■

Example 10.6 Solve the following inequality.

$$-3(2x+5) < -6x$$

Solution:

$-3(2x+5) < -6x$	Remove parentheses.
$-6x-15 < -6x$	Add $6x$ to both sides.
$-6x-15+6x < -6x+6x$	Simplify.
$-15 < 0$	

Since the obtained inequality $-15 < 0$ is always true, we conclude that all real numbers are solutions of the inequality $-3(2x+5) < -6x$.

Common mistakes and misconceptions

Mistake 10.1

DO NOT forget to reverse the inequality sign any time you multiply or divide both sides of an inequality by a negative number.

Mistake 10.2

When solving an inequality DO NOT change the inequality sign to an equation sign.

Mistake 10.3

DO NOT forget to write the final answer. If you are getting $-3 > 5$ that means there is no solution. You must write that there is 'no solution'. Likewise, if you get something like 5>3, which is true regardless of the value(s) of the variable(s), that means all numbers are solutions. You must write "all real numbers".

Exercises with Answers (For answers see Appendix A)

Ex.1 List two numbers satisfying $x < 5$. List two numbers satisfying $a < 5$. Is the second question different from the first one? How about asking: List two numbers satisfying $5 > x$?

Ex.2 Which of the following statements has the same meaning as $x < -2$.

a) $-2 > x$
b) $-2 \le x$
c) $x > -2$

d) $x \le -2$ and $x \ne -2$
e) All numbers x that are at least -2.
f) All numbers x that are no more than -2.

Ex.3 Name three numbers that satisfy the condition $x \le -1$

Ex.4 Find a number that satisfies $x \ge \frac{2}{3}$ but does not satisfy $x > \frac{2}{3}$.

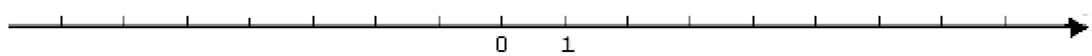
Ex.5 Circle all numbers that satisfy the following inequality $x > -3$
-3, -2, -1, 0, 1, 2, 3, 4, 5

Ex.6 Determine which of the following numbers do not satisfy the inequality $x > -3$
-3, -2, -1, 0, 1, 2, 3, 4, 5

Ex.7 Determine which of the following numbers satisfies the inequality $x \le -0.6$.
$\frac{-1}{2}$, $-.666$, -6, -0.6 $-.5999$, 0

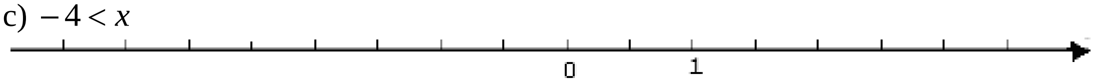
Ex.8 Describe the following sets of numbers using inequality signs.
a) All negative numbers x.
b) All non-positive numbers x.
c) All numbers x that are at least equal to 6.
d) All numbers x that are at most equal to 6.
e) All numbers x that are not more than 6

Ex.9 Graph the following sets on a number line provided. Assume that all marks on the line are equally spaced.
a) $x < -4$

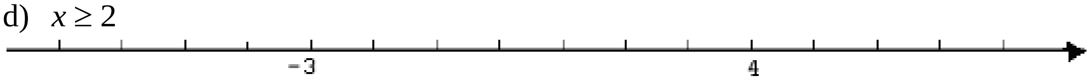


b) $x \ge 1\frac{2}{3}$

c) $-4 < x$

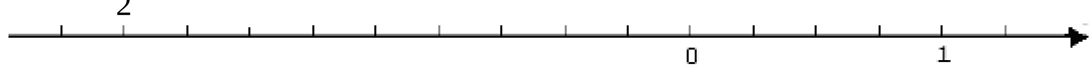


d) $x \ge 2$



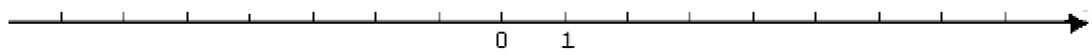
e) $x > \frac{4}{3}$

f) $x \le -\frac{5}{2}$



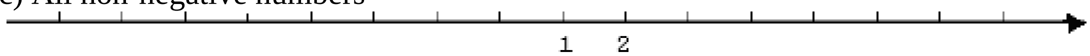
Ex.10 Graph the following number sets on a number line.

a) All numbers that are no less than -2



b) All numbers no more than 4

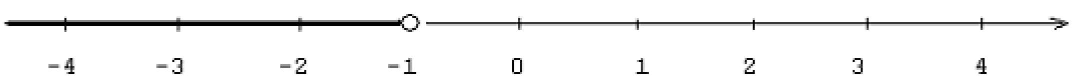
c) All non-negative numbers



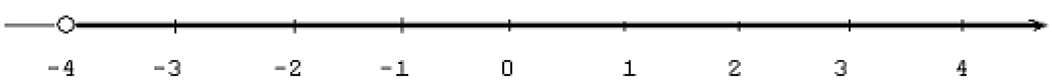
d) All numbers that are at most -1

Ex.11 Using inequality symbols, describe the set that is graphed below.

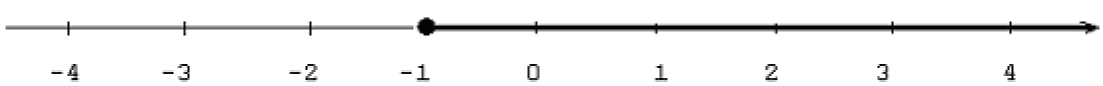
a)



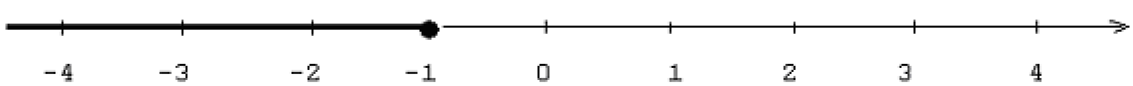
b)



c)



d)



Ex.12 Graph $x \geq \frac{2}{3}$ and $x \leq \frac{2}{3}$ on one number line, and then find a number that satisfies $x \geq \frac{2}{3}$ and also satisfies $x \leq \frac{2}{3}$.

Ex.13 Plot the points in part (a) and (b) on separate number lines. Then write an inequality for each that is satisfied by all points from the set.

a) 3, 4, 6

b) -2, -1, 3

Ex.14 Plot the points in part (a) and (b) on separate number lines. Then write an inequality for each that is *not* satisfied by any of these points.

a) 0, 1, 4   b) $-\frac{1}{2}$, $1\frac{1}{2}$ 2

Ex.15 Find an inequality that is satisfied by -1 but not by 3 (if it helps, you might plot the points).

Ex.16 Find an inequality that is satisfied by $\frac{3}{4}$ but not by 5 (if it helps, you might plot the points).

Ex.17 The number -5 is a solution of which of the following inequalities. Determine your answer without solving the inequality. Show your work.

a) $x+2<4$ b) $-x+8\geq 13$

c) $-x+3<1$ d) $3x>-15$

Ex.18 The solution of an inequality is $x\geq 2$
Here are the answers given by students.

Student A: $x>2$
Student B: $2\leq x$
Student C: $-x\geq -2$
Student D: $-2\leq x$
Student E: $x>3$

List all students who correctly solved the inequality.

Ex.19 The solution of a given inequality is $x\geq 0$.

a) Is $x=2$ *a* solution of this inequality?
b) Is $x=2$ *the* solution of this inequality? Why?
c) How many solutions does this inequality have?
d) List three solutions of the inequality.
e) List three numbers that are not solutions of this inequality.
f) If we write the solution of the inequality as $0\leq x$, would that also be correct? Why?
g) If we write the solution of the inequality as $0<x$, would that also be correct? Why?
h) Would it be right to say that the solution consists of all positive numbers?
i) Would it be right to say that the solution consists of all non-negative numbers?

Ex.20 Determine which of the following operations requires the change of inequality sign.

a) Multiplying both sides of an inequality by -2.
b) Multiplying both sides of an inequality by 2.
c) Adding 2 to both sides of an inequality.
d) Subtracting $\frac{1}{2}$ from both sides of an inequality.
e) Dividing both sides of an inequality by -2.

Ex.21 Name the operation that must be performed on both sides of an inequality to isolate z on one side. Determine if the operation requires the change of inequality sign (indicate it in writing), and then perform the operation, reversing the inequality sign, if needed (for example, to isolate z in the inequality $z-1<3$, 1 must be added to both sides, the operation of adding 1 does not require the change of sign, the resulting inequality is $z<4$).

a) $z+5<8$
b) $-2+z<1$
c) $-12<4z$

d) $-z > -3$

e) $-\frac{z}{3} > 1$

Ex.22 Knowing $x < -10$, which of the following inequalities must also be true?

a) $x + 10 < 0$

b) $-2x > -20$

c) $\frac{x}{2} > -5$

d) $-\frac{3}{10}x > 3$

Ex.23 Knowing $-2x \geq 0$, which of the following inequalities must also be true?

a) $x \geq 2$

b) $x \leq 0$

c) $x \leq -2$

d) $2x \leq 0$

Ex.24 Solve the following inequalities and graph their solutions.

a) $-2x > 8$

b) $x - 1 \leq x$

c) $-0.3x \leq 0.6$

d) $3 > 2a - 11$

e) $-6 - x < 4$

f) $2a - 1 \leq 3$

Ex.25 Solve the following inequalities. Each time you perform the operation on both sides of inequality, name the operation together with the operand (for example write "adding 2 to both sides", "dividing both sides by 3", and so on).

a) $4 > -3 - a$

b) $3x + 1 < 6$

c) $\frac{3x}{4} < -15$

d) $-5 \geq \frac{-x}{4}$

e) $\frac{x}{4} + 1 \geq -1$

f) $\frac{x+1}{4} \geq -1$

Ex.26 Solve the following inequalities.

a) $-3x > 18$

b) $\frac{-x}{2} > 1$

c) $2 - 3x \leq 14$

d) $x \leq x$

e) $x < x$

f) $-3 - a > 4a$

g) $4x - 1 \leq 7x - 5$

h) $-x + 1 > 6x - 2$

i) $\frac{3}{2}x > -1$

j) $x - 8 > x - 9$

k) $2 - a > 4a - 5a - 2$

l) $4x - 1 \leq 2(x - 5)$

m) $-3(x + 2) \leq -3x + 2$

n) $3x + 1 < 3(x + \frac{1}{3})$

o) $3(2x + 5) > 6x + 16$

p) $3(2x - 1) < 2(x - 2)$

q) $\frac{-x-2}{4} \leq -5$

r) $\frac{-y}{2} - \frac{y+3}{3} \leq -1$

s) $\frac{2a-3}{5} - 1 > -5a + \frac{2}{3}$

t) $3 - 2(3a - 5) < -6a$

Lesson 11

Topics: Recognizing and matching patterns; Writing expressions in a prescribed way; Definition of a linear equation; Factorization of the difference of squares.

This lesson is devoted to developing the ability to rewrite expressions according to a prescribed rule or pattern, as well as to match variables in the formula with appropriate parts of the expression.

Matching expressions

If we are asked to remove parentheses in the expression, let us say $(x+3)^2$, we might do it by remembering that raising an expression to the second power means to multiply it by itself two times and thus

$$(x+3)^2 = (x+3)(x+3) = x^2 + 3x + 3x + 9 = x^2 + 6x + 9 .$$

Alternatively, in this case, we could recall and use the equation that says that for any two numbers *a* and *b*, $(a+b)^2 = a^2 + 2ab + b^2$. To this end, we have to match the expression $(x+3)^2$ to the left-hand side of the equation $(a+b)^2$ and identify *a* and *b* in this representation. It might be helpful to write the two expressions one under the other.

$$(a+b)^2$$
$$(x+3)^2$$

This makes it easier for us to see that $a = x$ and $b = 3$ in this representation. Now, we rewrite the right side of the equation by replacing *a* with *x* and *b* with 3 in the right-hand side of the equation.

$$a^2 + 2ab \ + b^2 =$$
$$x^2 + 2x \cdot 3 + 3^2 = x^2 + 6x + 9$$

Thus $(x+3)^2 = x^2 + 6x + 9$.

In order to use the formulas, we need to learn how to match expressions.

Example 11.1 The expression $\dfrac{2}{z-1} + \dfrac{x}{z-1}$ is written in the form $\dfrac{a}{c} + \dfrac{b}{c}$. What algebraic expressions represent *a*, *b*, and *c*, respectively.

Solution:

We match $$\frac{2}{z-1} + \frac{x}{z-1}$$

with $$\frac{a}{c} \quad + \quad \frac{b}{c}$$

to see that $a = 2, \quad b = x$, and $c = z - 1$.

Example 11.2 The expression $(x-2)^2+(y-1)^2=3^2$ is written in the form

$$(x-p)^2+(y-q)^2=r^2$$

What are the values of *p*, *q*, and r?

Solution:

Compare $\quad (x-p)^2+(y-q)^2=r^2$

with $\quad (x-2)^2+(y-1)^2=3^2$

to find that $p=2,\ q=1$, and $r=3$.

Rewriting expressions in a prescribed form

It is quite often that in order to match expressions we must **rewrite a given expression in a prescribed form**.

Example 11.3 Write the expression

a) $x-y$ as a sum of two expressions, that is in the form $A+B$.

b) $x+y$ as a difference of two expressions, that is in the form $A-B$.

Solution:

a) The difference of any two expressions always can be written as a sum
$x-y=x+(-y)$

b) The sum of any two expressions always can be written as a difference
$x+y=x-(-y)$

To see that (a) and (b) are true, recall the rules for signed numbers $x+(-y)=x-y$, $x-(-y)=x+y$. Now, interchange the left-hand side of each of the equations with its right-hand one (it can always be done!), and we obtain exactly what we claimed in part (a) and (b) of the solutions.

Example 11.4 Write the expression $(x-2)^2+(y+1)^2=16$ in the form

$$(x+p)^2+(y-q)^2=r^2$$

What are the values of *p*, *q*, and r?

Solution:

We rewrite $(x-2)^2+(y+1)^2=16$ as $(x+(-2))^2+(y-(-1))^2=4^2$

Compare $\quad (x+\quad p)^2+(y-\quad q)^2=r^2$ with

$$(x+(-2))^2+(y-(-1))^2=4^2$$

to find that $p=-2, \mathrm{q}=-1,$ and $r=4$.

Example 11.5 Write the following expression in the form A^2, where A is any algebraic expression. Each time identify A.

a) $81x^2$ b) x^8 c) $\dfrac{0.09x^2}{y^6}$

Solution:

a) Express 81 as a square of a number $9^2 = 81$. We write $81x^2 = 9^2 x^2 = (9x)^2$ and thus $A = 9x$.

b) Recall that $\left(a^m\right)^n = a^{mn}$, and so $x^8 = \left(x^4\right)^2$. Notice that the power of x inside the parentheses multiplied by 2 must be equal to 8. Since $4 \times 2 = 8$, x needs to be raised to the fourth power. In the expression $\left(x^4\right)^2$ we identify A as $A = x^4$.

c) Write all factors that are not already written in this form, as a square of some expression $0.09 = 0.3^2$, $y^6 = \left(y^3\right)^2$ (the power of y inside the parentheses multiplied by 2 must be equal to 6, thus we have y^3). As a result, we have

$$\frac{0.09x^2}{y^6} = \frac{0.3^2 x^2}{\left(y^3\right)^2} = \left(\frac{0.3x}{y^3}\right)^2.$$

The expression $\left(\dfrac{0.3x}{y^3}\right)^2$ is written in the form A^2, with $A = \dfrac{0.3x}{y^3}$.

Example 11.6 Write the following expressions in the form $Ax + By$, where A, B are any numbers. Identify A and B in your representation.

a) $\dfrac{-x+8y}{4}$ b) $3x-(2y+x)$ c) $x-(y+x)$

Solution:

a) Recall that $\dfrac{-x+8y}{4} = \dfrac{-x}{4} + \dfrac{8y}{4} = -\dfrac{1}{4}x + 2y$. Compare the obtained expression $-\dfrac{1}{4}x + 2y$ with $Ax + By$ to recognize that $A = -\dfrac{1}{4}$, $B = 2$.

b) We simplify the expression (hoping to obtain the desired form of the expression). $3x-(2y+x) = 3x-2y-x = 2x-2y = 2x+(-2)y$. Notice how, in the last step, we expressed subtraction as the addition. Comparing $2x+(-2)y$ to $Ax+By$ gives us $A = 2$, $B = -2$.

c) We simplify the expression $x-(y+x) = x-y-x = -y = 0x+(-1)y$.

Notice that after the simplification the variable x has 'disappeared' from the expression. Since zero multiplied by any quantity is equal to zero, in such situation, we can write $0x$ to match the prescribed form. This 'trick' is often used, and thus worth remembering. The comparison of $0x+(-1)y$ with $Ax+By$ gives us $A=0$, $B=-1$.

