Math 016 Lessons

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LESSON 1

Natural Numbers

The set of *natural numbers* is given by

$$\mathbb{N} = \{0, 1, 2, 3, 4...\}.$$

Natural numbers are used for two main reasons:

1. counting, as for example, "there are 10 sheep in the herd",
2. or ordering, as for example, "Los Angeles is the second largest city in the USA."

Before we talk about the arithmetic of natural numbers we need to have a basic understanding of the way our number system works. To form any natural number, we use the 10 digits: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. We can represent numbers as large as we want by using their place position to represent different values (place value). The first digit right to left corresponds to the ones place, the next digit (**right to left**) corresponds to the tens place, the next digit to the hundreds place, and so on. We use addition to explain the place value.

For example, consider the number $45, 321$. Then we have

$$\underbrace{4}_{\text{ten-thousands}} \quad \underbrace{5}_{\text{thousands}} \; , \; \underbrace{3}_{\text{hundreds}} \quad \underbrace{2}_{\text{tens}} \quad \underbrace{1}_{\text{ones}}$$

$$45, 321 = 40, 000 + 5000 + 300 + 20 + 1$$

Addition and Multiplication of Natural Numbers.

When adding or multiplying natural numbers, the result is always a natural number.

The symbols '$=$' (is equal to) and '$\neq$' (is not equal to) are used between two objects in order to indicate whether they are or are not the same.

1.3)Examples:

1. $3 + 10 = 13$
2. $3 \times 10 = 30$

3. $3 + 10 \neq 3 \times 10$

4. $5000 + 400 + 90 + 2 = 5492$ (Notice the place value of the digits)

Notice that:

1. Any object is equal to itself, for example:
   (a) $12 = 12$
   (b) $a = a$

2. **Equals can always be substituted for equals**.
   (a) $2 + 3 = 5$ and $1 + 4 = 5,$ then $2 + 3 = 1 + 4$
   (b) $3 \times 10 = 30$ and $30 = 6 \times 5,$ then $3 \times 10 = 6 \times 5$

Some important properties of addition and multiplication of natural numbers are the following:

1. If a is a natural number then:
   (a) $a + 0 = 0 + a = a$
   For example: $5 + 0 = 0 + 5 = 5$
   (b) $1 \times a = a \times 1 = a$
   For example: $1 \times 21 = 21 \times 1 = 21$
   (c) $0 \times a = a \times 0 = 0$
   For example: $0 \times 1231 = 1231 \times 0 = 0$

2. Addition and multiplication of natural numbers are commutative; in other words, changing the order of the numbers does not change the result. That is, if a and b are natural numbers, then:
   (a) $a + b = b + a$
   (b) $a \times b = b \times a$

1.4)Examples:

1. $15 + 123 = 123 + 15; 15 + 123 = 138$, then $123 + 15 = 138$

2. $2 \times 3 = 3 \times 2, 3 \times 2 = 6$, then $2 \times 3 = 6$

When performing operations many times grouping symbols are used. Grouping symbols are symbols used to group together a combination of numbers and operation symbols. Examples of grouping symbols are parentheses ( ); brackets [ ] or braces {}. When performing arithmetic operations, the expression inside the grouping symbol must be computed first.

1.5) Examples:

1. $(3+2)+6=5+6=11$ (Notice the correct use of the '=' sign).

   WARNING: It is incorrect to write

$$(3+2)+6=5=11$$

   Because $5 \neq 11$!

2. $3+(2+6)=3+8=11$

3. $(2\times 3)\times 4=6\times 4=24$

4. $2\times(3\times 4)=2\times 12=24$

Another important property of addition and multiplication is the associative property. Basically it means that when adding or multiplying more than two numbers, no matter how we group them, the result will be the same. Formally: If $a, b \in \mathbb{N}$ then:

1. $(a+b)+c=a+(b+c)$

2. $(a\times b)\times c=a\times(b\times c)$

1.6)Examples:

1. $(2\times 5)\times 10=2\times(5\times 10)$ (Check it!).

2. $(4+8)+8=4+(8+8)$ (Check it!).

Notice that when we need to add or multiply more than two numbers and we realize that it can be done either from left to right or from right to left, we are actually making use of the associative property.

1.7)Examples:

1. $1 + 2 + 3 = (1 + 2) + 3 = 3 + 3 = 6$
2. $1 + 2 + 3 = 1 + (2 + 3) = 1 + 5 = 6$
3. $2 \times 5 \times 10 = (2 \times 5) \times 10 = 10 \times 10 = 100$
4. $2 \times 5 \times 10 = 2 \times (5 \times 10) = 2 \times 50 = 100$

1.1) ***Exercises***

1. Fill in the blank using either '=' or '≠' as appropriate.
   (a) $3 \quad\quad 3 + 0$
   (b) $1 \times 3 \quad\quad 2 \times 2$
   (c) $64 \quad\quad 604$
   (d) $56 \quad\quad 7 \times 8$
   (e) $4 + 3 + 2 \quad\quad (4 + 3) + 2$
   (f) $3 \times 2 \times 4 \quad\quad 8 \times 4$
2. Determine, without calculating, which of the following are true and in that case determine the property or properties of addition or multiplication that makes the statement true.
   (a) $1298 + 7459 = 7459 + 1298$
   (b) $1298 + 7459 = 7460 + 1298$
   (c) $358 \times 498 = 498 \times 359$
   (d) $987 \times 988 = 988 \times 987$
   (e) $(547 + 1250) + 3 = 547 + (1250 + 3)$
   (f) $(547 + 1250) + 3 = 547 + (3 + 1250)$

Let's concentrate for a little while on multiplication. It is important to mention that the numbers in a multiplication are called "*factors*". Notice that some multiplications are actually very easy, in fact we can guess the result without any computations. One particular case of this is the multiplication of a natural number and 10, 100, 1000, 10000 and so on (in other words we are considering natural numbers whose first digit left to right is one, followed by zeros). Basically the result of multiplying a natural number by 10 is the number resulting of adding one zero to the right of the number, if it were multiplying by 100 then we need to add two zeros and so on.

1.8) Examples

1. $58 \times 100 = 5800$

2. $10 \times 345 = 3450$

Related to multiplication we have the exponential notation.

Exponential Notation.

An exponential is an expression of the form

$$a^n$$

where, at this point both a and n are natural numbers. The number a is called the "*base*" of the exponential and n is called the "*exponent*". Notice the relative positions of the base and the exponent. To explain the meaning of the exponential notation let's consider the possibilities for n.

1. If $n = 0$, then $a^0 = 1$.

   For example

   (a) $31^0 = 1$

   (b) $3^0 = 1$

   (c) $0^0 = 1$

2. $a^1 = a$.

   For example

   (a) $3^1 = 3$

   (b) $1^1 = 1$

   (c) $0^1 = 0$

3. $a^2 = a \times a$. For example

   (a) $3^2 = 3 \times 3 = 9$

   (b) $1^2 = 1 \times 1 = 1$

(c) $0^2 = 0 \times 0 = 0$

4. $a^3 = a \times a \times a$.

   For example

   (a) $3^3 = 3 \times 3 \times 3 = 27$

   (b) $1^3 = 1 \times 1 \times 1 = 1$

   (c) $0^3 = 0 \times 0 \times 0 = 0$

5. In general, if n is more than 0, then

$$a^n = \underbrace{a \times a \times ..... \times a}_{n \quad times}$$

   WARNING: When evaluating an exponential it is incorrect to multiply the base times the exponent, it is actually the base times itself as many times as the exponent says.

   For example

   (a) $3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$

   (b) $1^6 = 1 \times 1 \times 1 \times 1 \times 1 \times 1 = 1$

   (c) $0^4 = 0 \times 0 \times 0 \times 0 = 0$

1.2)***Exercises***

Compute (use the '=' sign).

1. 1000×512

2. 10^4

3. 15^1

4. 6^0

5. 100^3

6. 325×100

7. Which one is larger, 2^3 or 3^2? Explain your answer.

8. Which one is larger, 10^0 or 0^{10}? Explain your answer.

9. Which one is larger, 1^3 or $1 + 1 + 1$? Explain your answer.

LESSON 2

Order of Operations

We have talked about addition and multiplication as well as exponential notation. Now let's consider the situation where these operations are combined, including the possible presence of grouping symbols. In that case the operations should be performed in the following order:

1. Operations inside grouping symbols.

2. Exponentials.

3. Multiplications.

4. Additions.

Also, be aware that when a number is followed by a grouping symbol (or vice versa) without an operation symbol in between, then the operation is actually multiplication.

WARNING:

$$(3)(5) \neq 35$$

In fact

$$(3)(5) = 3 \times 5 = 15$$

2.1)Examples

1. $3(5) = 3 \times 5 = 15$

2. $(7 \times 3)2 = (7 \times 3) \times 2 = 21 \times 2 = 42$

3. $3 \times (4 + 2) = 3 \times 6 = 18$

4. $3 + 4 \times 2 = 3 + 8 = 11$

5. $2 + 3^0 = 2 + 1 = 3$

6. $(2 + 3)^0 = 5^0 = 1$

7. $3 \times 2^3 = 3 \times 8 = 24$

8. $(3 \times 2)^3 = 6^3 = 6 \times 6 \times 6 = 216$

WARNING: Do not confuse an expression like

$$3 \times 3^0$$

with

$$(3 \times 3)^0$$

They are not the same!

In fact,

$3 \times 3^0 = 3 \times 1 = 3$, where the exponential is evaluated first, as there are no grouping symbols.

On the other hand,

$(3 \times 3)^0 = 9^0 = 1$, evaluating the expression inside the grouping symbol first.

2.1)***Exercises***

1. Compute. Use the '=' sign and when performing more than one operation show one step for each operation.

   (a) $1 + 0 \times 4$

   (b) $3 + 3^2$

   (c) $3(5 + 2)$

   (d) 4×10^2

   (e) $(4 \times 10)^2$

   (f) $(3 + 3)^2$

   (g) $5 \times 10^4 + 3 \times 10$

   (h) $342 + 23 \times 2$

   (i) $2 \times 200 + 2$

   (j) $398 + 2 + 10$

   (k) $45 \times 100 + 3 \times 15$

   (l) 0×278

   (m) 100×981

   (n) $2345 \times 678 \times 91 \times 0 \times 77$

(o) 1^{8731}

(p) $3^0 \times 1^{3298}$

(q) 6×10^2

(r) $20^4 \times 10$

2. Determine for which of the following removing the parentheses would not change the value of the expression.

(a) $(2+3)+4$

(b) $2 \times (3+4)$

(c) $2 \times (3 \times 4)$

(d) $2+(3)^2$

(e) $2+(3+3)^2$

(f) $(3 \times 3)^0$

(g) $2+(3 \times 4)$

3. Fill in the blank using either '$=$' or '$\neq$' as appropriate. Justify your answer.

(a) $2^3 \qquad 2 \times 3$

(b) $3^2 \qquad 2^3$

(c) $3 \times 3^2 \qquad 3^3$

(d) $4 \times 2^5 \qquad (4 \times 2)^5$

(e) $92 \times 34^3 \qquad 34^3 \times 92$

(f) $4^3 \qquad 3 \times 3 \times 3 \times 3$

(g) $75^5 \qquad 75 \times 75 \times 75 \times 75 \times 75$

(h) $12+45 \times 56 \qquad 12+(45 \times 56)$

(i) $9 \times 7+12 \qquad (9+7) \times 12$

(j) $(5+2) \times 4 \qquad 5+(2 \times 4)$

(k) $(235+789)+99 \qquad 235+(789+99)$

(l) $6543 \times 548 \qquad 548 \times 6543$

Subtraction of Natural Numbers and Order of Operations.

We now consider subtraction of natural numbers; however, the result would be another natural number as long as the first number in the subtraction is bigger than or equal to the second number. If the second number in the subtraction is bigger than the first number, the result is not a natural number

In other words, the result of subtracting two natural numbers **is not necessarily** another natural number.