Example 11.7 Write the following equation in the form $y = mx + b$, where m and b are any numbers. In the expression you obtained, identify m and b.

a) $y = 2x - 4$ b) $y - x = 0$ c) $2y = -4x + 3$

Solution:

a) To match the form $y = mx + b$, subtraction of 4 must be expressed as an addition. We rewrite $y = 2x - 4$ as $y = 2x + (-4)$; $m = 2$, $b = -4$.

b) To match the form $y = mx + b$, we must isolate y on one side of the equation (which means solving the equation for y). To this end, we add x on both sides and get $y = x$, or equivalently, $y = 1x + 0$; We see that $m = 1$, $b = 0$.

c) We must solve the equation for y, and then rewrite the right-hand side in such way that it matches $mx + b$.

$2y = -4x + 3$	Divide each side by 2.
$y = \dfrac{-4x+3}{2}$	Write the fraction as a sum of two fractions.
$y = \dfrac{-4x}{2} + \dfrac{3}{2}$	Simplify.
$y = -2x + \dfrac{3}{2}$	

As a result, $m = -2$, $b = \dfrac{3}{2}$.

<u>The definition of a linear equation</u>

The ability to rewrite an expression in a prescribed form is needed to determine if a given equation is linear. We use the following definition.

Linear equation in one variable **A linear equation in one variable *x* is an equation that can be written in the form**

$$\boldsymbol{ax+b = 0,}$$

where *b* is any real number, and *a* any real number different from zero ($a \neq 0$).

The equation $2x + 3 = 0$ is already in the form $ax + b = 0$.

$$2x + 3 = 0$$
$$ax + b = 0$$

In this representation $a = 2$, $b = 3$, and thus $2x + 3 = 0$ is an example of a linear equation.

To show that the equation $5x + 1 = x$ is also an example of a linear equation, one must be able to ***rewrite it in the prescribed form*** given in the definition above, namely $ax + b = 0$. To this end, we subtract x from both sides of the equation $5x + 1 - x = x - x$, collect like terms, and obtain the equation $4x + 1 = 0$ that matches the form $ax + b = 0$

$$4x + 1 = 0$$
$$ax + b = 0$$

We can now see that $a = 4$ and $b = 1$, and we have matched to the pattern given in the definition. Therefore, we can conclude that $5x + 1 = x$ is a linear equation in one variable.

Equations $x^2-3=0$ and $4x^3=5$ are not linear equations. No algebraic operation would 'remove' x^2 or x^3 from the equation, and hence each of the equations can never be written to match the desired form $ax+b=0$.

Example 11.8 Determine if the following equations are linear in one variable. If so, express them in the form $ax+b=0$, $a\neq 0$. Determine the values of a and b in your representation.

a) $\frac{x}{4}-1=0$ b) $7x+2=3x$ c) $2x(x+3)=2x^2$ d) $3x^2-1=0$

Solution:

a) It is a linear equation since it can be rewritten it in the form $\frac{1}{4}x+(-1)=0$; $a=\frac{1}{4}$ $\left(\frac{1}{4}\neq 0\right)$, $b=-1$. Notice, that this representation is not unique. If we multiply both sides of the equation $\frac{x}{4}-1=0$ by 4, we get $x-4=0$. This can be rewritten as $1x+(-4)=0$, and then $a=1$, $b=-4$. Both answers would be correct. There are other representations possible.

b) Subtract $3x$ from both sides $7x+2-3x=0$. Collect like terms $4x+2=0$. This matches the form $ax+b=0$, with $a=4$ $(4\neq 0)$, $b=2$. Thus $7x+2=3x$ is a linear equation in one variable.

c) Remove parentheses $2x^2+6x=2x^2$, subtract $2x^2$ from both sides $2x^2+6x-2x^2=0$, and simplify to obtain $6x=0$. We rewrite $6x=0$ as $6x+0=0$ (adding 0 does not change the value of the expression, but allows us to match the form). Thus $a=6$ $(6\neq 0)$, $b=0$ in our representation, and $2x(x+3)=2x^2$ is a linear equation in one variable.

d) Since no operation can remove x^2 from the equation, it is not a linear equation.

Factoring with the use of the Difference of Squares Formula

Consider the expression $(a-b)(a+b)$. When we remove parentheses and simplify, we get

$$(a-b)(a+b)=a^2+ab-ba-b^2=a^2-b^2$$

It is called the Difference of Squares Formula.

The Difference of Squares Formula

$$\boxed{a^2-b^2=(a-b)(a+b)}$$

Let us consider the expression x^2-9. Since $9=3^2$, we can rewrite x^2-9 in its equivalent form, x^2-3^2. The expression x^2-3^2 matches the left-hand side of the Difference of Squares Formula with

$a=x$ and $b=3$, so now we can rewrite the right-hand side of the formula, substituting x in place of a and 3 in place of b. We will write the formula directly above the expression x^2-3^2 to make it easier to see how x must replace a and 3 must replace b on the right hand side.

$$a^2-b^2=(a-b)(a+b)$$
$$x^2-3^2=(x-3)(x+3)$$

Notice, that by applying the Difference of Squares formula we were able to rewrite the expression x^2-9 in a form of a product, namely $x^2-9=(x-3)(x+3)$. Recall that **the process of rewriting an expression as a product is called factorization** (we used factorization (*) to simplify algebraic fractions). So far the only method of factorization we learned was a method of factoring a common factor (see, Lesson 5). The example above provides us with another method of factorization, a method that can be applied to any expression that can be rewritten to match the form a^2-b^2. Let us summarize the above considerations.

If we need to factor the expression x^2-9, we must notice that it can be rewritten as a difference of two squares, x^2-3^2 and then the Difference of Squares formula can be applied to get the desired factorization

$$x^2-9=(x-3)(x+3).$$

.

Example 11.9 Using the formula $a^2-b^2=(a-b)(a+b)$, factor the following expressions.

a) $25-x^2$ b) $x^2-\frac{1}{9}$

c) $36m^2-49n^4$ d) $m^2-(1-3m)^2$

Solution:

a) The expression $25-x^2$ must be written in the form a^2-b^2. Since $25=5^2$, we have 5^2-x^2. We recognize that $a=5$ and $b=x$, and substitute those values in the right-hand side of the equation

$$(a-b)(a+b)=(5-x)(5+x).$$

The expression $25-x^2$ is factored: $25-x^2=(5-x)(5+x)$.

b) Write the expression $x^2-\frac{1}{9}$ in the form a^2-b^2. To this end, notice that $\frac{1}{9}=\frac{1}{3^2}=\left(\frac{1}{3}\right)^2$, and thus $x^2-\frac{1}{9}=x^2-\left(\frac{1}{3}\right)^2$. We recognize that $a=x$, $b=\frac{1}{3}$, and substitute those values in the right-hand side of the equation

$$(a-b)(a+b)=\left(x-\frac{1}{3}\right)\left(x+\frac{1}{3}\right).$$

Thus $x^2-\frac{1}{9}=\left(x-\frac{1}{3}\right)\left(x+\frac{1}{3}\right)$.

(*) We should make sure that we understand that factorization of an expression does not change its "meaning". The obtained expression, although written in a different form, is still equal to the original one. In the example, we can convince ourselves that the two expressions are equal by removing parentheses in $(x-3)(x+3)$ and collecting like terms.

$$(x-3)(x+3)=xx+3x-3x-3\cdot 3=x^2-9.$$

c) Since $36m^2 = (6m)^2$, $49n^4 = (7n^2)^2$, thus $36m^2 - 49n^4 = (6m)^2 - (7n^2)^2$. The left-hand side of the equation is written in the form $a^2 - b^2$ with $a = 6m$ and $b = 7n^2$. Substituting those values in $(a-b)(a+b)$ gives us the desired factorization $36m^2 - 49n^4 = (6m - 7n^2)(6m + 7n^2)$.

d) The expression is already written in the form $a^2 - b^2$, with $a = m$ and $b = 1 - 3m$. We replace a and b in $(a-b)(a+b)$ with m and $1-3m$, respectively, and simplify.

$(a-b)(a+b) = [m-(1-3m)][m+(1-3m)] = (m-1+3m)(m+1-3m) = (4m-1)(1-2m)$

. This gives us the desired factorization: $m^2 - (1-3m)^2 = (4m-1)(1-2m)$.

Exercises with Answers (For answers see Appendix A)

Ex.1 The expression $\left(y - \frac{3}{4}\right)^2$ is written in the form $(y-a)^2$. What is the value of *a*?

Ex.2 The following expressions are written in the form $ax + by$. What are the values of *a* and *b* in these representations?

a) $3x + 4y$ b) $-4x + \frac{2}{3}y$

Ex.3 The expression $-3 + x$ is written

a) in the form $p + x$. Determine the value of p in this representation.

b) in the form $-p + x$. Determine the value of p in this representation

Ex.4 The expression x^{-2+5} is written in the form x^{a+b}. What are the values of *a* and *b*?

Ex.5 The expression $(-7)^2 x$ is written in the form $a^2 x$. What is the value of *a*?

Ex.6 The expression $4x - 2y$ is written in the form $A - 2B$. What algebraic expression represents *A* and *B.*

Ex.7 The following expressions are written in the form XY^2 For each of them determine what algebraic expression represents X and Y.

a) $3ab^2$ b) $3(ab)^2$

Ex.8 The following expressions are written in the form $\frac{c}{b}^9$. For each of expressions identify (without rewriting the expression) what algebraic expression represents *c* and b.

a) $\frac{(x-1)^9}{2}$ b) $\frac{x^9}{y^9}$

Ex.9 For each of the following expressions (1-6) indicate if they match A, B, C, D, E or F. Each time identify a and b.

(1) $x^3 - 1^3$	(A) $(a-b)^3$
(2) $(x-1)^2$	(B) $a^2 - b^2$
(3) $(8+x)(64-8x+x^2)$	(C) $(a-b)(a+b)$
(4) $(3x-5y)^3$	(D) $(a+b)(a^2-ab+b^2)$
(5) $(3x)^2-(5y)^2$	(E) a^3-b^3
(6) $(3-y)(3+y)$	(F) $(a-b)^2$

Ex.10 Among the expressions below identify those that are written in the form $A-B$ and those that are in the form $A+B$, where A and B are any expressions except 0. Those that are in the form $A-B$ rewrite as $A+B$, those that are in the form $A+B$ rewrite as $A-B$ (In other words, rewrite sums as differences and differences as sums).

a) $5-(-n)$ b) $5+n$

c) $5+(-n)$ d) $5-n$

Ex.11 Rewrite the following expressions in the form ax^3+b, where a and b are any numbers. Identify a and b in your representation.

a) $-x^3+3$ b) $2x^3-3$

c) $1-\frac{x^3}{2}$ d) $\frac{4x^3+3}{2}$

e) $\frac{3-2x^3}{2}$ f) $(2x)^3+4$

Ex.12 Write the following expressions in the form a^2, where a is any algebraic expression or a number. In each case state what a is equal to.

a) 36 b) 400

c) 0.16 d) $\frac{9}{49}$

e) $25y^2$ f) $\frac{b^2}{100}$

g) $0.49c^2$ h) X^4

i) $4x^6$ j) $81x^2y^8$

Ex.13 Write the following expressions in the form a^3, where a is any algebraic expression or a number. In each case state what a is equal to.

a) -1 b) 27

c) 0.027 d) $\frac{8}{125}$

e) $-z^3$ f) $64x^3$

g) $\frac{-x^3}{8}$ h) y^6

i) $1000x^9$ j) $\frac{x^{15}}{8y^3}$

Ex.14 Write the expressions x^{24} in the following forms.

a) a^4
b) a^6
c) a^{12}

Ex.15 Write the following expressions in the form A^7. Identify A in your representation.

a) $-x^7$ b) x^{14}

c) $x^{14}y^{21}$ d) $\frac{y^{21}}{z^{70}}$

Ex.16 Write the following expressions in the form a^m, where *a* is any algebraic expression and *m* is a positive integer different from 1. Identify *a* and *m* in your representation.

a) x^2x^3 b) $s^7(tv)^7$

c) $b(b^3)^2$ d) $\frac{B^3}{C^3}$

e) $\left(\frac{x+y}{z}\right)^3\frac{x+y}{z}$ f) $64x^2$

Ex.17 Write the following equations in the form $ax^2+by^2=0$, where a and b are any numbers. Identify a and b in your representation.

a) $x^2-2y^2=0$ b) $3x^2=y^2$

c) $\frac{x^2}{2}-(y^2-6)=6$ d) $x^2=0$

e) $\frac{3x^2+y^2}{4}=0$ f) $-x^2=\frac{8x^2-5y^2}{2}$

Ex.18 Write in the form $Ax+By+Cz$, where A,B,C are any numbers. Identify A,B, and C.

a) $-x+\frac{3}{2}z-\frac{y}{4}$ b) $-3(2x+y)+z$

c) $\frac{3x-2y+z}{4}$ d) $x-y$

e) $\frac{x+3y}{4}-x+2z$ f) $\frac{z}{3}-\frac{2}{5}z$

Ex.19 Write the following equations to match the form $y=mx+b$, where m and b are any numbers. In each case determine the value of m and b.

a) $y=3x-2$ b) $y=x$

c) $y = \frac{x}{5}$

d) $y + 3x - 2 = 0$

e) $3y = 6x - 1$

f) $y - x = -x - 5$

g) $3 - y = 3$

h) $\frac{-y + 4x + 2}{2} = 4$

Ex.20 Recall the formula for the square of the sum $(a+b)^2 = a^2 + 2ab + b^2$. The following expressions are written in the form $(a+b)^2$. For each such expression, identify *a* and *b,* and then substitute their values in $a^2 + 2ab + b^2$. Simplify.

a) $(3+b)^2$

b) $(2x+3y)^2$

c) $\left(y^2 + \frac{2}{5}\right)^2$

Ex.21 Determine if the following equations are linear equations in one variable. If so, express them in the form $ax + b = 0$, where *b* is any real number, *a* is any real number except zero, and x is unknown. Determine the values of *a*, and *b* in your representation.

a) $3x^2 - 9 = 0$

b) $\frac{x}{3} - 1 = 0$

c) $-x = 0$

d) $\frac{x+8}{8} = 0.1$

e) $x(x+1) = x^2 - 2$

f) $-3x + 7 = x$

g) $-(x+2) = 0.5$

h) $\frac{x}{5} = 2x - \frac{1}{2}$

Ex.22 The equation $4x + 2 = 0$ is a linear equation written in the form $ax + b = 0$. Record the values of a and b in this representation. Obtain an equivalent equation by multiplying both side of the equation by 3. What are the values of a and b in the new representation?

Ex.23 Using the formula for the difference of two squares $a^2 - b^2 = (a-b)(a+b)$, factor the following expressions. Simplify your answer, if possible.

a) $x^2 - 1$

b) $4 - 9x^2$

c) $y^2 - 100a^2$

d) $25s^2 - 64t^2$

e) $x^4 - \frac{1}{9}$

f) $x^2y^2 - 0.25$

g) $\frac{a^2}{b^2} - 36$

h) $m^2 - (2m+1)^2$

i) $(x+1)^2 - (3x+5)^2$

j) $(2a-3)^2 - 9a^2$

Ex.24 The following formula is true $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$. Factor each of the following expressions using the above formula. To this end

- rewrite each expression to match the left-hand side of the equation

- identify the value of a and b in your representation
- replace a and b in $(a-b)(a^2+ab+b^2)$ with their representation
- simplify, if possible

a) x^3-64 b) $1-8y^3$

c) $8x^3-27y^3$ d) $x^6-\frac{1}{27}$

Ex.25 The following formulas are true

$$a^2-b^2=(a-b)(a+b)$$
$$a^3+b^3=(a+b)(a^2-ab+b^2).$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

Factor each of the following expressions using one of the above formulas. To this end, you must first match each expression with one of the above formulas, then identify the value of a and b in your representation, and finally replace a and b in the right-hand side of the used formula, Please, simplify your answer..

a) $8x^3+y^3$ b) $36x^2-0.01y^2$

c) x^3-27y^3 d) $64a^3-b^3c^3$

Ex.26 Calculate $10.7^2-9.3^2$ using $a^2-b^2=(a-b)(a+b)$ (Show how you matched to the given identity in order to arrive at your answer).

Ex.27 The following formula is true (you are not asked to check it, although you certainly can; you have enough knowledge to do so)

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

Use the above formula to calculate 11^4. Hint: Use the fact that $11=10+1$ to write 11^4 to match the form $(a+b)^4$.

APPENDIX A: ANSWERS TO EXERCISES

Lesson 1

1 $3x+2$, y^2, $\frac{a+bc}{2}$, $(-2a+1)^3$ are examples of algebraic expressions. Ψ x, y, a, b, c are examples of variables but also examples of algebraic expressions . Variables represent unknown numbers. If we know the value of x, we can evaluate $3x+2$, and as a result we get a number.