If we need to perform more than one operation with natural numbers (additions, subtractions, multiplications, exponentials, and the possible presence of grouping symbols), we adopt the following convention regarding the order in which the operations are to be performed:

1. Operations inside grouping symbols.
2. Exponentials.
3. Multiplications.
4. Additions and subtractions are associated from LEFT TO RIGHT.

2.2)Examples

1. $6 - 4 + 2 = 2 + 2 = 4$

   WARNING: It is INCORRECT to say $6 - \underbrace{4 + 2}_{6} = 0$. Remember that when we are performing combinations of addition and subtractions we should go LEFT TO RIGHT.
2. $6 - (4 + 2) = 6 - 6 = 0$
3. $9 - 2^3 = 9 - 8 = 1$
4. $(9 - 2)^3 = 7^3 = 7 \times 7 \times 7 = 343$
5. $10 - 5 \times 2 = 10 - 10 = 0$
6. $(10 - 5) \times 2 = 5 \times 2 = 10$

2.2) ***Exercises***

1. Compute. Use the '=' sign and show one step for each operation.

   (a) $7 - 5 + 2$

(b) $8 - 3 - 2$

(c) $7 + 3 - 10$

(d) $9 - (5 + 4)$

(e) $(8 - 3) \times 10^2$

(f) $12 - 2 \times 3$

(g) $(12 - 2) \times 3$

(h) $8 - 2^3$

(i) $10 \times (5 - 3 + 1)$

(j) $10 \times 5 - 3 + 1$

(k) $10 + 5 \times 3 - 3$

2. For each of the following determine if removing the parentheses would change the value of the expression. Justify your answer.

   (a) $(7 - 5) + 2$

   (b) $7 - (5 + 2)$

   (c) $(5 - 4)^{10}$

   (d) $9 - (2 \times 3)$

   (e) $(9 - 2) \times 3$.

LESSON 3

REVIEW

Homework/Class Work/Quiz 1

1. Fill in the blank using either '=' or '≠' as appropriate.

   (a) $0 + 5 \qquad 0 \times 5$

   (b) $1^2 \qquad 2$

   (c) $2 \times 2 \times 2 \qquad 2^3$

   (d) $1 + 1 + 1 \qquad 1 + (1 + 1)$

   (e) $3 \times 9 \qquad 3 \times (3 \times 3)$

2. Determine whether the following are true or false. Justify your answer.

   (a) $948 + 3221 = 3221 + 948$

   (b) $(3 + 2) \times 4 = 3 + 2 \times 4$

   (c) $218 + 49 \times 85 = 218 + (49 \times 85)$

   (d) $(9 + 15) + 18 = 9 + (15 + 18)$

   (e) $(35 \times 10) \times 41 = 35 \times (10 \times 41)$

   (f) $(4 \times 3) + 5 = 4 \times (3 + 5)$

blank

Homework/Class Work/Quiz 2

1. Fill in the blank using either '=' or '$\neq$' as appropriate.

   (a) $3^3 \quad\quad 3 \times 3$

   (b) $1^3 \quad\quad 2^0$

   (c) $2 \times 4 \quad\quad 2 + 4$

   (d) $340 \quad\quad 34 \times 10$

   (e) $3(20) \quad\quad 320$

2. Determine whether the following are true or false. Justify your answer.

   (a) $585 \times 324 = 324 \times 585$

   (b) $(3+2)^0 = (3+2)0$

   (c) $(98+25)+33 = 98+(33+25)$

   (d) $4+(3 \times 7) = (4+3) \times 7$

   (e) $5+6 \times 2 = 5+2 \times 6$

   (f) $6 \times 6 \times 6 = 6^3$

blank

Homework/Class Work/Quiz 3

1. Compute. Use the '=' sign.
   (a) 10^3
   (b) 35^1
   (c) 9^0
   (d) 100×21
   (e) 2^4

2. Which one is larger 1^4 or 4^1 ? Explain your answer.

3. Which one is larger 5^0 or 5×0 ? Explain your answer.

4. Compute. Use the '=' sign. Show one step for each operation.
   (a) $2 + 3 \times 0$
   (b) $2 \times 3 \times 0$
   (c) $2 + 2^3$
   (d) $(2 + 2)^3$
   (e) $4(1 + 2)$
   (f) $3 + 10^2$
   (g) 3×10^2

blank

Homework/Class Work/Quiz 4

1. Compute. Use the '=' sign.

   (a) 5^2

   (b) 2^5

   (c) 35^0

   (d) 1^5

   (e) 18×100

2. Which one is larger 4^0 or 0^4 ? Explain your answer.

3. Which one is larger 2^1 or 1^2 ? Explain your answer.

4. Compute. Use the '=' sign. Show one step for each operation.

   (a) $1 + 0 \times 2$

   (b) $1 \times 0 + 2$

   (c) $4 + 4^1$

   (d) $(4 + 4)^1$

   (e) $(3 + 2)2$

   (f) $3 + 3^2$

   (g) $10^3 \times 4$

blank

Homework/Class Work/Quiz 5

1. Determine for which of the following removing the parentheses would change the value of the expression. Explain your answer.

   (a) $(4+5)+3$

   (b) $4 \times (5+3)$

   (c) $4 \times (5 \times 3)$

   (d) $(2 \times 4)^0$

   (e) $1 + (3 \times 5)$

2. Compute. Use the '=' sign. Show one step for each operation.

   (a) $3 \times 100 + 2 \times 5$

   (b) $93 \times 0 \times 81$

   (c) $2^3 \times 1^{23}$

   (d) $9 - 5 - 4$

   (e) $9 - 5 \times 1$

   (f) $10 \times (7 - 4 + 2)$

   (g) $10 \times 7 - 4 + 2$

blank

Homework/Class Work/Quiz 6

1. Determine for which of the following removing the parentheses would change the value of the expression. Explain your answer.

   (a) $7 \times (3 + 2)$

   (b) $(9 - 3) + 1$

   (c) $9 - (3 + 1)$

   (d) $(2 \times 3)^1$

   (e) $(2 \times 3)^2$

2. Compute. Use the '=' sign. Show one step for each operation.

   (a) $3 \times 200 + 3$

   (b) $3 \times 2 \times 100$

   (c) $2 \times 10^2 + 1 \times 10$

   (d) $1 + 0^{321}$

   (e) $(1 + 0)^{321}$

   (f) $19 - 4^2$

   (g) $10 - 3 \times 2$

blank

Homework/Class Work/Quiz 7

Compute. Use the '=' sign. Show one step for each operation.

1. $5(2 \times 5)$

2. $5 + 2 \times 5$

3. $9^0 \times 10^3$

4. $(9 \times 10)^0$

5. $2 \times 10 + 10 \times 3$

6. $2 \times (10 + 2) \times 3$

7. $7 - 3 - 1$

8. $7 - (3 - 1)$

9. $10 - 2 \times 5$

10. $(10 - 2) \times 5$

11. $12 - 2^2$

12. $(9 - 2) \times 10^3$

13. $4 + 5 \times 2 - 10$

blank

LESSON 4

INTEGERS

We already mentioned that when subtracting natural numbers, the result may not be a natural number. To fix this problem we will extend the set of natural numbers to the set of integers.

The set of integers is

$$\mathbb{Z} = \{..., -4, -3, -2, -1, 0, 1, 2, 3, 4...\}$$

It is important to be aware of the following:

1. Any natural number is an integer.
2. Any natural number different from zero will be called a "*positive integer*".

   The set of positive integers is

   $$\{1, 2, 3, 4, 5, 6, ...\}$$

3. Any integer that is not a natural number will be called a "*negative integer*".

   The set of negative integers is

   $$\{... - 7, -6, -5, -4, -3, -2, -1\}$$

4. According with our definitions of positive integer and negative integer, 0 is the only integer that is neither positive nor negative.
5. For any integer a, we have that

   $$a = +a$$

   For example:

   (a) $3 = +3$

   (b) $-2 = +(-2)$ Notice the use of parentheses.

   (c) $0 = +0$

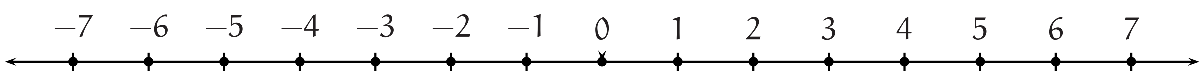
6. If a is a positive integer, then $-a$ is a negative integer and also

   $$-a = -(a) = -(+a)$$

4.1) Examples

1. -2 is a negative integer.
2. -2 is not a natural number.
3. $+124 = 124$
4. $-124 = +(-124)$
5. $-10 = -(+10)$
6. $-6 \neq 6$
7. $+9 = 9$

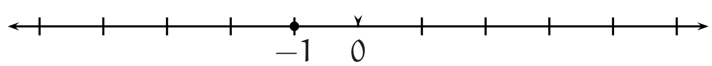
In order to have a graphic picture idea of the integers, we use a number line. It is a straight line that can be made horizontal, with equally spaced points corresponding to integers.



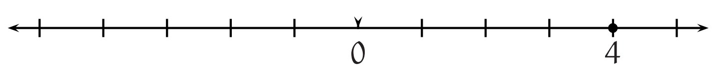
4.2)Examples.

Plot and label the numbers on a number line

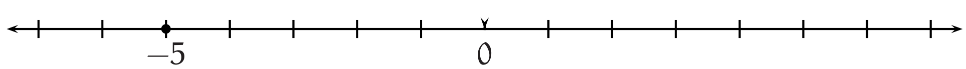
1. -1



2. 4



3. -5



The number line gives us a sense of a number being to the left or to the right of another. For example, 0 is to the right of any negative integer and to the left of any positive integer. This idea can be used to introduce an "*order*" on the integers. This order would be expressed by using the symbols '<' (less than) or '>' (greater than).

If a and b are integers then $a < b$ (a is less than b) or $b > a$ (b is greater than a) means that if we locate a and b on a number line, then a is to the left of b or b is to the right of a.

Notice that

1. If a and b are integers then either

   (a) $a = b$ or

   (b) $a < b$ or

   (c) $a > b$.

2. If a is a positive integer then $a > 0$ (or $0 < a$).

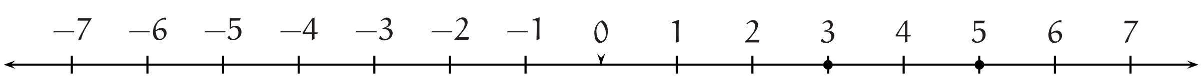
3. If a is a negative integer then $a < 0$ (or $0 > a$).

4. If a is a negative integer and b is a positive integer, then $a < b$.

   Notice:

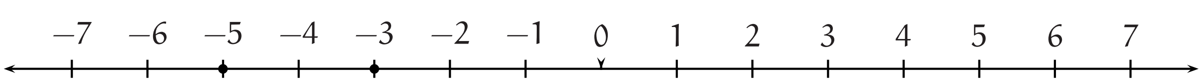
   If a and b are positive integers and $a < b$ then $-a > -b$. Also, if $a > b$ then $-a < -b$.

   For example, $3 < 5$



Therefore

$-3 > -5$



4.3) Examples

1. $3 > 0$

2. $-2 < 0$

3. $-100 < 1$

4. $2 < 67$

5. $-67 < -2$

ADDITION OF INTEGERS

We understand that addition of integers should be an extension of addition of natural numbers. Adding two integers will result in another integer.

We treat grouping symbols with integers exactly the same way that we treat them with natural numbers.

Let's set some basic rules:

1. If a is an integer, then

$$a + 0 = 0 + a = a$$

2. If a is a positive integer then

$$a + (-a) = (-a) + a = 0$$

   Here it is important to notice the use of parentheses.