2 a) "a squared" or "a raised to the second power." b) "a cubed" or "a raised to the third power." c) "a raised to the twelfth power." d) "2 to the m" or " 2 raised to the m-th power." e) "minus y" or "the opposite of y" f) "the product of *c* and *d*" or "c times d" g) "a minus b" h) "two-fifth times x" or "two-fifth x"

3 a) $7\times n$ b) $-5\times k\times m$ c) "there is no multiplication performed" d) $-x\times(-y)$ e) $\frac{3\times x}{2}$ f) $2\times x - y\times z + w\times(-t)$

4 a) multiplication b) division c) exponentiation d) division e) subtraction f) multiplication

5 a) Any time two operation signs are next to each other, parentheses are needed. b) If parentheses are removed, only *m* would be raised to the fourth power. c) Any time two operation signs are next to each other, parentheses are needed, even if the multiplication sign is not explicitly displayed d) If parentheses are removed, only *a* would be raised to the fourth power. e) Any time two operation signs are next to each other, parentheses are needed, even if the multiplication sign is not explicitly displayed f) Any time two operation signs are next to each other, parentheses are needed

6 a) $-s$ b) s c) st d) t e) x f) $\frac{x}{y}$ g) st h) s i) $y-s$

7 a) x b) $-x$ c) multiplication

8 a) $1\cdot x = x$; b) $-1\cdot x = -x$ c) $0\cdot x = 0$

9 a) $-(-x)$ b) $-\frac{x^3}{y}$ c) $-\left(-\frac{x^3}{y}\right)$ d) $-\left(\frac{-x^3}{-y}\right)$

10 a) $\frac{1}{2}y$ b) $\frac{3}{4}y$ c) $y+5$ d) $v-y$ e) y^2 f) $y+3$ g) $y-x$ h) xy i) $2y$

11 a) $a+(-b)$ b) $a-(-b)$ c) $a(-b)$ d) $-C$ f) $-(-C)$ g) $-\frac{-a}{-b}$

h) $v(-t)(-p)$ i) $\frac{c}{-B}$ j) $(-x)^m$ k) $\left(\frac{x}{y}\right)^m$

12 a) $x+10$ b) $\frac{2}{3}x$ c) $\frac{x}{3}$ d) $\frac{100}{x}$ e) $x-2$ f) $30x$ g) $x-3$ h) $x-5$

13 $\frac{d}{t}$

14 ma

15 $\frac{1}{2}(bh)$ or $\frac{1}{2}bh$

16 a) $3+5=8$ b) $3-2=1$ c) $\frac{3}{3}=1$ d) $4\times 3=12$ e) $3^2=9$ f) $\frac{6}{3}=2$

17 $\frac{1}{x}$ can not be evaluated with $x=0$, because the denominator of a fraction can not be 0.

If $x=0$, $\frac{1}{x-5}$ can be evaluated: $\frac{1}{x-5}=\frac{1}{0-5}=-\frac{1}{5}$, but if $x=5$ then $\frac{1}{x-5}=\frac{1}{5-5}=-\frac{1}{0}$ is undefined. Another example could be: $\frac{3}{y+4}$ cannot be evaluated with $y=-4$. (answers vary)

18 a) $3\cdot 0=0$ b) $0-2=-2$ c) undefined d) $\frac{0}{7}=0$ e) $\frac{2}{0-3}=-\frac{2}{3}$ f) undefined

19 a) $3^2=9$ b) $2^3=8$ c) $2^2=4$

20 a) $-A=-2$ b) $-A=2$

21 a) $6-8=-2$ b) $-10-6=-16$ c) $-4+6=2$ d) $6-6=0$ e) $-2+6-6=-2$

22 a) $2+(-2)=0$ b) $2-(-2)=4$ c) $-2-(-2)=0$ d) $-5-(-2)+4=1$
e) $6+(-2)-10-(-2)=-4$

23 a) $3(10)=30$ b) $-5(10)=-50$ c) $\frac{-200}{10}=-20$ d) $-\frac{10}{2}=-5$
e) $-5\div 10=-0.5$ or $-\frac{1}{2}$ f) $10^4=10{,}000$

24 a) $-1000(-12)=12{,}000$ b) $\frac{-12}{6}=-2$ c) $-5(-12)=60$
d) $\frac{6}{-12+12}$ "not possible" e) $-24\div(-12)=2$ f) $(-12)^2=144$

25 a) $\frac{7}{3}$ or $2\frac{1}{3}$ b) $\frac{13}{15}$ c) $-\frac{8}{21}$ d) $-\frac{13}{12}$ e) $\frac{8}{3}$ or $2\frac{2}{3}$ f) $-\frac{11}{3}$ or $-3\frac{2}{3}$

26 a) $\frac{9}{10}$ b) $\frac{16}{35}$ c) $\frac{8}{5}$ *or* $1\frac{3}{5}$ d) $-\frac{13}{20}$ e) $-\frac{29}{10}$ *or* $-2\frac{9}{10}$

27 a) $\frac{4}{7}$ b) -2 c) $-\frac{4}{3}$ d) $\frac{5}{8}$ e) $\frac{35}{2}$ *or* $17\frac{1}{2}$ f) $-\frac{1}{7}$

28 a) $\frac{9}{16}$ b) -1 c) $\frac{9}{20}$ d) $\frac{3}{2}$ e) $\frac{1}{4}$ f) 0

29 a) 3.41 b) 34. 81 c) 0.05 d) −8 e) 0.06 f) −5

30 a) −3.9 b) −2.1 c) 2 d) 360 e) −0.0006 f) −0.216

31 a) −1.58 b) −1.1 c) −10 d) 0.3 e) 20

32 a) $\frac{19}{15}$ *or* $1\frac{4}{15}$ b) $-\frac{5}{14}$ c) -1.28

33 a) $-\frac{1}{15}$ b) $\frac{13}{14}$ c) 0.88

34 a) $\frac{4}{9}$ b) $\frac{18}{5}$ *or* $3\frac{3}{5}$ c) -0.002

35 a) $\frac{9}{121}$ b) $\frac{40}{9}$ c) -20

36 a) $-10{,}000{,}000$ b) 16 c) $-\frac{1}{8}$ d) 0.00001 or $\frac{1}{100{,}000}$

37 a) $2x=2\left(-\frac{1}{2}\right)=-1$ b) $\frac{3}{4}x=\frac{3}{4}\left(-\frac{1}{2}\right)=-\frac{3}{8}$ c) $\left(-\frac{1}{2}\right)^2=\frac{1}{4}$

38 $-t=-1$ if $t=1$, $-t=-(-1)=1$ if $t=-1$; $-t$ may be positive or negative depending on the value of t

Lesson 2

1 a) $x-(m-2n)$ b) $-(m-2n)$ c) $(m-2n)\cdot 7$ d) $m-2n-3a$ e) $-(k^2-3k+1)$ f) $4\div(-4x+y)$

2 a) $3x+y$ b) $4(a+b)$ c) $(-x)^6$ d) $z(y-3)$ *or* $(y-3)z$ e) $9x^3$
f) $(9x)^3$ g) $\frac{a-b}{c}$ h) $\frac{3}{y}+x$ i) $-(M+3)$ j) $(-x)^3+y^7$

3 a) $2(x-7)$ b) $\frac{2}{3}(x+c)$ c) $\frac{1}{4}x-5$ d) $c-9x$ e) $\left(\frac{x}{2}\right)^3$ f) $4(-x)$
g) x^3+6 h) $y(x-4)$ i) $-(y-x)$ j) $x(x+5)$ k) $(-x)^{121}$ l) $-x^2$

4 $\frac{9}{5}C+32$

5 $2(L+W)$

6 mc^2

7 a) $a+b^5$ exponentiation b) $(a+b)^5$ addition

c) $-x^8$ exponentiation d) $(-x)^8$ opposite of x

e) $\frac{a-b}{c}$ subtraction f) $a \div b \times c$ division

g) $4-7y$ multiplication h) $3+a \div b$ division

8 a) Multiply x by 4 and then subtract y. b) Add *a* and 3 and then divide the result by *x*.
c) Add *x* and 3 and then multiply the result by *y*. d) Divide *s* by *t* and then add 2 to the result.
e) Square x and then multiply by 3. f) Multiply *x* by 3 and then square the result.
g) Add *a* and *c* and then raise the result to the 4th power. h) Raise *c* to the 4th power and then add the result to *a*.

9 a) parentheses are needed b) $c-3-a$ c) $3a+x$ d) parentheses are needed e) parentheses are needed
f) $\frac{-c+d}{a}$ g) parentheses are needed h) yx^8 i) parentheses are needed j) parentheses are needed

10 a) 2 b) 2 c) 2 d) Yes, because we performed the same operations. e) −2

11 a) $-2\times 3-5=-11$ b) $-4+3^2=5$ c) $\frac{3}{3-3}$ cannot be performed

d) $(-3)^2=9$ e) $-3^2=-9$ f) $\frac{3-3}{4+3}=0$ g) $3^3=27$

12 a) $-2^4=-16$ b) $(-2)^4=16$ c) $(-4)^2=16$ d) $-4^2=-16$

13 a) $(-(-1))+(-1)=0$ b) $(-(-1))-(-1)=2$ c) $(-(-1))(-(-1))=1$ d) $(-1)^2=1$ e) $-(-1)^2=-1$

14 a) $\frac{1}{\frac{1}{2}}=2$ b) $\frac{1}{\frac{1}{2}}+\frac{1}{2}=2\frac{1}{2}=\frac{5}{2}$ c) $\left(-\frac{1}{2}\right)^2=\frac{1}{4}$ d) $-\left(\frac{1}{2}\right)^2=-\frac{1}{4}$

15 a) 0.39 b) −5 c) −1 d) −273

16 b) c) and f) We would have 0 in the denominator in these cases.

17 a) −19 b) 60 c) −34 d) −35 e) −50

18 a) $8\times\left(-\frac{1}{8}\right)-10\times\frac{4}{5}=-9$ b) $10\left(-\frac{1}{8}\right)\left(\frac{4}{5}\right)=-1$ c) $2\left(\frac{4}{5}+\left(-\frac{1}{8}\right)\right)=\frac{37}{20}$

d) $-8\left(-\frac{1}{8}\right)^2+\frac{4}{5}=\frac{27}{40}$ e) $\frac{4}{5}\div\left(\frac{1}{8}+\left(-\frac{1}{8}\right)\right)$ cannot be performed .f) $\frac{4}{5}\div\frac{3}{10}+\left(-\frac{1}{8}\right)=\frac{61}{24}$

19 a) $2\times\left(\frac{1}{3}\right)^4=\frac{2}{81}$ b) $\left(-\frac{2}{3}\right)^4=\frac{16}{81}$ c) $-\left(-\frac{2}{3}\right)^4=-\frac{16}{81}$ d) $\frac{\frac{1}{3}-\frac{2}{3}}{\frac{1}{3}+\frac{2}{3}}=-\frac{1}{3}$ e) $\frac{\frac{1}{3}\times\left(-\left(-\frac{2}{3}\right)\right)}{\frac{1}{3}\div\left(-\frac{2}{3}\right)}=-\frac{4}{9}$

20 a) −1 b) 64 c) $-\frac{1}{9}$ d) $\frac{1}{9}$ e) 1 f) −1

21 a) − 2.2 b) −2.1

22 a) −0.1 b) 0.1 c) −0.2

23 a) −2 b) −2

24 a) −5 b) 0 c) 1.3 d) $1\frac{1}{2}=\frac{3}{2}$ e) $2\frac{13}{15}=\frac{43}{15}$ f) $\frac{4}{35}$

25 a) 4 b) 7 c) 0.5 d) 7 e) $\frac{3}{14}$ f) $\frac{17}{3}=5\frac{2}{3}$

26 a) 10 b) −50 c) −0.05

27 a) 0 b) -6

28 a) addition, $7+x$ b) multiplication c) multiplication, $12x$ d) addition, $7x$
e) exponentiation, x^8 f) subtraction $\frac{-1}{x}$

29 a) $(3x)^2 = 81$ b) $\frac{x-(-4)}{0.2} = 5$ c) $\left(\frac{9}{x}\right)^3 = -27$

Lesson 3

1 In the expression $4x^2 \times 2y$, $4x^2$ and $2y$ are called factors. In the expression $4x^2 + 2y$, $4x^2$ and $2y$ are called terms.

2 a) $3, \quad x$ b) $ab, \quad -cd$ c) $\frac{xy}{2}, \quad 2y^2, -1$ d) $-(2-b)^2, \quad \frac{x}{y}, \quad -z$

3 All these expressions are equal (equivalent) because of the commutative property of addition.

4 a) $m-n=-1$, $n-m=1$; they are not equivalent. b) True

5 a) Terms: $2m, z$; $2m+z=z+2m$ b) Terms: $x, -2$; $x-2=-2+x$ c) Terms: $-3c, 2$; $-3c+2=2-3c$
d) Terms: $-2x^2, -\frac{y^3}{2}$ $-2x^2-\frac{y^3}{2}=-\frac{y^3}{2}-2x^2$ e) Terms: $c(d-f), y^2$; $c(d-f)+y^2=y^2+c(d-f)$
f) Terms: $-(x-y)^2, \quad \frac{s+2}{3}$; $-(x-y)^2+\frac{s+2}{3}=\frac{s+2}{3}-(x-y)^2$

6 a) $x-mn+2=-mn+2+x$ b) $3-(2a-3b)+4x=4x+3-(2a-3b)$

7 a) Terms: $-x^2, x, -x^3$; $-x^2+x-x^3=x-x^2-x^3=-x^3+x-x^2$ (answers vary)
b) Terms: $-a^2, -2bc, \frac{3x}{2}$; $-a^2-2bc+\frac{3x}{2}=\frac{3x}{2}-a^2-2bc=\frac{3x}{2}-2bc-a^2$ (answers vary)

8 (1) -- (C), (2) -- (E), (3) --(A) , (4) --(B), (5)-- (D

9 a) $2\times a$; factors: 2, a b) $3\times(a+b)$; factors: $3, (a+b)$
c) $-3\times x\times\frac{2}{y}$; factors: $-3, x, \frac{2}{y}$ d) $4\times(x+y)\times(b-c)$; factors: $4, (x+y), (b-c)$

10 a) $mn=nm$ b) $-5\times 7=7(-5)$ c) $-cd=d(-c)$ d) $-c(a+d)=(a+d)(-c)$

11 a) $vst=stv=svt=tsv$ b) $v(x-y)t=vt(x-y)=(x-y)vt=(x-y)tv$.

12 a) All these expressions are equal (equivalent) because of the commutative property of multiplication
b) $-3AB=-3\times1\times2=-6$; $BA-3=2\times1-3=-1$ when $A=1$ and $B=2$, thus they are not equivalent,

13 They are all equivalent because of the commutative property of addition and multiplication

14 a) $\frac{2-5}{7}=\frac{2}{7}-\frac{5}{7}$ b) $\frac{a+6}{3}=\frac{a}{3}+\frac{6}{3}$ c) $\frac{a-2}{a+b}=\frac{a}{a+b}-\frac{2}{a+b}$ d) $\frac{b^2+c}{ab^2-c}=\frac{b^2}{ab^2-c}+\frac{c}{ab^2-c}$

15 a) $\frac{m+n}{4}$ b) $\frac{7m-n^2}{4}$ c) $\frac{5m-2n^2}{4c-2}$ d) $\frac{A-B+2C}{x}$

16 a) $\frac{4-7+2}{5}$ b) $\frac{7m+n^2-3}{4x}$ c) $\frac{m-3-t}{s-1}$

17 a) $\frac{2}{3}x$ b) $2\frac{x^2}{y}$ c) $\frac{-2}{3}x^2$ d) $-2\cdot\frac{(a+2b)}{y}$ e) $\frac{1}{3}x$ f) $\frac{1}{3}(a+2b)$

18 a) $\frac{3m}{n}$ b) $\frac{3(-m)}{n}$ c) $\frac{a(-b)}{4}$ d) $\frac{-a(-b)}{4}$ e) $\frac{(s-4)t}{n}$ f) $\frac{4(-a)}{n-1}$

g) $\frac{3(m+n)}{t}$ h) $\frac{a(-x+1)}{x^2}$

19 The opposite to $\frac{s}{t}$ is $-\frac{s}{t}$ or equivalently $\frac{-s}{t}$, $\frac{s}{-t}$. All students were right.