Two numbers are *opposites* or additive inverses of each other if when adding the two numbers, the result is zero. In particular:

1. If a is an integer, then

   (a) $-a$ is the opposite of a

   (b) a is the opposite of $-a$.

   In view of this, we can say that the opposite of the opposite of a number is the original number or

$$-(-a) = a$$

2. Also notice that if a is a negative integer then $-a$ is a positive integer.

3. $0 + 0 = 0$, therefore the opposite of 0 is 0, or

$$-0 = 0$$

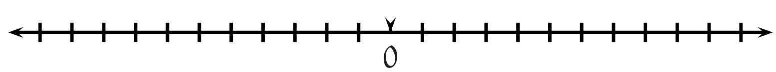
   .

4.4)Examples

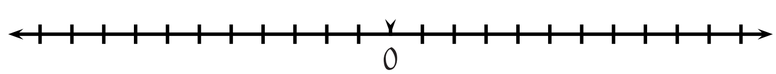
1. $0 + (-3) = -3$
2. $-2 + 2 = 0$
3. $-53 + 0 = -53$
4. $100 + (-100) = 0$
5. $-(-8) = 8$
6. $0 = +0 = -0$
7. $-(-7) > -7$
8. $+(-3) < -(-3)$

Exercises

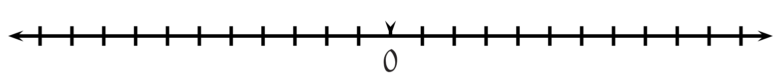
1. Fill in the blank using either '=', '<', or '>', as appropriate.
   (a) $-101 \qquad 100$
   (b) $-(-6) \qquad 6$
   (c) $+125 \qquad 125$
   (d) $-450 \qquad 0$
   (e) $-(-5) \qquad 0$
   (f) $-(-7) \qquad -7$
   (g) $-5 \qquad -4$
2. Plot and label the numbers on a number line.
   (a) -2



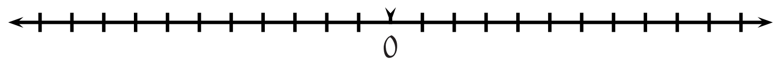
   (b) 10



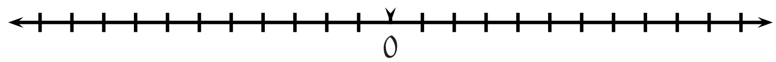
   (c) -7



(d) -9



(e) 5



3. Write the following numbers from the smallest to the largest.

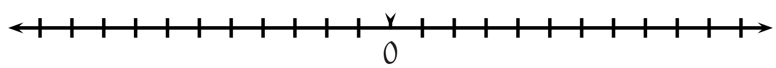
$$-2, 10, -7, -9, 5, 0$$

4. Among the following numbers, identify all pairs of opposite numbers.

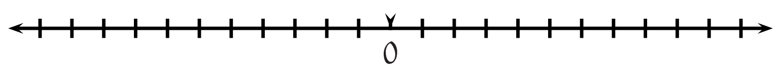
$$9, 7, -9, -(-8), 8, -8, 0, 12, -(+3), +(+3)$$

5. Find the opposite of each of the following integers. Then plot both numbers on a number line.

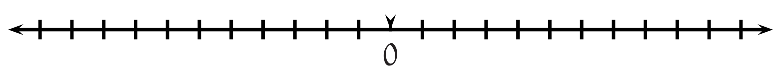
(a) 8



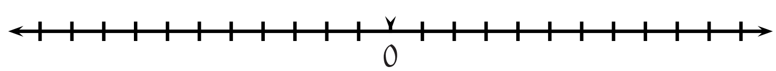
(b) $-(+4)$



(c) $-(-2)$



(d) -6



LESSON 5

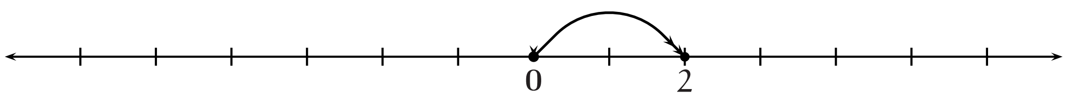
We can add integers by using "movement on a number line". Namely, if we start on the point corresponding to zero, we will move on the line using the numbers in the addition. If we have a positive integer in the addition, that indicates movement to the right. If we have a negative integer, it indicates movement to the left. If we have zero, then we do not move.

Let's see some examples.

1. $2+3$

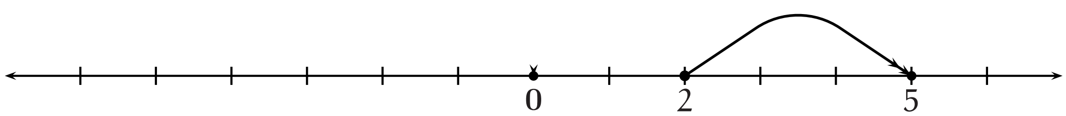
   STEP ONE

   Starting on 0 we move to the right 2 units



   STEP TWO

   Now we move again to the right 3 units



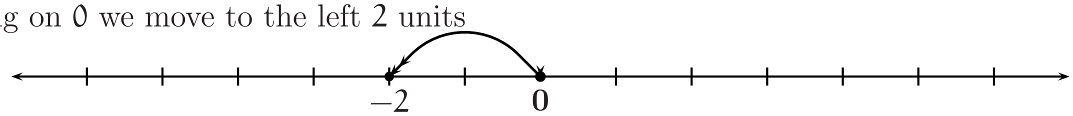
   Therefore,

$$2+3=5$$

2. $(-2)+(-3)$

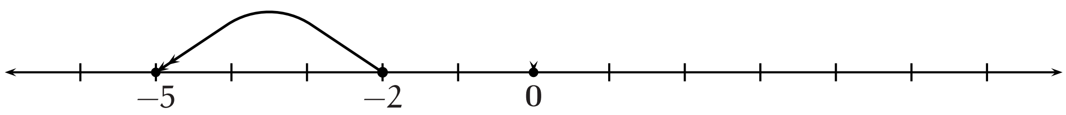
   STEP ONE

   Starting on 0 we move to the left 2 units



STEP TWO

Now we move again to the left 3 units



Therefore,

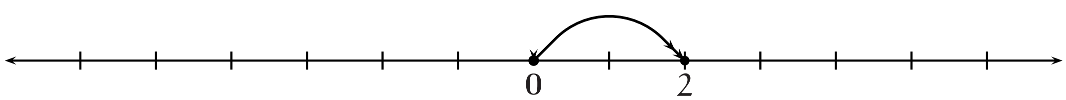
$$(-2) + (-3) = -5$$

Notice the use of parentheses. Also, since $2 + 3 = 5$ and **equals can be substituted for equals** then $-2 + (-3) = -(2 + 3)$

3. $2 + (-3)$

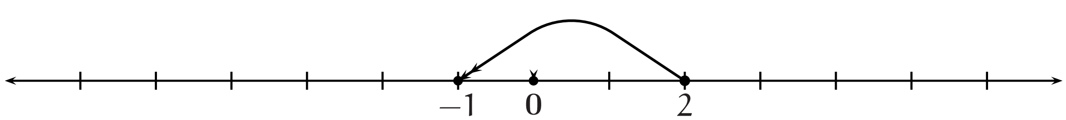
STEP ONE

Starting on 0 we move to the right 2 units



STEP TWO

Now we move again to the left 3 units



Therefore,

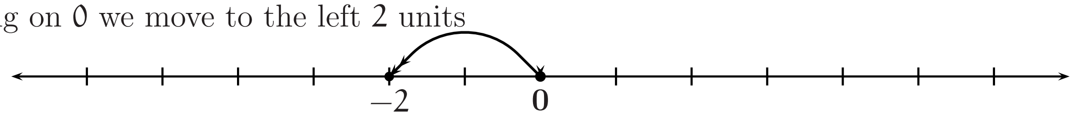
$$2 + (-3) = -1$$

Notice that since $3 - 2 = 1$ (this a subtraction of natural numbers!) and **equals can be substituted for equals**, then $2 + (-3) = -(3 - 2)$

4. $-2 + 3$

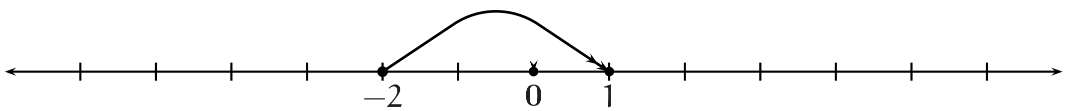
   STEP ONE

   Starting on 0 we move to the left 2 units

   STEP TWO

   Now we move again to the right 3 units

   Therefore,

   $$-2 + 3 = 1$$

   Notice that, since $3-2 = 1$ and **equals can be substituted for equals**, then $-2+3 = 3-2$.

In light of these examples, we can find the following rules reasonable:

If a and b are positive integers and $b > a$ (in the previous example $a = 2$, $b = 3$ and it is true that $3 > 2$), then:

1. $-a + (-b) = -(a + b)$

2. $a + (-b) = -(b - a)$

3. $-a + b = b - a$

Commutative and associative properties are valid for addition of integers. That is, if x, y and z are integers, then:

1. $x + y = y + x$

2. $(x + y) + z = x + (y + z)$

5.1) Examples

1. $-13+0=-13$
2. $3+5=8$
3. $-3+(-5)=-(3+5)=-8$
4. $-5+3=3+(-5)=-(5-3)=-2$
5. $5+(-5)=0$
6. $2+(-7)=-(7-2)=-5$
7. $0+(-7)=-7$
8. $-3+5=5-3=2$
9. $5+(-3)=5-3=2$
10. $-3+3=0$

Exercises

1. Determine which of the following are true and in that case determine the property of addition that makes the statement true.
   (a) $-235+789=789+(-235)$
   (b) $-87+(-56)=87+56$
   (c) $-99+(-33)=-33+(-99)$
   (d) $54+(-67)=67+(-54)$
   (e) $(-6+4)+(-2)=-6+[4+(-2)]$
   (f) $-1+(-2+3)=(-1+2)+3$
2. Compute
   (a) $-6+(-2)$
   (b) $-8+0$
   (c) $-135+135$
   (d) $0+(-189)$

(e) $-15 + 9$

(f) $15 + (-9)$

(g) $-1 + 1$

(h) $-1 + (-1)$

(i) $6 + (+5)$

(j) $-(-4) + 0$

(k) $-18 + 8$

(l) $-25 + 29$

(m) $+1 + (+2)$

(n) $(-9) + (-11)$

(o) $-9 + 11$

(p) $9 + (-11)$

LESSON 6

Multiplication of Integers.

Multiplication of integers will also be a consistent extension of the multiplication of natural numbers. Therefore, we already know how to multiply integers greater than or equal to zero. We need to explain how to multiply if any of the numbers in the multiplication is negative.

First notice that the following properties will remain:

1. For any integer a
$$a \times 1 = 1 \times a = a$$
$$a \times 0 = 0 \times a = 0$$
In fact, the only way the multiplication of two integers could be equal to zero is when one of them is equal to zero.

   In particular $(-1) \times 1 = 1 \times (-1) = -1$. This suggests that the multiplication of a negative number and a positive number should be negative. We codify this as a rule:
$$(-) \times (+) = (+) \times (-) = (-)$$

2. Multiplication is commutative:
$$a \times b = b \times a \text{ for any integers } a \text{ and } b$$

3. Multiplication is associative:
$$(a \times b) \times c = a \times (b \times c)$$
We understand that the presence of parentheses indicates that the multiplication inside should be performed first.

   Notice also that one way to think about the associative property is that now, if we want to multiply more than two numbers, we can group them in any way we want. For example, we can perform the multiplications left to right or right to left.

Now let's try to summarize some of what we know about multiplication of integers, keeping in mind that we are trying to extend the multiplication of natural numbers.

1. Since positive integers are natural numbers, multiplying two positive integers will result in another positive integer.