20 a) $-\frac{2a}{b}=\frac{-2a}{b}=\frac{2a}{-b}$ b) $\frac{2a+c}{-2d}=\frac{-(2a+c)}{2d}=-\frac{2a+c}{2d}$

21 a) $\frac{2x+4y}{3}$ b) $\frac{2x-y}{3}$ c) $\frac{2x-7y}{3}$ d) $\frac{3x-y}{t}$ e) $\frac{-2x+3y}{t}$ f) $\frac{-1-2n}{k+t}$

g) $\frac{3(a-b)+2(cd-1)}{t}$ h) $\frac{-(2+a)-3(m-n)}{xy}$

22 a) $\frac{3x}{y}=x\cdot\frac{3}{y}$ b) $\frac{3x}{y}=3\cdot\frac{x}{y}$ c) $\frac{3x}{y}=\frac{1}{y}\cdot 3x$ d) $\frac{a+b}{y}=\frac{1}{y}\cdot(a+b)$

23 a) $-4+a$ b) $-32a$ c) $2a$ d) $64x$ e) not possible f) not possible g) $2x$ h) $2x$

i) 3^m j) x^2 k) $-xy^2$ l) not possible m) $2x$ n) $2x$

o) $\frac{8}{5}-x$ p) $\frac{6y}{xz}$ q) $-1-x$ r) $\frac{bd}{3ac}$ s) $-0.06xyz$ t) $-4x$

24 Yes, both are equivalent.

25 $(x+y)(1+a)$ and $x+y(1+a)$ are not equivalent. In $(x+y)(1+a)$ the entire expression of $(x+y)$ is multiplied with $(1+a)$ and in $x+y(1+a)$ only y is multiplied with $(1+a)$.

26 a) $a-2b+c=c-2b+a$ b) $x=\frac{x}{4}\cdot 4$ c) $\frac{xy}{4}=\frac{1}{4}xy$ d) $a\div 2=\frac{a}{2}$

e) $\frac{x+y}{3}=\frac{x}{3}+\frac{y}{3}$ f) $z-c=-c+z$ g) $-xyz=yz(-x)$ h) $\frac{ab}{2}=a\cdot\frac{b}{2}$

27 $\frac{m}{-1}$, $-\frac{m}{1}$, $m(-1)$ are equivalent to $-m$.

28 $m-(n)$ and $-n+m$ are equivalent to $m-n$:

29 $-a+b-c+d$ and $d+b-c-a$ are equivalent to $-a-c+b+d$.

30 $\frac{-a}{b}$, $-a\cdot\frac{1}{b}$, $-\frac{2a}{2b}$, $\frac{a}{-b}$ are equivalent to $-\frac{a}{b}$.

31 All of the expressions are equal to $\frac{5x}{6}$ except $\frac{5}{6x}$

32 $\frac{x}{2}$, $\frac{4x}{8}$, and $\frac{-x}{-2}$ are equivalent to $x\div 2$

33 $8a+3$, $3+a\cdot 8$, $2\cdot\frac{3+8a}{2}$ are equivalent to $3+8a$

34 $\frac{1}{6}(3a-b)$, $\frac{3a}{6}-\frac{b}{6}$, $\frac{-b+3a}{6}$, and $(3a-b)\frac{1}{6}$ are equivalent to $\frac{3a-b}{6}$.

35 $\frac{m-n}{3}$, $\frac{1}{3}\cdot m-\frac{1}{3}n$, $(m-n)\cdot\frac{1}{3}$ *and* $\frac{-n+m}{3}$ are equivalent to $\frac{m}{3}-\frac{n}{3}$.

36 Both John and Mary are right.

37 $(-x)^4=1$, $-x^4=-1$. Since $-1\neq 1$ the expressions are not equivalent.

38 $x^2+y^2=5$, $(x+y)^2=1$, when $x=-1$, $y=2$. Since $5\neq 1$ the expressions are not equivalent.

39 $m-n+p=-2$, $m-(n+p)=-4$, when $m=2$, $n=5$ $n=5$, and $p=1$. Since $-2\neq -4$ the expressions are not equivalent.

40 a) $(-1)^m=-1$, $-1^m=-1$, when $m=1$ b) $(-1)^m=-1$, $-1^m=-1$, when $m=3$

c) $(-1)^m=-1$, $-1^m=-1$, when $m=5$ d) $(-1)^m=-1$, $-1^m=-1$, when $m=7$

e) No, we cannot. Even if the expressions have the same answers in a-d, we cannot conclude that the expression will always be equivalent. f) $(-1)^m = 1$, $-1^m = -1$, when $m=2$. Yes, we can, they are not equivalent. It is enough to find **one** set of values of variables for which two expressions are not equal to determine that they are not equivalent.

Lesson 4

1 coefficient, exponent or power, the base.

2 a) first b) zero

3 a) b b) ab c) de d) $-a$ e) a f) $\left(\frac{2x}{y}\right)$

4 a) $\left(\frac{a}{2}\right)^4$ b) $\left(\frac{2}{3}a\right)^3$ c) $\frac{c^3}{7}$ d) $(5a)^2$ e) $8a^2$ f) $(-x)^5$ g) $-x^{10}$

5

	base	exponent	coefficient
a)	x	4	3
b)	x	m	-1
c)	x	3	$\frac{2}{3}$
d)	$a-bc$	2	-1
e)	$\frac{x}{y}$	m	1
f)	$x+y$	7	$\frac{1}{4}$
g)	$\frac{3x+z}{w}$	7	$\frac{3}{4}$
h)	ab	5	$-\frac{1}{2}$

6 a) 6^5 b) z^4 c) $3^3 \cdot a^4$ d) $-x^5y^2$ e) $-a-a^4$ f) x^2y-xy^2 g) $(a+b)^3$ h) $(2t^3)^4$

i) $m+n^2+m$ j) $\frac{k^3}{n}\cdot k = \frac{k^4}{n}$ k) $\frac{-z-z-z}{z^4}$ l) $\frac{(-z)^3}{z+z+z}$ m) $x^3 \cdot \frac{x}{2} = \frac{x^4}{2}$ n) $(3-x)^3$

o) $\left(\frac{2a}{b}\right)^3$ p) $\left(-\frac{3}{x}\right)^3$ q) $\frac{1}{(w+2v)^3}$ r) $\left(\frac{x-y}{m}\right)^3$ s) $(m+n)^2m+n$ t) $(m+p-n)^2(n+m-p)$

7 a) $(-4)(-4)(-4)(-4)(-4)$ b) $-4\cdot4\cdot4\cdot4\cdot4$ c) $(-m)(-m)(-m)$ d) $-m\cdot m\cdot m$

e) $(2a)(2a)(2a)$ f) $2\cdot a\cdot a\cdot a$ g) $(a+b)(a+b)$ h) $a+b\cdot b$

8 a) 1 b) 3 c) x d) a e) ab f) 1 g) $ab+1$ h) 1

9 a) x^3 b) $(-x)^4$, necessary c) $-x^7$ d) $a+(2b)^3$, necessary

e) $(a+2b)^3$, necessary f) $a+2b^3$ g) $a(bc)^m$, necessary h) $\left(\frac{2x}{y}\right)^4$, necessary

10 a) 2,000,000 b) 8,000,000

The answers are different, since the order of operations is different. In part b, we must first complete operations within parentheses.

11 a) To multiply exponential expressions with the same bases one needs to add the exponents.

b) To divide exponential expressions with the same bases one needs to subtract their exponents

c) To raise an exponential expression to another power one needs to multiply exponents.

12 a) n^{23} b) s^{14} c) x^6 d) b^8 e) $16x^2$ f) $2m^9$ g) b h) $x^0 = 1$

i) $9a^{16}$ j) x^{12} k) $\frac{1}{2}a^2$ l) $3x^3$ m) $6a^{32}$ n) $5s^{11}$ o) $-5t^{12}$ p) $8x^{21}$

13 a) $m^2(-2m)=-2m^3$ b) $\frac{3x^5}{18x^3}=\frac{1}{6}x^2 \ or\ \frac{x^2}{6}$ c) $(2a^5)^3=8a^{15}$

14 a) $a=2$ b) $2a^4=32$ c) $-a^3=-8$ d) $a^2=4$

15 a) $m^8=1$ b) $-\frac{1}{2}m=\frac{1}{2}$ c) $\frac{1}{3}m^{28}=\frac{1}{3}$ d) $2m^{57}=-2$

16 a) $-x^2=-49$ b) -49 c) $-\frac{4}{9}$ d) -0.0049

17 a) $-B^5$ nc: -1 b) B^8 nc: 1 c) $-B^8$ nc: -1 d) $-B^4$ nc: -1

18 a) $8x^3$ b) $-32x^3$ c) $-6x^4$ d) $12x^8$ e) $\frac{1}{4}a^2$ f) $\frac{1}{3}x^3$ g) $-64x^2$ h) $-\frac{1}{2}a$

i) $-\frac{1}{2}a^6$ j) $-2x^4$ k) $27x^2$ l) 1 m) v^5 n) $9a^6$ o) $-\frac{2}{5}x^6$ p) $-\frac{x^6}{0.16}$

19 a) $(-3xx^2)^3=-27x^9$ b) $\left(\frac{4a^{12}}{a^2}\right)^2=16a^{20}$ c) $(-a^3)^7\cdot(a)=-a^{22}$

20 a) $x^{10}y^6$ b) x^3y^3 c) $3ab^{10}$ d) $3a^2b^4$ e) $-\frac{a^6}{16b^4}$ f) $-x^5y^5$

g) -1 h) $\frac{1}{16}x^3$ i) $2x^5y^3$ j) $4(m-n)^5$ k) $\frac{1}{2}(a+b)^4$ l) a^8bc^2

21 a) $\frac{(xy^7)^5}{y^3}=x^5y^{32}$ b) $(3ab^3)^2b=9a^2b^7$

22 a) $mn^4=-2$ b) $m^2=4$ c) $(m+n)^2=1$ d) 1

23 $\frac{4}{25}y^2$, $\frac{2y^2}{5}$, $y\cdot\frac{2}{5}\cdot y$, $\left(\frac{2}{5}y\right)^2$, $\frac{2}{5}yy$

24 $\frac{a^{20}}{3}$, $\frac{a^{20}}{3^{20}}$, $\frac{(a^{12})^8}{3^{20}}$, $\frac{a^{12}a^8}{3^{20}}$, $\frac{(a^4)^5}{3^{20}}$

25 $4y^2x^3$, $(2y)^2x^3$, $(-2)y^2(-2)x^3$ $(4xy)^2x$, $4(xy)^2x$

26 $2a^6c^2b^3$, $2a^2b^3c^2a^3$, $2(abc)^{11}$, $\frac{7c^2b^3a^6}{14}$, $2(a^3c)^2b^3$

27 a) $\Psi=8$ b) $\Psi=10$ c) $\Psi=400$ d) $\Psi=12$ e) $\Psi=14$ f) $\Psi=10$

28 a) 15 b) $3^2=9$ c) -12 d) $0.5^2=0.25$ e) $4^2=16$ f) $-2^3=-8$

Lesson 5

1 Based on the Commutative Law of Multiplication: $(a+b)c=c(a+b)$, and from here we can apply the Distributive Law: $c(a+b)=ca+cb$.

2 a) $2L+2R$ b) $R-Rx$ c) $P+Prt$ d) R^2s-r^2s e) $-3-xy$ f) x^3-7zx g) $2ac+c^2-c^6$ h) $a-a^2+2$

3 Based on the Commutative Law of Addition $xb+yb+xc+yc=xb+xc+yb+yc$, so the two answers are equivalent. Some other ways (answers vary):

$(x+y)(b+c)=xb+xc+yc+yb=yc+yb+xb+xc=yc+xc+yb+xb=xc+xb+yc+yb$

4 a) $\frac{5}{9}F-10$ b) $18y^5+24y^4$ c) x^6-2x^5 d) $-2c+22d$ e) $a^3+a^2b+a^2b^2$

f) $2x^4y^2+\frac{3}{7}xy-x^2y^5$ g) $xz-yz+xw-yw$ h) $10a-4b-a^4+\frac{2}{5}a^3b$

i) $ac+bc-gc-ad-bd+gd$ j) $x^3y+x^2y+xy+y-x^3-x^2-x-1$

k) $2xy-4y+3x-6$ l) $2ac-c-2abc+bc-4a+2+4ab-2b$

5 a) $(x^2-y)\cdot 5=5x^2-5y$ b) $-(-4x+1)=4x-1$ c) $a(-c+1)=-ac+a$

d) $2y(-a+2b+d)=-2ya+4yb+2yd$ e) $(x-1)(y^3+2)=xy^3-y^3+2x-2$

f) $-(x-x^2+2x^4)=-x+x^2-2x^4$

6 a) $3a+3b=3b+3a=(a+b)\cdot 3=a\cdot 3+b\cdot 3=b\cdot 3+a\cdot 3$ (answers vary)

b) $2z-yz=-yz+2z=z(2-y)=z\cdot 2-zy=2z-zy$ (answers vary)

7 $mn+mp$ $\quad (n+p)m$ $\quad pm+nm$ $\quad mn+p$ $\quad mp+nm$

8 $(-1)(x-y+z)$ $\quad (x-y+z)(-1)$ $\quad -1(x-y+z)$ $\quad (x-y+z)-1$

$-x+y-z$ $\quad -x-y+z$ $\quad -x+y+z$ $\quad -(x+z-y)$

9 a) 2 b) 3 c) m

10 a) $5(x+y)$ b) $7(1-7a)$ c) $c(3-2c)$ d) $y(-8xy^5+1)$

e) $11t(-1+4t^2)$ f) $2(lw+lh+wh)$ g) $x(4x^2-5x+5)$

11 a) $xy(2-a^2)$ b) $xy(-x+y)$ c) $xy(a+b-1)$

12 a) $-1(-3-x)$ b) $-1(a-b-1)$ c) $-1\left(-a+\frac{x+y-z}{2}\right)$

13 a) $5a(2-3a)$ b) $11t\left(-\frac{1}{2}t+4\right)$ c) $5x^3(3x^2+1)$ d) $-4y^5(2xy-1)$

e) $c^2d^2(8-ad)$ f) $3x^2y(-1+3xy)$ g) $\frac{x}{y}(2+a)$ h) $\frac{1}{a+b}(1+s)$

14 a) $\frac{2}{3}(x^2y-2z)$ b) $\frac{1}{5}\left(x-\frac{1}{5}\right)$ c) $0.6(6x-10y+1)$ d) $\frac{1}{6}(5a+6b)$

e) $\frac{2}{7}(2x^2-1)$ f) $0.1(20a+3b-1)$ g) $\frac{3}{11}(11xy+2x-\frac{11}{3})$ h) $0.02(20m-50n)$

15 a) $(a+b)(6-x)$ b) $(a+b)[4-3(a+b)]$

16 a) $2xy((1-y+2xy)$ b) $a^3b^3(b+5a^4-1)$ c) $17xy(x^4y^2+2x^2y+3)$ d) $m^4n(1-5n^2+m^4)$

e) $4ac^3(-4+2c^4-3c^5d)$ f) $7ab^2(5b-2ab^2+3)$ g) $(x-2y)(-2+z^2)$ h) $(c+d)^2[1-4a(c+d)]$

17 $a\left((1-a+\frac{3}{a}\right)$

18 a) $2(2-\frac{x}{2})$ b) $x\left(\frac{4}{x}-1\right)$ c) $2x\left(\frac{2}{x}-\frac{1}{2}\right)$ d) $4x\left(\frac{1}{x}-\frac{1}{4}\right)$

19 a) Numerator: One term, *t*. (can be viewed as two factors, 1 and *t*) Denominator: two terms: 2*t* (with factors 2 and *t*) and −*ty* (with factors −1, *t* and *y*). Therefore *t* is a common factor of ALL terms (numerator and denominator.) We can divide the numerator and the denominator by *t*.

b) Numerator: two terms, *x* (with factors 1 and *x*) and *xy* (with factors *x* and *y*) Denominator: one term, 2a*x* (with factors 2, *a* and *x*). ALL terms in the numerator AND in the denominator have a common factor, *x*. We can therefore divide both the numerator and denominator by *x*.

c) Numerator: one term, $3ab$ (with three factors 3, a and b). Denominator: two terms, ab (with factors a and b) and $-a^2$ (with factors -1 and a^2). ALL terms in the numerator AND in the denominator have a common factor, a. We can therefore divide both the numerator AND the denominator by a.

20 x is NOT a factor in the denominator, but it can be viewed as a factor in the numerator. We can NOT cancel x, because it is not a factor in the denominator.

21 x IS a factor in BOTH the numerator and the denominator, therefore we can cancel x. The result is $\frac{7}{xy}$

22 a IS a factor in BOTH the numerator and denominator, thus we can cancel it. The result: $a+x$.

23 ab IS a factor in the denominator, but NOT in the denominator, therefore we cannot cancel it.