2. Multiplying a positive integer and a negative integer will result in a negative integer.

3. Now we wonder what should happen if we multiply two negative integers. To try to figure it out, let's recall that we said that the statement that claims that "the opposite of the opposite of a number is the original number", will be expressed by writing:

$$-(-a) = a$$

In particular $-(-1) = 1$. We also said before that when a number is in front of a grouping symbol and there is no operation sign in between, it means that the operation is actually multiplication. Then, it sounds reasonable to think that we can interpret $-(-1)$ as indication that to eliminate the parentheses, we should "multiply" the signs. Since we know that the result is a positive number, this is leading us to think that the rule should be that multiplication of negative numbers will result in a positive one. In order to remember this rule, we can write:

$$(-) \times (-) = +$$

Now we should be ready to state how to multiply integers if any of the numbers in the multiplication is not a natural number.

If a and b are positive integers (therefore $-a$ and $-b$ are negative integers), then

1. $-a \times b = a \times (-b) = -(a \times b)$

   In particular, notice that

$$-1 \times a = -(1 \times a) = -a$$

2. $-a \times (-b) = a \times b$

6.1) Examples.
Compute.

1. $3 \times (-5) = -(3 \times 5) = -15$

2. $-21 \times 10 = -(21 \times 10) = -210$

3. $-4 \times (-7) = 4 \times 7 = 28$

4. $0 \times (-310) = 0$

5. $-1 \times 529 = -529$

6. $9(-8) = -(9 \times 8) = -72$

7. $-(-35) = 35$

8. $-1 \times 3 \times 7 = -3 \times 7 = -21$

9. $-10 \times (-1) \times (-15) = 10 \times (-15) = -150$

10. $(-6) \times (-7) \times 100 = 42 \times 100 = 4200$

Exponential Notation.

Now, for the exponential notation we will allow the base to be an integer, keeping the same definition we gave before.

6.2) Examples

1. $(-3)^2 = (-3) \times (-3) = 9$

   Notice the presence of the parentheses indicating that the base is -3.

   WARNING: $(-3)^2 \neq -3^2$. In fact

   $$-3^2 = -(3^2) = -(3 \times 3) = -9$$

   In other words, without the parentheses the negative sign is "external" to the exponential.

2. $(-2)^3 = (-2) \times (-2) \times (-2) = 4 \times (-2) = -8$

3. $(-1)^4 = (-1)(-1)(-1)(-1) = 1(-1)(-1) = (-1)(-1) = 1$

   Notice that an even exponent with a negative base will result in a positive result and an odd exponent with a negative base will result in a negative result.

4. $0^{210} = 0$

5. $1^{325} = 1$

6. $(-1)^{424} = 1$

7. $(-1)^{1231} = -1$

Exercises

Compute

1. $-1 \times (-2)$
2. $3 \times (-9)$
3. $-5 \times 0 \times 9$
4. $(-2)^5$
5. -2^5
6. -10×251
7. $(-10)^4$
8. -10^4
9. $(-100)^3$
10. $(-1)^{53}$
11. $(-1)^{62}$
12. -1^{48}
13. -1^{49}
14. $2 \times (-3) \times 3$
15. $-5 \times 10 \times (-3)$

LESSON 7

Subtraction of Integers.

When we talked about addition of integers we stated that if a and b are positive integers with $b > a$, then

$$b + (-a) = -a + b = b - a$$

So we can say that

$$b - a = b + (-a)$$

In other words, subtraction of natural numbers can be understood as addition of integers. We will extend this idea to define subtraction of integers.

If a and b are integers, then

$$a - b = a + (-b)$$

7.1) Examples.

1. $-3 - 2 = (-3) + (-2) = -5$

   WARNING: Do not confuse $-3 - 2$ with $-3(-2)$.

   $-3(-2)$ indicates multiplication not subtraction!

2. $5 - 8 = 5 + (-8) = -3$

3. $9 - 5 = 4$

   This example can be understood as subtraction of natural numbers where the result will be another natural number, so we don't need to rewrite it as an addition.

Also notice that

$$a - (-b) = a + [-(-b)] = a + [b] = a + b$$

One way we can think of what happens here is that, in order to **eliminate** parentheses we need to multiply the sign in front of the parentheses times the sign inside. Since $(-) \times (-) = (+)$, then

$$a - (-b) = a + b$$

7.2) Examples.

1. $8 - (-4) = 8 + 4 = 12$

2. $-8 - (-4) = -8 + 4 = -4$

3. $0 - (-5) = 0 + 5 = 5$

Exercises

Compute

1. $12 - 3$

2. $3 - 12$

3. $12 - (-3)$

4. $-3 - 12$

5. $-3 - (-12)$

6. $1 - (-1)$

7. $(+1) - [+(-1)]$

8. $-5 - 0$

9. $5 + (-0)$

10. $0 - 2$

11. $0 - (-2)$

12. $+(-4) - 2$

Division of Integers and Fraction Notation.

The only operation that we have not mentioned so far is division. Let's start with natural numbers.

If a and b are natural numbers with $b \neq 0$ we can perform the ordinary process of long division of a divided by b . In this case a is called the dividend and b the divisor. The process ends with a remainder r (smaller than b), and a quotient q. Both r and q are natural numbers.

We can express the result of the division in the form

$$a = b \times q + r$$

For example, when dividing 14 by 3, the quotient is 4 and the remainder is 2.

That means that:

$$14 = 3 \times 4 + 2$$

In the case when the remainder is zero (for example when dividing 12 by 3), then we have

$$a = b \times q$$

In this case we say that

$$a \div b = q$$

For example

$$12 = 3 \times 4$$

therefore

$$12 \div 3 = 4$$

What we are trying to show is that we can say that the result of dividing one natural number by another natural number is a natural number, only when remainder of the division is zero. Since this is not always true, then we can conclude that the result of dividing natural numbers is not always another natural number.

This problem is not fixed by considering the integers. To divide integers we must extend the division of natural numbers. In other words, if a and b are integers with $b \neq 0$, and q is another integer, then $a \div b = q$ if and only if $a = b \times q$. Again, having such an integer q is not always possible. This means that the result of dividing two integers is not always another integer.