24 a) $\frac{1}{3}$; 3xy   b) $\frac{c}{4}$; 2ab   c) $\frac{-1}{b^2}$; a^2 d) $\frac{xy}{4}$; $5y^3$ e) $\frac{3(a-b)}{5}$; $5x$

f) $\frac{a(b-c)}{2}$; a   g) $-20x^2$; x h) $c(b+e)$; b i) $\frac{1}{7}$; $(a+b)$ j) not possible

25 a) $\frac{1}{x+y}$ b) $\frac{y+z}{3}$ c) $4-5x$ d) $\frac{1+x}{2}$ e) $\frac{1}{1-3x}$ f) $a^2b-4b^4a^3$ g) 1

h) $\frac{1}{2-x^3}$ i) not possible   j) $\frac{1}{1+y}$ k) not possible   l) $-x+3y+2z$

m) $\frac{1-y+x}{3}$ n) -1 o) $2v-3u+1$ p) $\frac{1}{3x^3-6xyz}$

Lesson 6

1 b) and d)

2 Yes.

3 No. (All three are unlike)

4 $6x^2y^2$ $-2x^2y$ $5xy$ $0.3yx^2$

5 $5(ab)^2$ $2a^2b$ $-\frac{3}{7}ab^2$ $\frac{ba^2}{7}$

6 y^3x^2 xy^3x $2x^3y^2$ y^2xx $yyxxy$

7 $a^2b^5a^3$ $-b^2a^5b$ $(ab)^3b^3$ $2(ab)^3a^2$ a^3b^5

8 a) $2x$ b) not possible   c) 0   d) not possible   e) $2st$ or $2ts$ f) $2ac^2$ g) not possible

h) $6hmv$ i) $3xy^3z^5$ j) $9m^3n$ k) not possible   l) $2a^2b^2$

9 a) $(3-4)x=-x$ b) $\left(\frac{1}{3}-\frac{2}{7}\right)x=\frac{1}{21}x$ c) $\left(\frac{2}{11}-\frac{3}{22}\right)x=\frac{1}{22}x$

d) $(0.3-0.5)x=-0.2x$ e) $\left(-\frac{7}{9}-\frac{2}{5}\right)x=-\frac{53}{45}x$ f) $\left(\frac{1}{5}-\frac{2}{3}+\frac{3}{10}\right)x=-\frac{1}{6}x$

10 a) $-x$ b) $-5a$ c) x d) $-x$ e) $-7ab$ f) not possible   g) not possible   h) $-1.2x^2y$

11 a) $-3x-8x-2x+8y=-13x+8y$ b) $-5a-7a-3b-2b=-12a-5b$

c) $-2ab-4ba+3ab+2-1=-3ab+1$

12 a) j b) $2+\frac{1}{3}a$ c) $0.7z$ d) $-2-7m$ e) $-2x^3+x^4$ f) $3x-y$ g) $-x-1$

h) $y-\frac{1}{2}x$ i) $\frac{2}{3}cd-d+c$ j) $-4b^2a^3-5a^2b^3$ k) $4yx^4-7x$ l) $-\frac{11}{38}ab+5\frac{7}{8}$

13 Student A and C.

14 $x+y$ a) -4 b) $-\frac{9}{5}$ c) -2.7

15 x; a) 4 b) -10 c) $\frac{2}{3}$.

16 a) $18a-2$ b) $27a+2$ c) $4y-6$ d) $2t+10$ e) $-\frac{8}{5}x+1$ f) $1.3a-0.2$ g) $-2x+1$ h) $-29q-48$ i) $-a-\frac{41}{5}b$ j) $-10x-9y$ k) $2a-5d$ l) $-a^2-3$ m) $3xy-3$

n) $-5bc+abc-2a$ o) $-\frac{1}{3}b$ p) $6x^4+3x^2+2$ q) $12x^2+6$ r) $-2.5m+3.97$

17 a) The student claiming that $(A+B)^2=A^2+2AB+B^2$ was right. b) $(A-B)^2=A^2-2AB+B^2$

18 a) x^8+4x^4+4 b) $9x^2-6x+1$ c) a^2+2a-8 d) $3b^2+\frac{1}{2}bc-\frac{1}{2}c^2$ e) $-c+\frac{29}{3}x$

f) $2x^3-5x^2+x+2$ g) $a^5-2a^3b+a^2b-2b^2$ h) $\frac{4}{9}-\frac{8}{3}x+4x^2$ i) $3b-2c$ j) $-13x+5y$

k) $-x^2+5x+8$ l) $-5a^2-12a-4$ m) $3x^2-x$ n) $-2m^2+3m+2$ o) k^2-2k+6 p) 0

19 a) $3yx-(-2xy)=5xy$ b) $(3x-1)+(-4x+2)=-x+1$

c) $(-4a^3+2a)-(4a^3-2)=-8a^3+2a+2$ d) $(-a+2+3b)-(-2b+a)=-2a+5b+2$

e) $\left(a-\frac{1}{3}\right)\left(\frac{2}{5}-2a\right)=-2a^2+\frac{16}{15}a-\frac{2}{15}$ f) $(2x^2-y)(3y-x^2)=-2x^4+7x^2y-3y^2$

g) $-mnk+4nmk+(-3mn)=3mnk-3mn$ h) $(3x+2)^2=9x^2+12x+4$

i) $(2a-b)^2=4a^2-4ab+b^2$ j) $(5x+2)+2(-4x+1)=-3x+4$

20 $-3a-3b$; a) 9 b) $-\frac{3}{2}$ *or* $-1\frac{1}{2}$ c) 5.4

21 $-3xy$ a) -9 b) 1 c) 0.3

Lesson 7

1 a) 20 b) $\frac{1}{27}$ c) -32

2 a) 6 b) -36 c) 4 d) -9

3 a) -4 b) -4 c) 4

4 a) -14 b) $-\frac{2}{7}$ c) 2 d) 4

5 a) $-\frac{2}{7}$ b) $\frac{2}{7}$ c) $\frac{2}{7}$ d) $\frac{4}{49}$

6 a) -7 b) 7 c) -7 d) -7 e) -7

7 a) 1 b) -1 c) -2 d) -1

8	a) 9 b) -3	c) 3	d) $\frac{1}{3}$	e) 6	f) 27	
9	a) $-\frac{1}{3}$	b) $\frac{1}{3}$	c) $-\frac{1}{9}$	d) $-\frac{1}{3}$	e) -3	f) $-\frac{1}{3}$
10	a) 1	b) 2	c) 39			
11	a) −3	b) −0.4	c) −0.12			
12	a) −12	b) −3	c) −2	d) −2		
13	a) $\frac{2}{3}$	b) $-\frac{2}{3}$	c) $\frac{2}{9}$	d) -4		
14	a) -0.2	b) $\frac{1}{2}$	c) 0.05 or $\frac{1}{20}$	d) 10		
15	a) $x+z=-1$	b) $-z=3$				
16	a) $\frac{6x}{2}=3x$					
17	a) $\frac{-3x^2}{2}$ *or* $-\frac{3}{2}x^2$	b) $-\frac{7}{2}x$	c) $\frac{27}{2}x^3$	d) $-x$		
18	a) x^2	b) $25x^2$	c) x^2	d) $25x^2$	e) x^2+2x+1	f) x^6
19	a) $1-2x+x^2$ b) x^2	c) $\frac{x^2}{4}+x^2+x^2=\frac{9}{4}x^2$	d) 0			
20	a) $4x-3$	b) $-3x^2+2x-1$	c) $6x^2-4x+2$	d) $3x^2-5x+2$		
	e) $9x^2-6x+1$	f) $3x^2-7x+2$	g) $8x-1$	h) $2x^2+2x-3$		
21	a) $2s^3-2s$	b) $2s^7$	c) $10s^3$	d) s^5		
22	x					
23	a) m^4-m^3	b) $4s^4-2s^3$				
24	a) $-10s^4$	b) $8s^{10}-8s^{12}$	c) $-s^4$			
25	a) $26s^2$	b) $4m^2$	c) $4m^2$			
26	a) $(2b)^4=16b^4$ b) $16b^4$	c) equal	d) 0	e) $b^4-4b^4+6b^4-4b^4+b^4=0$	f) equal	
27	$4x$					
28	a) $-3m$	b) $\frac{m}{2}$	c) 6^m	d) m^3	e) m	f) m
29	a) $-3y$	b) $-3y^2$	c) y	d) $-5y$		
30	$(6x-2)(4x+2)$					

Lesson 8

1 One can solve an equation but not an algebraic expression. If the left hand side of an equation is equal to the right hand side of the equation for $x=7$, then 7 is called a solution. The solutions of an equation are all values of variables that make the equation true. The statement that contains two quantities separated by an equal sign is called an equation. A solution always makes the equation true.

2 Equations: b) $5x=2$ c) $x^2=36$ e) $x=-4+2x$

3 Both Tom's and Mary's answers are correct, because $x=3$ is equivalent to both $3=x$ and $-x=-3$.

4 False. 7 is not a solution of $2(x+1)-x=7$.

5 None of the numbers is a solution of $-x^4=16$. The number 2 and −2 are solutions of $x^4=16$.

6 a) No b) Yes

7 a) Yes b) No c) Yes

8 a) No b) Yes c) No d) Yes

9 a) Yes b) Yes c) No

10 a) algebraic expression b) equation; −3 is a solution c) equation; −3 is not a solution
d) equation; −3 is not a solution e) algebraic expression f) equation; −3 is a solution

11 a) algebraic expression b) equation; 0.6 is a solution c) equation; 0.6 is a solution
d) algebraic expression e) equation; 0.6 is a solution f) equation; 0.6 is not a solution

12 a) equation; −1 b) algebraic expression c) equation; 0 or 1 d) equation; 1 e) algebraic expression f) equation;

13 a) −2 b) 3 (answers vary; any number different from −2)

14 a) -5 b) 5 (answers vary; any number different from −5.)

15 a) a solution b) a solution c) a solution d) not a solution e) not a solution f) a solution

16 For example, −2, −1, 0, 1 (answers vary; every number is a solution of this equation)

17 We can only divide by a variable if we assume it is not 0. It is better if we always try to avoid dividing by a variable or by an algebraic expression. A better way to solve this equation is: $x = 2x$; $x - 2x = 2x - 2x$; $x = 0$

18 a) $x = -3$ b) $x = 6$ c) $x = 5$ d) $x = -20$ e) −50 f) $x = -3$ g) $x = -2$ h) $x = -8$

19 a) $x = -4$ b) $x = 3$ c) $x = -3$ d) $x = \frac{4}{5}$ e) $x = -18$ f) $x = -5$

20 a) $x = -\frac{12}{5}$ b) $x = \frac{5}{7}$ c) $x = 0$ d) $x = \frac{2}{7}$ e) $x = \frac{3}{8}$ f) $x = -1$

21 f) $x = 6$ g) $x = -1$ c) $x = -8$ h) $x = -1$ e) $x = 6$ f) $x = 1$

22 a) no solution b) $x = 0$ (exactly one solution) c) all real numbers d) no solution e) all real numbers

23 a) $x = -2$ b) $a = -60$ c) $x = 7$ d) $x = 0$ e) $a = 0$ f) $x = 5$ g) $y = 2$
h) $y = \frac{22}{6} = \frac{11}{3}$ i) $x = 0$ j) $x = -6$ k) $x = -\frac{2}{5}$ l) $a = 10$ m) $B = \frac{1}{6}$ n) $x = 3$ o) all real numbers
p) $x = -2$ q) $x = \frac{1}{3}$ r) no solution s) no solution t) no solution u) $x = \frac{7}{5}$ v) $x = 2$

Lesson 9

1 a) $x = -18$ b) $x = -\frac{2}{15}$ c) $x = -\frac{5}{16}$ d) $x = \frac{8}{5}$ e) $y = -2$ f) $x = \frac{1}{6}$

2 a) no solution b) $x = -\frac{2}{5}$ c) $a = \frac{1}{4}$ d) $x = -\frac{1}{6}$ e) no solution f) $x = -\frac{1}{4}$
g) $x = 0$ h) $x = \frac{10}{21}$ i) all real numbers j) $x = \frac{4}{9}$ k) $x = \frac{13}{8}$ l) $x = -\frac{9}{2}$

3 a) $x = 0$ b) $x = \frac{1}{5}$ c) $x = 0$

4 a) $x = -8$ b) $x = -\frac{2}{7}$ c) $x = \frac{11}{2}$

5 a) no solution b) $a = \frac{1}{7}$ c) $a = -\frac{2}{7}$

6 No. $\frac{x+1}{2} - 5y$ is not an equation, so we can not solve it.

7 a) $x = 14$ b) $x = ab$

8 a) $x = 5$ b) $x = b - a$

9 a) $x = 6$ b) $x = \frac{c+b}{a}$

10 a) $x = 2$ b) $x = \frac{c}{a-b}$

11 a) $x=1$ b) $x=\dfrac{bc}{a-b}$

12 a) $x=-a$ b) $b=a^2c$ c) $a=\dfrac{b^4}{c}$ d) $a=\dfrac{3b}{x}$ e) $u=a$ f) $b=a^2c$ g) $m=2n^3$

h) $y=t+x$ i) $y=x^2$ j) $A=\dfrac{1-x}{x-1}=\dfrac{-(x-1)}{x-1}=-1$ k) $x=s+1$ l) $a=-\dfrac{5by}{2x}$

m) $v=\dfrac{s}{3}+t$ or $v=\dfrac{s+3t}{3}$ n) $x=\dfrac{1}{2a+2}$ o) $x=\dfrac{m-2n}{2n-m}=-1$ p) $m=\dfrac{3b}{b+b^2}=\dfrac{3}{1+b}$

13 a) $d=\dfrac{x}{y}-e$ b) $x=y(d+e)$ c) $y=\dfrac{x}{d+e}$

14 a) $b=t-at$ b) $a=\dfrac{t-b}{t}$ or $a=1-\dfrac{b}{t}$ c) $t=\dfrac{b}{1-a}$ or $t=\dfrac{-b}{a-1}$

15 $x=a^9=-1$

16 $n=-m=4$

17 a) $h=\dfrac{2A}{b}$ b) $h=4$ inches

18 a) $W=\dfrac{P-2L}{2}$ or $W=\dfrac{P}{2}-L$ b) $W=1$ c) $W=2$ inches

Lesson 10

1 For example $x=3$ or $x=-2$ (answers vary). The same numbers could be solutions for the other inequality. Both x<5 and 5>x state the same condition (are equivalent)

2 a) and d)

3 For example $x=-1$ or $x=-3$ or $x=-10$ (answers vary.)

4 $x=\dfrac{2}{3}$

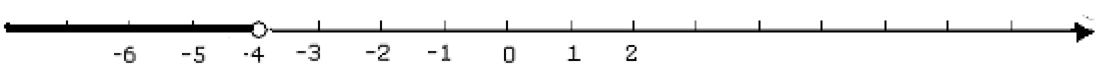
5. -3, -2, -1, 0, 1, 2, 3, 4, 5

6. -3, -2, -1, 0, 1, 2, 3, 4, 5

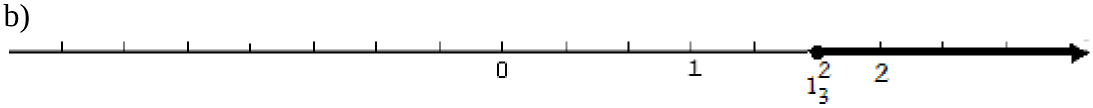
7 $-.666$, -6, -0.6

8 a) $x<0$ b) $x\le 0$ c) $x\ge 6$ d) $x\le 6$ e) $x\le 6$

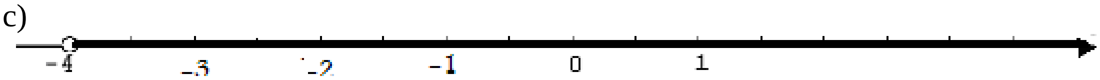
9



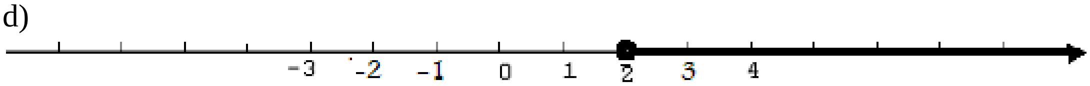
b)



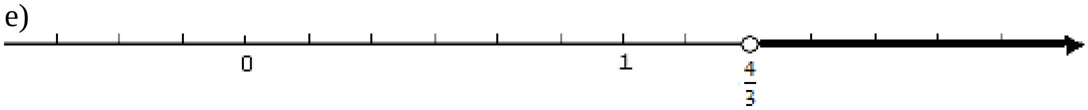
c)



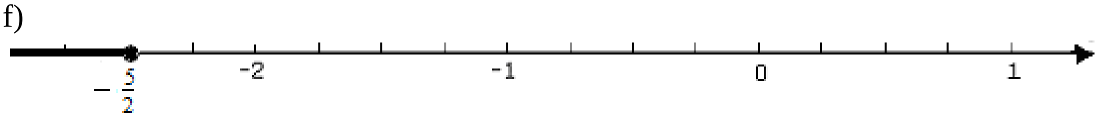
d)



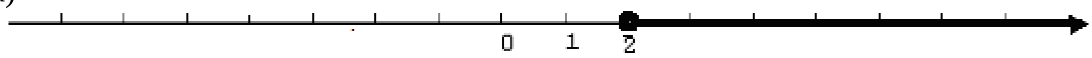
e)



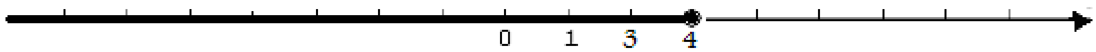
f)



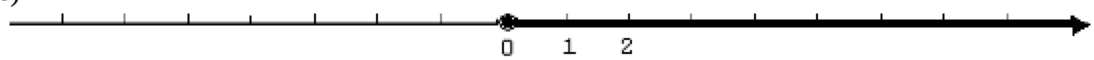
10 a)



b)



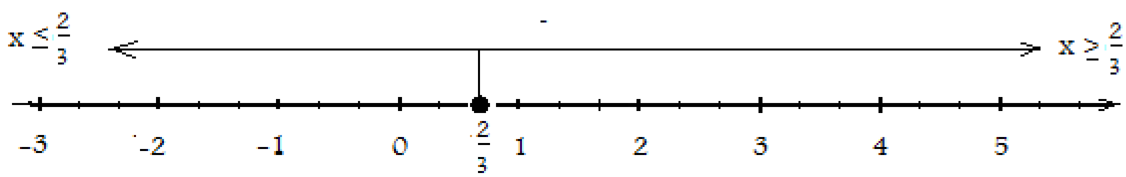
c)



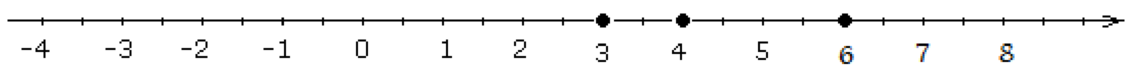
d)

11 a) $x < -1$ b) $x > -4$ c) $x \geq -1$ d) $x \leq -1$

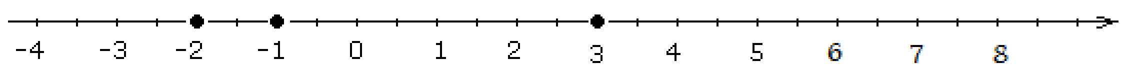
12 $x = \frac{2}{3}$ is the value that satisfies both inequalities.



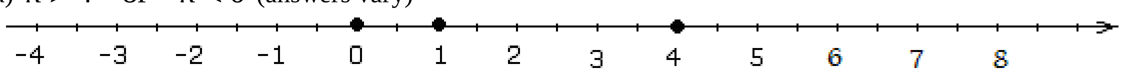
13 a) $x \geq 3$ or $x < 7$ (answers vary)



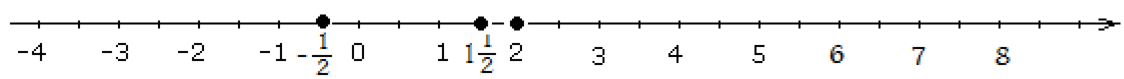
b) $x \geq -2$ or $x \leq 3$ (answers vary)



14 a) $x > 4$ or $x < 0$ (answers vary)



b) $x > 2$ or $x < -\frac{1}{2}$ (answers vary)



15 $x \leq 0$ (answers vary.)

16 $x \leq 4$ (answers vary)

17 a) $-5+2 < 4$ and b) $5+8 \geq 13$; -5 is not a solution of (c) and (d) $-3<4$ $13 \geq 13$

18 Only student B

19 a) yes b) no (there are infinitely many solutions) c) infinitely many d) 1, 10, 100
e) $-1, -3, -5$ (answers vary.) f) yes g) no h) no i) yes

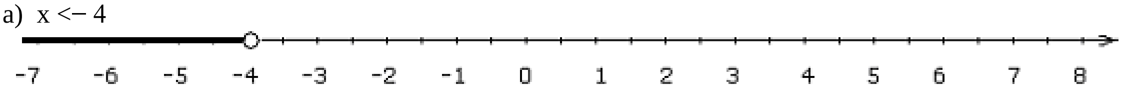
20 a) and e)

21 a) subtract 5; no sign change; $z < 3$ b) add 2; no sign change; $z < 3$ c) divide by 4; no sign change; $z > -3$
d) divide or multiply by -1; sign changes; $z < 3$ e) multiply by -3; sign changes; $z < -3$

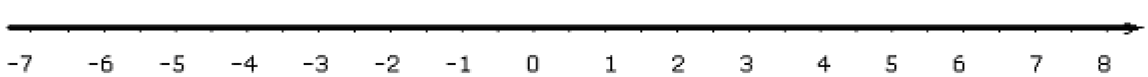
22 a) and d)

23 b) and d)

24 a) $x < -4$

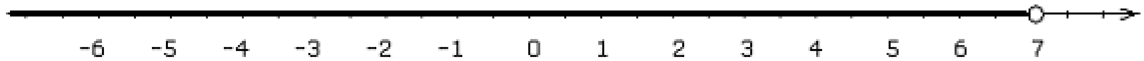


b) All real numbers



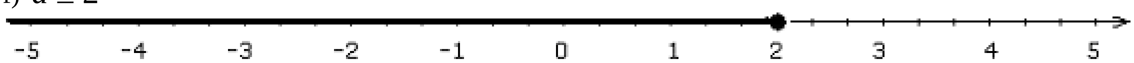
c) $x \geq -2$

d) $a < 7$



e) $x > -10$

f) $a \leq 2$



25 a) "add 3 to both sides", "divide each side by -1"; $a > -7$

b) "subtract 1 from both sides", "divide each side by 3"; $x < \frac{5}{3}$

c) "multiply each side by 4", "divide each side by 3"; $x < -20$

d) "multiply each side by 4", "divide each side by -1"; $x \geq 20$

e) "subtract 1 from both sides", "multiply each side by 4"; $x \geq -8$

f) "multiply each side by 4", "subtract 1 from both sides"; $x \geq -5$

26 a) $x < -6$ b) $x < -2$ c) $x \geq -4$ d) all real numbers e) no solution f) $a < -\frac{3}{5}$ g) $x \geq \frac{4}{3}$

h) $x < \frac{3}{7}$ i) $x > -\frac{2}{3}$ j) all real numbers k) all real numbers l) $x \leq -\frac{9}{2}$ m) all real numbers

n) no solution o) no solution p) $x < -\frac{1}{4}$ q) $x \geq 18$ r) $y \geq 0$ s) $a > \frac{34}{81}$ t) no solution

Lesson 11

1 $a = \frac{3}{4}$

2 a) $a = 3$ and $b = 4$ b) $a = -4$ and $b = \frac{2}{3}$

3 a) $p = -3$ b) $p = 3$

4 $a = -2$ and $b = 5$

5 $a = -7$

6 $A = 4x$ and $B = y$

7 a) $X = 3a$ and $Y = b$ b) $X = 3$ and $Y = ab$

8 a) $c = x - 1$ and $b = 2$ b) $c = x$ and $b = y^9$

9 (1)-E; $a = x$ and $b = 1$ (2)-F; $a = x$ and $b = 1$ (3)-D; $a = 8$ and $b = x$

(4)-A; $a = 3x$ and $b = 5y$ (5)-B; $a = 3x$ and $b = 5y$ (6)-C; $a = 3$ and $b = y$

10 a) form of $A - B$; $5 - (-n) = 5 + n$ b) form of $A + B$; $5 + n = 5 - (-n)$

c) form of $A + B$; $5 + (-n) = 5 - n$ d) form of $A - B$; $5 - n = 5 + (-n)$

11 a) $a = -1$ *and* $b = 3$ b) $a = 2$ *and* $b = -3$ c) $-\frac{1}{2}x^3 + 1$; $a = -\frac{1}{2}$ *and* $b = 1$

d) $2x^3 + \frac{3}{2}$; $a = 2$ *and* $b = \frac{3}{2}$ e) $-x^3 + \frac{3}{2}$; $a = -1$ *and* $b = \frac{3}{2}$ f) $8x^3 + 4$ $a = 8$, $b = 4$

12 a) $6^2; a=6$ b) $20^2; a=20$ c) $0.4^2; a=0.4$ d) $\left(\frac{3}{7}\right)^2; a=\frac{3}{7}$ e) $(5y)^2; a=5y$ f) $\left(\frac{b}{10}\right)^2; a=\frac{b}{10}$

g) $(0.7c)^2;\ a=0.7c$ h) $(X^2)^2;\ a=X^2$ i) $(2x^3)^2;\ a=2x^3$ j) $(9xy^4)^2;\ a=9xy^4$

13 a) $(-1)^3;\ a=-1$ b) $3^3;\ a=3$ c) $0.3^3;\ a=0.3$ d) $\left(\frac{2}{5}\right)^3;\ a=\frac{2}{5}$ e) $(-z)^3;\ a=-z$

f) $(4x)^3; a=4x$ g) $\left(-\frac{x}{2}\right)^3; a=-\frac{x}{2}$ h) $(y^2)^3; a=y^2$ i) $(10x^3)^3; a=10x^3$ j) $\left(\frac{x^5}{2y}\right)^3; a=\frac{x^5}{2y}$

14 a) $(x^6)^4;\ a=x^6$ b) $(x^4)^6;\ a=x^4$ c) $(x^2)^{12};\ a=x^2$

15 a) $(-x)^7;\ A=-x$ b) $(x^2)^7;\ A=x^2$ c) $(x^2y^3)^7; A=x^2y^3$ d) $\left(\frac{y^3}{z^{10}}\right)^7;\ A=\frac{y^3}{z^{10}}$

16 a) $x^5;\ a=x; m=5$ b) $(stv)^7;\ a=stv; m=7$ c) $b^7;\ a=b; m=7$

d) $\left(\frac{B}{C}\right)^3;\ a=\frac{B}{C}; m=3$ e) $\left(\frac{x+y}{z}\right)^4;\ a=\frac{x+y}{z}; m=4$ f) $(8x)^2;\ a=8x; m=2$

17 a) $x^2+(-2)y^2=0;\ a=1; b=-2$ b) $3x^2+(-1)y^2=0;\ a=3; b=-1$

c) $\frac{1}{2}x^2+(-1)y^2=0; a=\frac{1}{2}; b=-1$ d) $1x^2+0y^2=0; a=1; b=0$

e) $\frac{3}{4}x^2+\frac{1}{4}y^2=0; a=\frac{3}{4}; b=\frac{1}{4}$ or $3x^2+y^2=0$ if you were to multiply both sides by 4, then $a=3, b=1$

f) $10x^2+(-5)y^2=0;\ a=10; b=-5$ or $-10x^2+5y^2=0;\ a=-10; b=5$

18 a) $(-1)x+\left(-\frac{1}{4}\right)y+\frac{3}{2}z; A=-1; B=-\frac{1}{4}; C=\frac{3}{2}$ b) $-6x+(-3)y+z; A=-6; B=-3; C=1$

c) $\frac{3}{4}x+\left(-\frac{1}{2}\right)y+\frac{1}{4}z; A=\frac{3}{4}; B=-\frac{1}{2}; C=\frac{1}{4}$ d) $1x+(-1)y+0z;\ A=1; B=-1; C=0$

e) $-\frac{3}{4}x+\frac{3}{4}y+2z; A=-\frac{3}{4}; B=\frac{3}{4}; C=2$ f) $0x+0y+\left(-\frac{1}{15}\right)z; A=0; B=0; C=-\frac{1}{15}$

19 a) $y=3x+(-2);\ m=3; b=-2$ b) $y=1x+0; m=1; b=0$ c) $y=\frac{1}{5}x+0;\ m=\frac{1}{5}; b=0$

d) $y=-3x+2; m=-3; b=2$ e) $y=2x+\left(-\frac{1}{3}\right);\ m=2; b=-\frac{1}{3}$

f) $y=0x+(-5);\ m=0; b=-5$ g) $y=0x+0;\ m=0; b=0$ h) $y=4x+(-6);\ m=4; b=-6$

20 a) $a=3; b=b;\ 3^2+2(3)b+b^2=9+6b+b^2$

b) $a=2x; b=3y;\ (2x)^2+2(2x)(3y)+(3y)^2=4x^2+12xy+9y^2$

c) $a=y^2; b=\frac{2}{5};\ (y^2)^2+2(y^2)\frac{2}{5}+\left(\frac{2}{5}\right)^2=y^4+\frac{4}{5}y^2+\frac{4}{25}$

21 a) not linear b) $\frac{1}{3}x+(-1)=0; a=\frac{1}{3}; b=-1$ c) $-1x+0=0; a=-1; b=0$

d) $\frac{1}{8}x+0.9=0; a=\frac{1}{8}; b=0.9$ or $x+7.2=0; a=1; b=7.2$ e) $x+2=0$, $a=1, b=2$

f) $-4x+7=0;\ a=-4;\ b=7$ or $4x+(-7)=0;\ a=4;\ b=-7$ g) $-x+(-2.5)=0;\ a=-1;\ b=-2.5$

h) $18x+(-5)=0;\ a=18;\ b=-5$ or $-18x+5=0;\ a=-18;\ b=5$

22 $a=4;\ b=2$; $12x+6=0$ where $a=12;\ b=6$

23 a) $(x-1)(x+1)$ b) $(2-3x)(2+3x)$ c) $(y-10a)(y+10a)$ d) $(5s-8t)(5s+8t)$

e) $\left(x^2-\frac{1}{3}\right)\left(x^2+\frac{1}{3}\right)$ f) $(xy-0.5)(xy+0.5)$ g) $\left(\frac{a}{b}-6\right)\left(\frac{a}{b}+6\right)$

h) $[m-(2m+1)][m+(2m+1)]=(-m-1)(3m+1)$

i) $[(x+1)-(3x+5)][(x+1)+(3x+5)]=(-2x-4)(4x+6)$

j) $[(2a-3)-3a][(2a-3)+3a]=(-a-3)(5a-3)$

24 a) $x^3-4^3;\ a=x;\ b=4;\quad (x-4)(x^2+4x+16)$ b) $1^3-(2y)^3;\ a=1;\ b=2y;\quad (1-2y)(1+2y+4y^2)$

c) $(2x)^3-(3y)^3;\ a=2x;\ b=3y;\quad (2x-3y)(4x^2+6xy+9y^2)$

d) $(x^2)^3-\left(\frac{1}{3}\right)^3;\ a=x^2;\ b=\frac{1}{3};\quad \left(x^2-\frac{1}{3}\right)\left(x^4+\frac{1}{3}x^2+\frac{1}{9}\right)$

25 a) $(2x+y)(4x^2-2xy+y^2)$ b) $(6x-0.1y)(6x+0.1y)$

c) $(x-3y)(x^2+3xy+9y^2)$ d) $(4a-bc)(16a^2+4abc+b^2c^2)$

26 $10.7^2-9.3^2=(10.7-9.3)(10.7+9.3)=1.4\times 20=28$

27 $(10+1)^4=10^4+4\times 10^3\times 1+6\times 10^2\times 1^2+4\times 10\times 1^3+1^4=14641$

APPENDIX B: SAMPLE TESTS

Sample Test 1 (Lesson 1-3)

1. Write the following statements as algebraic expressions. Remember to place parentheses where needed (please, place them only when needed).
   a) two seventh of xy
   b) the product of x and $y+2$
   c) xy raised to fourth power
   d) Subtract A from B, and then multiply by C
   e) take the opposite to x, and then raise it to the third power

2. By placing the minus sign differently, write the expression $-\dfrac{a+b}{c}$ in two additional equivalent ways. Use parentheses when needed.

3. Let $m=-2$. Rewrite the expression replacing the variable with its value (Remember to place parentheses) and then evaluate, if possible. Otherwise write "undefined".
a) $-m$ b) $5m$ c) $-m^4$ d) $\dfrac{1}{2+m}$

4. Let $a=0.1$, $b=-0.3$. Rewrite the expression replacing the variable with its value (remember to place parentheses) and then evaluate, if possible. Otherwise, write "undefined". a) $a-b$ b) $100ab$ c) $\dfrac{b^2}{a}$

5. Write an algebraic expression representing the opposite of (do not remove parentheses): $\dfrac{-A}{-B}$

6. Let $x=\dfrac{3}{4}$, $y=\dfrac{1}{5}$. Rewrite the expression replacing the variable with its value (remember to place parentheses) and then evaluate, if possible. Otherwise write "undefined".
a) $8x-5y$ b) $5xy$ c) $-\dfrac{x}{y}$

7. Determine which of the following expressions are equal to $-x$

$$x(-1) \quad -1\cdot x \quad \frac{x}{-1} \quad \frac{-x}{1} \quad 1-2x$$

8. Write the expressions $-\dfrac{2}{3}x-\dfrac{1}{3}y$ as a single fraction. .

9. Fill in the blanks to make a true statement.
   a) $-\dfrac{A}{y}=-A\cdot$ ____________
   b) $\dfrac{2a+b}{c}=\dfrac{2a}{c}+$ ____________
   c) $\dfrac{3a}{4}=a\cdot$ ____________

10. Perform all numerical operations that are possible. If no simplification is possible, write "not possible". a) $-4+2x$ b) $(-4x)(-2)$
c) $-(-2x^2)$ d) $\dfrac{3x}{9}$ e) $-12\cdot\dfrac{x}{3}$

Sample Test 2 (Lesson 1-3)

1. The expression $\dfrac{x+3}{-x+2}$ cannot be evaluated for which of the following values of *x*? List all.
a) $x=2$ b) $x=-2$ c) $x=0$ d) $x=-3$

2. Write the following statements as algebraic expressions:
a) x raised to the 4^{th} power, then subtracted from y.
b) the sum of 5 and three-forth of a number.
c) the opposite of $b+4$
d) 6 less than the square of a number.

3. Let $m = -3$. Evaluate the following expressions if possible (otherwise, write "undefined"):

a) m^2 b) $\frac{m+5}{2}$ c) $\frac{m+1}{m+3}$ d) $3-2m$

4. Evaluate uv^2 if possible (otherwise, write "undefined"), when $u = \frac{2}{3}$ *and* $v = \frac{1}{3}$.

5. Evaluate $\frac{m}{p} - n$ if possible (otherwise, write "undefined"), when $m = -0.1,\ n = 0.02,\ p = -1.$

6. Let F be a variable representing the temperature. Use F to write the following statement as an algebraic expression:
five-ninth times the difference of F and 32.

7. Perform all numerical operations if possible (if performing a numerical operation is not possible, write "not possible").

a) $\frac{10b}{-5}$ b) $2+3c$ c) $x-(-3)^2$ d) $\frac{2x^3}{3}\left(-\frac{9y}{5}\right)$

e) $-2x^3(2-5)-4$ f) $2\cdot 4^n(-5)$

8. Write $\frac{-5a}{2b}$ in three different ways.

9. Determine which of the following expressions are equivalent to $\frac{2a}{b}$

$2\cdot\frac{a}{b},\quad 2a\div b,\quad \frac{1}{b}\cdot 2a\quad \frac{-2a}{-b},\quad (1+1a)\div b,\quad \frac{(1+1)a}{b}$

10. Determine which of the following expressions are equivalent to $mn-k$:

$k-mn \quad nm-k \quad mn+(-k) \quad m(n-k) \quad -k+mn$

11. Replace Ψ with expressions such that the resulting statement is true. Use parentheses when needed.

a) $-xyz = yz\Psi$

b) $A = 7\cdot\frac{A}{\Psi}$

c) $\frac{a+b}{c} = \frac{\Psi}{c}(a+b)$

Sample Test 3 (Lesson 1-3)

1. Use the letter x to represent a number and write the following statements as algebraic expressions.
   a) Half of a number is subtracted from 7.
   b) Twice a number is added to 5, and then the result is multiplied by 3.
   c) The number is tripled, and then squared.

2. Write the opposite (do not simplify).

a) $-2x+3y$ b) $\frac{x-y}{x+y}$

3. Name the operation that should be performed first.

a) $3(x+y)$ b) $x\div yz$ c) $-4x^2$ d) $(-4x)^2$

4. If possible (otherwise, write "undefined"), evaluate $-3xy$ when a) $x = -6$ and $y = 2$

b) $x = \frac{5}{6}$ and $y = \frac{3}{4}$ c) $x = -0.1$ and $y = -0.2$.

5. If possible (otherwise, write "undefined"), evaluate $-\frac{x}{y}$ when a) $x = -6$ and $y = 2$

b) $x = \frac{5}{6}$ and $y = \frac{3}{4}$ c) $x = -0.1$ and $y = -0.2$.

6. If possible (otherwise, write "undefined"), evaluate $\frac{x}{1-x}$ when

a) $x = 0$ b) $x = 1$ c) $x = -1$

7. State which expressions are equivalent to $\frac{x-y}{2}$

a) $x-y\div 2$ b) $(x-y)\div 2$ c) $\frac{x}{2}-\frac{y}{2}$

d) $\frac{-y+x}{2}$ e) $\frac{1}{2}(x-y)$

8. Replace X with expressions such that the resulting statement is true. Use parentheses when needed.

a) $-x+2y-3z=-3z+2y+X$

b) $3\cdot\frac{a}{y}\cdot b=\frac{X}{y}$

c) $\frac{m+n}{v}=\frac{n}{v}+X$.

9. Perform all possible numerical operations. If none is possible, write "not possible"

a) $-5-(6-2)x$ b) $-6(2x)$ c) $(6\div 2)x$

d) $\frac{2}{3}-\frac{5}{6}x$ e) $\frac{2}{3}-\frac{5}{6}-x$

10. Show that $(-x)^2$ is not equivalent to $-x^2$ by evaluating both expressions when $x=-\frac{2}{3}$

Sample Test 1 (Lesson 4-6)

1. Write the following statements as an algebraic expression using parentheses where appropriate, and then remove parentheses:
   a) The product of A and $B-A+3$
   b) The opposite of $-x+2$

2. Factor $2a^3$ from the expression $-8a^5b+10a^3$.

3. Factor $\frac{2}{7}$ from the expression $\frac{4}{7}a-\frac{2}{7}$

4. If possible, add (or subtract) the following expressions. If not possible, write "not possible".

a) $2hk-\frac{1}{2}kh$ b) $ab-ab^2$

c) mn^2-nmn d) $\frac{x}{3}-\frac{1}{6}x$

5. Collect like terms in $-4yx-y+3xy+8y-7y$, and then evaluate it when $x=-2,\ y=\frac{1}{4}$.

6. Remove parentheses and then collect like terms

a) $-7b-\frac{1}{3}(3b-6)$ b) $-4x-(x+2)$

c) $(3x-b)^2$ d) $2x^2-(2x-1)(x+2)$

7. Simplify, if possible (assume that all denominators are different from zero). If not possible, write "not possible".

a) $\frac{ax-b}{ax}$ b) $\frac{xy^3}{x}$

c) $\frac{2(a+b)}{b+a}$ d) $\frac{2c}{xy}\cdot xy^2$

e) $\frac{x-2}{2-x}$

8. Write as a single exponential expression. For each expression, identify the numerical coefficient.

a) $\frac{(-a)^5}{a^3}$ b) $x^7(2x^2)^3$ c) $\frac{(2ab)^3}{2ab^3}$

9. Evaluate the following expressions:

a) $\frac{2^{45}}{2^{43}}$ b) $3a^0$

10. Find such X that the following is true:

a) $4^7=2^X$

b) $\left(\frac{1}{3}\right)^X=\left(\frac{1}{9}\right)^5$

Sample Test 2 (Lesson 4-6)

1. Write the following statements as algebraic expressions using parentheses if appropriate, and then remove parentheses. Simplify, if possible:

a) The product of $2a-3$ and $-3a^3+4a$
b) 2 times the opposite of $-3y+9$
c) 3 subtracted from *x*, then squared.

---

2. Factor *b* from the expression: $9b^2-5b$

---

3. Factor $5x^3$ from the expression: $15x^7+5x^3$

---

4. Collect like terms:
a) $-3ab+4ba-5ab$
b) $3x+4x^2-5+3x^2-6$

---

5. If possible, add the following expressions, otherwise write “unlike terms”.

a) $-\frac{1}{3}ab^2+\frac{1}{2}b^2a$ b) $-mn^2+(mn)^2$

c) $-x^2y^2+(4xy)^2$

6. Subtract $2a-b$ from $-3a+2b$ and simplify.

---

7. Remove parentheses. Simplify.

a) $-7x+2\left(3x-\frac{1}{4}\right)$ b) $-2b-(1-b)$

---

8. Simplify if possible. If not possible, clearly say so:

a) $\frac{3a-2b}{2b-3a}$ b) $\frac{3}{6b-3}$ c) $\frac{x-2y}{x+2y}$ d) $\frac{a^6}{a^2}$

---

9. Write as a single exponential expression.

a) $2x(-3x^2)\left(x^3\right)^4$ b) $\left(\frac{x^8}{x^4x}\right)^2$ c) $\frac{a^5}{2a}\cdot 4a^3$

---

10. Factor $\frac{2}{3}$ from the expression: $\frac{4}{3}y^2+\frac{2}{9}y-\frac{8}{3}$

Sample Test 3 (Lesson 4-6)

---

1. Circle all expressions equivalent to x^2y^6

xy^6x $\left(xy^3\right)^2$ $x\left(y^3\right)^2$ $y^3x^2y^2$

---

2. Write as a one exponential expression.

a) $\frac{3x^4}{(2x)^2}$ b) $\frac{2}{a^3}\times a^9$ c) $3m(-3m)$

---

3. Replace Ψ with a number so the following is equal: a) $9^7=3^{\Psi}$ b) $25^8=5^{\Psi}$

---

4. Remove parentheses and identify numerical coefficient:

a) $(-2x^3)(-3x^2)x$ b) $(2a^2b^4)^3$ c) $(-5b)\cdot\frac{7b^3}{10b^2}$

---

5. Write each statement as an algebraic expression using parentheses, and then rewrite without parentheses and collect like terms.
a) opposite of $-3x+5y$
b) product of x^2+5 and $2x-3$
c) sum of $5x-4y$ and $2x+7$
d) difference of $2x^2-7x+3$ and $5x^2-8x-6$

6. Factor
a) *x* from the expression $5x^3-4x^2+7x$.
b) $-3x$ from the expression $-15x^2+3x$
c) $a+z$ from the expression $4(a+z)-x(a+z)^2$
d) -1 from the expression $\frac{c}{d}-2$

---

7. Simplify, if possible. If not possible then write "not possible". a) $\frac{6xy}{9yx}$ b) $\frac{xy+xz}{4x}$

c) $\frac{5x-3y}{-3y+5x}$ d) $\frac{x+4}{x+6}$

8. Evaluate the following expression: $\frac{5^{27}}{5^{25}}$.

---

9. Collect like terms, and then evaluate, when $a=-3$.
a) $-2a-7a-(4-8a)$ b) $(3-a)^2+6a$
c) $6a+9-(a+3)^2$

Sample Test 1 (Lesson 7-9)

1. Evaluate the expressions, if $x=-1$, and $y=\frac{3}{4}$

a) xy^2 b) $\frac{x}{y}$

2. Evaluate a^2bc if

a) $a=-2, bc=\frac{1}{4}$ b) $a^2c=-0.4, b=0.5$

3. Let $x=3-m$. Express the following expression in terms of m and simplify: $\frac{5+x}{3-x}$

4. Write the following expression $2x-y+z$ in terms of m, if $z+2x=3m$ and $y=-m$. Simplify your answer.

5. Is $x=\frac{2}{3}$ a solution of the following equations? a) $\frac{1}{x}=\frac{3}{2}$ b) $-x^2=\frac{4}{9}$

3. Evaluate $2a-(3b+c)$ if

a) $a=-3, b=-2, c=2$ b) $a=2, c+3b=0.6$

7. If $P=-x+3$ and $Q=2x+4$, find such x, that a) $P=Q$ b) $\frac{P}{2}=Q+1$

8. Solve the following equations:

a) $b+\frac{1}{3}=2$ b) $5-2x=1$ c) $2-x=2x-4$

d) $-(x-4)=2-x$ e) $\frac{6y-4}{2}=3y-2$

9. Solve the following equations for x. Any time you must perform the operation of division, assume that the devisor is different from zero. Simplify whenever possible:

a) $\frac{x}{b}=2b$ b) $ax-b=2b$ c) $ax+a^2x=a$

10. If $l+d=8$, evaluate

a) $2(l+d)$ b) $2l+2d$ c) $-(l+d)^2$

Sample Test 2 (Lesson 7-9)

1. Evaluate the following expressions, if

$abc=-2$ a) $\frac{-abc}{4}$ b) $a^2b^2c^2$

2. Evaluate $x-2+3^{x-2}$ if

a) $x=4$ b) $x-2=1$

3. Write the following expressions in terms of a, if $x+y=a$ and $xy=a+1$. Simplify your answer.

a) $\frac{x}{3}+\frac{y}{3}-xy$ b) $\frac{x^2y^2}{x+y}$

4. Determine whether the following mathematical sentences represent an equation or an algebraic expression. In the case of an equation, circle the right-hand side of the equation. $3(x-2)=3x+4$, $5x+\frac{1}{x-2}$, $5x-7=8$, $2+3=5$

5. Let $m=2y$ and $n=3y$. Find y so that the following is true.

a) $m+1=n$ b) $m=n$

6. Solve the following equations:

a) $-0.6y=-0.4$ b) $-3x+1=2$

c) $-\frac{2}{5}a=4$ d) $2-x=2x-4$

e) $2(2c-1)-4c=-2$ f) $\frac{x}{5}-\frac{x-7}{3}=\frac{1}{3}$

---

7. Solve the following equations for d (assume that all devisors are different from zero). Simplify your answer whenever possible.

a) $d-c=c^2-2c$ b) $\frac{b}{d}=-a$ c) $db^2=b^2-b$

8. Is $y=0$ a solution of the following equations:

a) $\frac{1}{y}=0$

b) $3y^2-y=0$

c) $\frac{3+y}{y-3}=-1$

---

9. Which of the following numbers are solutions of $-x^2=9$?

a) -3

b) 3

Sample Test 3 (Lesson 7-9)

---

1. If $A+B=-\frac{3}{8}$, evaluate:

a) $\frac{2}{A+B}$ b) $B+A$ c) $-A-B$

---

2. Evaluate the following expressions, if $x-y=0.2$ and $\frac{1}{v}=-0.6$:

a) $\frac{1}{v}+x-y$ b) $\frac{x-y}{v}$

---

3. Is $x=-2$ a solution of $x^2-x=2$? Explain how you arrived at your answer.

---

4. Rewrite the expression $2a-b^2$ in terms of x. Write your answer without parentheses. Simplify.

a) $a=x+2, b=x+1$

b) $a=3x, \quad b=2x$

---

5. Write the following expression $3m(m-2n)$ in terms of t, if $m=n=t$. Simplify your answer.

6. Which of the following are solutions of $xy=24$?

a) $x=-6, \ y=-4$

b) $x=-\frac{1}{2}, \ y=-48$ c) $x=0, \ y=24$

---

7. Solve the following equations.

a) $x-3=27$ b) $2x=7x$ c) $3(x-1)=1+3x$

d) $-2x+1=5$ e) $\frac{3x}{2}-x=1$

---

8. Let $A=3x$, $B=x$, and $C=-x$. Find x so that the following is true: $\frac{A}{2}-\frac{B}{3}=C$

---

9. Solve for x. Any time you must perform the operation of division, assume that the devisor is different from zero. Simplify your answer, if possible

a) $-ax=b$ b) $\frac{x}{3d^2}=d$ c) $ax-a=cx$

Sample Test 1 (Lesson 10-11)

---

1. Describe the following set of numbers using inequality sign and then graph the set of on the number line: all numbers x at least equal to -1.

2. Which of the following numbers:

$-1, \quad 0, \quad 2.99, \quad \frac{7}{4}, \quad \frac{10}{3}, \quad \frac{9}{3}$

satisfy the inequality: $x \geq 3$?

3. Solve the following inequalities.

a) $-3x \leq 2$ b) $3(x+1) > 4x$ c) $3-2x < \frac{1}{4}$

4. The following expression $\frac{x-3}{4}$ is written in the form $\frac{x-p}{2}$. Identify p in this representation.

5. Write the expression $x-7$ in the form: $x+p$, where x is unknown and p any number. Then identify p in your representation

6. Write the following expressions in the form A^3, where A is any algebraic expression. Identify A in your representation a) $\frac{y^3}{8}$ b) x^3y^6

7. Write the following expressions in the form $y = ax^3 + bx^2 + c$, where $a, b,$ and c are any numbers. Then identify $a, b,$ and c?

a) $y = -2x^2 + \frac{x^3}{4}$ b) $2y - 2x^3 = 4x^2 + 7$

8. Factor the following expressions:

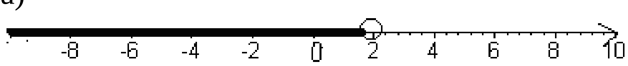
a) $x^2 - 100$ b) $\frac{1}{25} - y^4$

Sample Test 2 (Lesson 10-11)

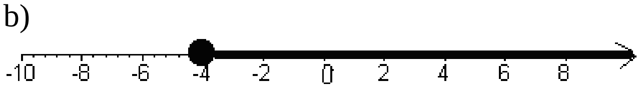
1. The number 4 satisfies which of the following statements? a) $x < 2$ b) $x \geq 4$ c) $x \leq 4$

2. Using inequality symbols, describe the set that is graphed below.

a)



b)



3. Solve the following inequalities.

a) $5x - 2 < 13$ b) $5 > -4 + x$ c) $-2x \leq 8$

d) $\frac{-x-6}{5} \geq -3$ e) $4x - 12 > 4(x-3)$

4. Write the following expression as a difference of two expressions, that is in the form $A - B$, where A and B are any expressions except zero. Then identify A and B

a) $a + 3b$ b) $\frac{m-n}{k}$

5. Write the following expressions in the form of a^n, where a and n is an algebraic expression and then identify a and n:

a) $2 \cdot 2^5$ b) $(2zy)^3(x)^3$

6. Write the following equations in the form $ax + by = c$, where a, b, c are any real numbers. Identify $a, b,$ and c in your representation.

a) $2x - 3y = 5$ b) $3y = 7(x-1)$ c) $x = 0$

7. Determine if the following equations are linear equations in one variable. If so, express it in the form $ax + b = 0$, where a is any real number except 0, b is any real number, and x is unknown. Then identify a and b. If not a linear equation, then state: *not a linear equation*

a) $x^2 + 1 = 0$ b) $\frac{x}{3} - 1 = 0$ c) $\frac{8x^4 - x}{8} = x^4$

8. Factor the following expressions:

a) $1-a^2$ b) $(2x-1)^2-x^2$

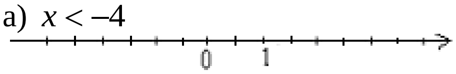
9. Using the formula $a^3-b^3=(a-b)(a^2+ab+b^2)$, factor $1-27x^3$.

Sample Test 3 (Lesson 10-11)

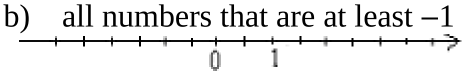
---

1. Graph the following sets on the number line:

a) $x<-4$



b) all numbers that are at least -1



---

2. Solve the following inequalities:

a) $-x<5$ b) $3x+3>-4$

c) $\frac{x-6}{5}\geq 0$ d) $-(2x+1)\geq 4-2x$

e) $\frac{3}{2}x-\frac{2}{3}<1$

---

3. If $x>1$, what inequality is true for

a) $x-2$ b) $\frac{x}{-2}$

---

4. The expression $3(a+2)+(-2)^2$ is written in the form: $3A+B^2$. What algebraic expression represents A and B?

---

5. Write the following expressions in the form x^3, where x is any algebraic expression. Identify x in your representation: a) $8y^3$ b) y^{18}

6. Write the following equations in the form $az^3=b$, where a and b are any numbers. Also Identify a and b in your representation.

a) $z^3-3z^3=-1$ b) $\frac{6z^3+3}{3}=1$

---

7. Factor the following expressions.

a) $0.25-9x^2$ b) $(3a+1)^2-(2a)^2$

c) $1-y^4$

---

8. Determine if the following equations are linear equations in one variable. If so, express it in the form $ax+b=0$, where a is any real number except 0, b is any real number, and x is unknown. Then identify the values of a and b

a) $5x-7=8$ b) $\frac{x}{4}=1$

c) $2(x-4)=3x$

---

9. Using the formula

$a^3+b^3=(a+b)(a^2-ab+b^2)$,

factor x^3y^3-1000

APPENDIX C: SAMPLE TESTS SOLUTIONS

Sample Test 1 (Lesson 1-3) Solutions

1. a) $\frac{2}{7}xy$ b) $x(y+2)$ c) $(xy)^4$
   d) $C(B-A)$ e) $(-x)^3$
2. $-\frac{a+b}{c} = \frac{-(a+b)}{c} = \frac{a+b}{-c}$
3. a) 2 b) −10 c) −16 d) undefined
4. a) 0.4 b) −3 c) 0.9
5. $-\left(\frac{-A}{-B}\right)$
6. a) 5 b) $\frac{3}{4}$ c) $-\frac{15}{4}$
7. all except $1-2x$
8. $\frac{-2x-y}{3}$
9. a) $\frac{1}{y}$ b) $\frac{b}{c}$ c) $\frac{3}{4}$
10. a) $-4+2x$ b) $8x$ c) $2x^2$ d) $\frac{x}{3}$ e) $-4x$

Sample Test 2 (Lesson 1-3) Solutions

1. $x=2$
2. a) $y-x^4$ b) $5+\frac{3x}{4}$ c) $-(b+4)$ d) x^2-6
3. a) 9 b) 1 c) undefined d) 9
4. $\frac{2}{27}$
5. 0.08
6. $\frac{5}{9}(F-32)$
7. a) $-2b$ b) not possible c) $x-9$
   d) $-\frac{6}{5}x^3y$ e) $6x^3-4$ f) -10×4^n
8. possible ways: $-\frac{5}{2}\cdot\frac{a}{b}$, $-5\cdot\frac{a}{2b}$, $-\frac{1}{2}\cdot\frac{5a}{b}$
9. all except: $(1+1a)\div b$
10. $nm-k$, $mn+(-k)$, $-k+mn$
11. a) replace with $(-x)$ b) replace with 7
    c) replace with 1

Sample Test 3 (Lesson 1-3) Solutions

1. a) $7-\frac{x}{2}$ b) $3(2x+5)$ c) $(3x)^2$
2. a) $-(-2x+3y)$ b) $-\frac{x-y}{x+y}$
3. a) addition of x and y b) division of x by y
   c) exponentiation of x d) exponentiation of $-4x$
4. a) 36 b) $-\frac{15}{8}$ c) −0.06
5. a) 3 b) $-\frac{10}{9}$ c) −0.5
6. a) 0 b) undefined c) $-\frac{1}{2}$
7. all are equivalent except (a): $x-y\div2$
8. a) $X=(-x)$ b) $X=3ab$ c) $X=\frac{m}{v}$
9. a) $-5-4x$ b) $-12x$ c) $3x$
   d) $\frac{2}{3}-\frac{5}{6}x$ e) $-\frac{1}{6}-x$
10. If $x=-\frac{2}{3}$ then $(-x)^2=\frac{4}{9}$ and
    $-x^2=-\frac{4}{9}$ thus $(-x)^2\neq -x^2$

Sample Test 1 (Lessons 4-6) Solutions

1. a) $A(B-A+3)=AB-A^2+3A$
   b) $-(-x+2)=x-2$
2. $2a^3(-4a^2b+5)$
3. $\frac{2}{7}(2a-1)$
4. a) $\frac{3}{2}hk$ b) not possible c) 0 d) $\frac{x}{6}$
5. $-xy$; after evaluating: $-xy=\frac{1}{2}$
6. a) $-7b-\frac{1}{3}(3b-6)=-7b-b+2=-8b+2$
   b) $-4x-(x+2)=-4x-x-2=-5x-2$
   c) $(3x-b)^2=9x^2-6bx+b^2$
   d) $-3x+2$
7. a) not possible b) $\frac{xy^3}{x}=y^3$ c) $\frac{2(a+b)}{b+a}=2$
   d) $\frac{2c}{xy}\cdot xy^2=2cy$ e) $\frac{x-2}{2-x}=\frac{-1(2-x)}{2-x}=-1$
8. a) $-a^2$; coefficient: -1
   b) $8x^{13}$;coefficient 8
   c) $4a^2$, coefficient 4
9. a) $\frac{2^{45}}{2^{43}}=2^2=4$ b) $3a^0=3(1)=3$
10. a) $4^7=2^X$; $X=14$
    b) $\left(\frac{1}{3}\right)^X=\left(\frac{1}{9}\right)^5$;$X=10$

Sample Test 2 (Lessons 4-6) Solutions

1. a) $(2a-3)(-3a^3+4a)=$
   $-6a^4+8a^2+9a^3-12a$
   b) $2(-(-3y+9))=2(3y-9)=6y-18$
   c) $(x-3)^2=x^2-6x+9$
2. $b(9b-5)$
3. $5x^3(3x^4+1)$
4. a) $-3ab+4ba-5ab=-4ab$
   b) $3x+4x^2-5+3x^2-6=7x^2+3x-11$
5. a) $\frac{1}{6}ab^2$ b) unlike terms c) $15x^2y^2$
6. $(-3a+2b)-(2a-b)=-5a+3b$
7. a) $-x-\frac{1}{2}$ b) $-b-1$
8. a) -1 b) $\frac{1}{2b-1}$ c) not possible d) a^4
9. a) $-6x^{15}$ b) x^6 c) $2a^7$
10. $\frac{2}{3}\left(2y^2+\frac{1}{3}y-4\right)$

Sample Test 3 (Lessons 4-6) Solutions

1. xy^6x $\quad (xy^3)^2$ $\quad x(y^3)^2$ $\quad y^3x^2y^2$
2. a) $\frac{3}{4}x^2$ b) $2a^6$ c) $-9m^2$
3 a) $\Psi=14$ b) $\Psi=16$
4. a) $6x^6$; coefficient: 6 b) $8a^6b^{12}$;coefficient: 8
   c) $-\frac{7}{2}b^2$ coefficient: $-\frac{7}{2}$
5. a) $-(-3x+5y)=3x-5y$
   b) $(x^2+5)(2x-3)=2x^3-3x^2+10x-15$
   c) $(5x-4y)+(2x+7)=7x-4y+7$
   d) $(2x^2-7x+3)-(5x^2-8x-6)=$
   $-3x^2+x+9$
6. a) $x(5x^2-4x+7)$ b) $-3x(5x-1)$
   c) $(a+z)(4-x)$ d) $-(-\frac{c}{d}+2)$
7. a) $\frac{2}{3}$ b) $\frac{1}{4}(y+z)$ c) 1 d) not possible
8. $\frac{5^{27}}{5^{25}}=5^2=25$
9 a) $-a-4$ when $a=-3$, $-a-4=-1$

b) $9+a^2$ when $a=-3$, $9+a^2=18$

c) $-a^2$ when $a=-3$, $-a^2=-9$

Sample Test 1 (Lessons 7-9) Solutions

1. a) $-\dfrac{9}{16}$ b) $-\dfrac{4}{3}$
2. a) 1 b) -0.2
3. $\dfrac{8-m}{m}$ or $\dfrac{8}{m}-1$
4. $4m$
5. a) yes, it is a solution b) $\dfrac{2}{3}$ is not a solution
6. a) -2 b) 3.4
7. a) $x=-\dfrac{1}{3}$ b) $x=-\dfrac{7}{5}$
8. a) $b=\dfrac{5}{3}$ b) $x=2$ c) $x=2$
   d) no solution e) all real numbers
9. a) $x=2b^2$ b) $x=\dfrac{3b}{a}$ c) $x=\dfrac{1}{1+a}$
10. a) 16 b) 16 c) -64

Sample Test 2 (Lessons 7-9) Solutions

1. a) $\dfrac{1}{2}$ b) 4
2. a) 11 b) 4
3. a) $\dfrac{-2a-3}{3}$ b) $\dfrac{a^2+2a+1}{a}$
4. $3(x-2)=3x+4$; equation
   $5x+\dfrac{1}{x-2}$; expression
   $5x-7=8$; equation 2+3 = 5 equation
5. a) $y=1$ b) $y=0$
6. a) $y=\dfrac{2}{3}$ b) $x=-\dfrac{1}{3}$ c) $a=-10$
   d) $x=2$ e) all real numbers f) 15
7. a) $d=c^2-c$ b) $d=-\dfrac{b}{a}$ c) $d=\dfrac{b-1}{b}$
8. a) no, not a solution b) yes, it is a solution
   c) yes, it is a solution
9. a) no, not a solution b) no, not a solution

Sample Test 3 (Lessons 7-9) Solutions

1. a) $-\dfrac{16}{3}$ b) $-\dfrac{3}{8}$ c) $\dfrac{3}{8}$
2. a) -0.4 b) -0.12
3. no, not a solution
4. a) $-x^2+3$ b) $6x-4x^2$
5. a) $-3t^2$
6. a) it is a solution set b) it is a solution set
   c) it is a solution set
7. a) $x=30$ b) $x=0$ c) no solution
   d) $x=-2$ e) $x=2$
8. $x=0$
9. a) $x=-\dfrac{b}{a}$ b) $x=3d^3$ c) $x=\dfrac{a}{a-c}$

Sample Test 1 (Lessons 10-11) Solutions

1. $x\geq -1$

2. $x\geq 3$ $\dfrac{10}{3}$ and $\dfrac{9}{3}$ satisfy this inequality
3. a) $x\geq -\dfrac{2}{3}$ b) $x<3$ c) $x>\dfrac{11}{8}$
4. $p=3$
5. $x-7=x+(-7)$; $p=-7$
6. a) $\dfrac{y^3}{8}=\left(\dfrac{y}{2}\right)^3$; $A=\dfrac{y}{2}$

b) $x^3y^6 = \left(xy^2\right)^3$; $A = xy^2$

7. a) $y = \frac{1}{4}x^3 + (-2)x + 0$; $a = \frac{1}{4}, b = -2, c = 0$

b) $y = x^3 + 2x^2 + \frac{7}{2}$; $a = 1,\quad b = 2,\quad c = \frac{7}{2}$

8. a) $x^2 - 100 = (x-10)(x+10)$

b) $\frac{1}{25} - y^4 = \left(\frac{1}{5} - x^2\right)\left(\frac{1}{5} + x^2\right)$

Sample Test 2 (Lessons 10-11) Solutions

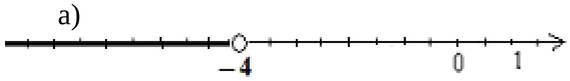
---

1. b and c
2. a) $x < 2$ b) $x \ge -4$
3. a) $x < 3$ b) $x < 9$ c) $x \ge -4$
   d) $x \le 9$ e) all real numbers
4. a) $a + 3b = a - (-3b)$; $A = a$ $B = -3b$
   b) $\frac{m-n}{k} = \frac{m}{k} - \frac{n}{k}$; $A = \frac{m}{k}$, $B = \frac{n}{k}$
5 a) $2 \cdot 2^5 = 2^6$; $a = 2$ *and* $n = 6$
   b) $(2xyz)^3$; $a = 2xyz$, $n = 3$
6. a) *2x + (-3)y = 5* so: $a = 2, b = -3, c = 5$
   b) either: $-7x + 3y = -7$ so
   $a = -7,\ b = 3, c = -7$ or $7x - 3y = 7$ so
   $a = 7,\ b = -3, c = 7$
   c) $1x + 0y = 0$ $a = 1, b = 0, c = 0$
7. a) not a linear equation
   b) $\frac{1}{3}x + (-1) = 0$; a linear equation with
   $a = \frac{1}{3}$, $b = -1$
   c) $-x + 0 = 0$; a linear equation with
   $a = -1,\ b = 0, c = 0$ (note: other results for a are possible but $b = 0, c = 0$ for all results)
8. a) $1 - a^2 = (1-a)(1+a)$
   b) $(2x-1)^2 - x^2 = (2x-1-x)(2x-1+x)$
   $= (x-1)(3x-1)$
9. $1 - 27x^3 = (1)^3 - (3x)^3$
   $= (1-3x)(1+3x+9x^2)$

Sample Test 3 (Lessons 10-11) Solutions

---

1. a)



   b) $x \ge -1$

2. a) $x > -5$ b) $x > -\frac{7}{3}$ c) $x \ge 6$
   d) no solution e) $x < \frac{10}{9}$
3. a) $x - 2$ says that $x > -1$
   b) $\frac{x}{-2}$ says that $x < -\frac{1}{2}$
4. $A = a + 2,\ B = -2$
5. a) $8y^3 = (2y)^3$; $x = 2y$
   b) $y^{18} = \left(y^6\right)^3$; $x = y^6$
6. a) $-2z^3 = -1$; $a = -2,\ b = -1$
   b) $6z^3 = 0$; $a = 6$, $b = 0$
7. a) $0.25 - 9x^2 = (0.5 - 3x)(0.5 + 3x)$
   b) $(3a+1)^2 - (2a)^2 = (a+1)(5a+1)$
   c) $1 - y^4 = (1)^2 - \left(y^2\right)^2 =$
   $\left(1+y^2\right)\left(1-y^2\right) = \left(1+y^2\right)(1+y)(1-y)$
8. a) $5x - 7 = 8$ can become $5x - 15 = 0$; linear equation $a = 5$ and $b = -15$
   b) $\frac{x}{4} = 1$ can become $x - 4 = 0$; linear equation with $a = 1, b = -4$
   c) $2(x-4) = 3x$ can become $x + 8 = 0$; linear equation $a = 1$, $b = 8$
9. $x^3y^3 - 1000 = (xy - 10)\left(x^2y^2 + 10xy + 100\right)$