LESSON 8

Understanding that we have limitations, let's mention some things regarding division of integers.

Based on the fact that $a \div b = q$ if and only if $a = b \times q$, then when a and b are positive, so is q (of course, it is the same division of natural numbers!). When a and b are both negative, then q has to be positive. If a is positive and b is negative(or vice versa), then q has to be negative.

We can write the following rules for "division of signs":

1. $(+) \div (+) = (-) \div (-) = (+)$
2. $(+) \div (-) = (-) \div (+) = (-)$

In particular, it happens that:

1. $-a \div b = a \div (-b)$
2. $-a \div (-b) = a \div b$

Also notice that:

1. $0 \div b = 0 \qquad (b \neq 0),$ since $0 = b \times 0$.
2. $a \div 0$ is "undefined".
3. $b \div b = 1 \qquad (b \neq 0),$ since $b = b \times 1$.

8.1)Examples.

1. $-250 \div (-10) = 250 \div 10 = 25$
2. $5 \div (-5) = -5 \div 5 = -1$
3. $-12 \div 4 = 12 \div (-4) = -3$
4. $0 \div 98 = 0$
5. $98 \div 0$ is undefined.
6. $1 \div (-1) = -1$
7. $-1 \div (-1) = 1$
8. $0 \div 0$ is undefined.

Fraction Notation.

So far we have been using the symbol '$\div$' to indicate division. Another way to express it is using fraction notation, that is, to indicate $a \div b$ we can write instead

$$\frac{a}{b}$$

. In this expression a (the dividend) is called the "numerator", and b (the divisor) is called the "denominator".

So we are saying:

$$a \div b = \frac{a}{b} \qquad b \neq 0$$

8.2) Examples

1. $\frac{0}{4} = 0 \div 4 = 0$
2. $\frac{-3}{0}$ is undefined.
3. $\frac{-4}{2} = \frac{4}{-2} = -\frac{4}{2} = -2$
4. $\frac{-12}{-3} = \frac{12}{3} = 4$
5. $\frac{68}{-68} = -\frac{68}{68} = -1$

Exercises

Compute if possible or write undefined.

1. $0 \div (-3)$
2. $\frac{16}{-2}$
3. $+(-18) \div (-3)$
4. $\frac{7}{0}$
5. $[-(-5)] \div (+1)$

6. $\frac{250}{-10}$

7. $-(+6) \div [+(-6)]$

8. $\frac{-5}{+(-5)}$

9. $\frac{0}{-(-8)}$

LESSON 9

REVIEW.

Homework/Class Work/Quiz 8

1. Fill in the blank using either '=', '<', or '>', as appropriate.

   (a) $-9 \qquad 1$

   (b) $0 \qquad -3241$

   (c) $-(-3) \qquad +3$

   (d) $+(-9) \qquad 0$

   (e) $-3 \qquad -1$

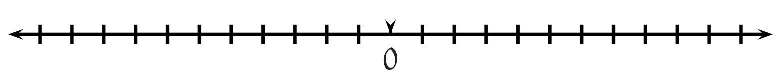
2. Write the following numbers from the smallest to the largest.

$$-134, 71, -1, -2, -(-1), +2$$

3. Find the opposite of each of the following integers. Then plot both numbers on a number line.

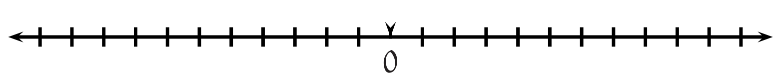
   (a) $-(-3)$

   Opposite:



   (b) $-(+2)$

   Opposite:



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Homework/Class Work/Quiz 9

1. Fill in the blank using either '=', '<', or '>', as appropriate.

   (a) $-35 \quad\quad - 1$

   (b) $+(-7) \quad\quad$ -(-7)

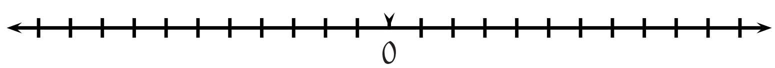
   (c) $0 \quad\quad -(-4)$

   (d) $+(-4) \quad\quad -4$

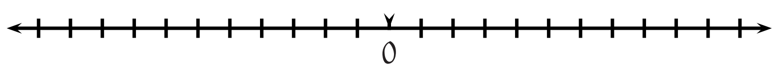
   (e) $-0 \quad\quad +0$

2. Plot and label the numbers on a number line.

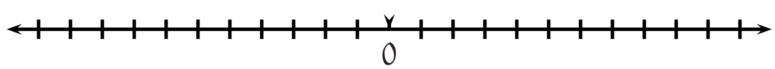
   (a) $-(+1)$



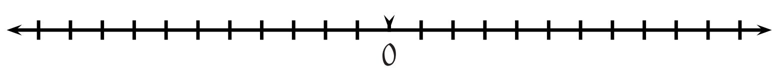
   (b) $+(+4)$



   (c) $-(-3)$



   (d) 3^0



3. Among the following numbers, identify all pairs of opposite numbers.

$$-2, 0, +(+2), +0, -(-4) - 3, -(+4) - (-3)$$

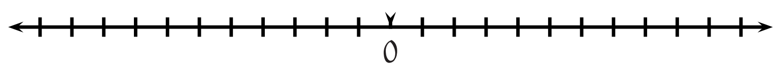
blank

Homework/Class Work/Quiz 10

1. Find the opposite of each of the following integers. Then plot both numbers on a number line.

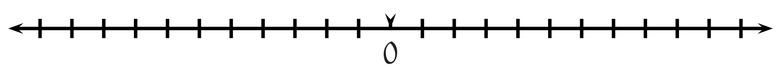
   (a) $+(-1)$

   Opposite:



   (b) $-(-5)$

   Opposite:



2. Determine which of the following are true and in that case determine the property of addition that makes the statement true.

   (a) $-23 + 38 = 38 + (-23)$

   (b) $15 + (-21) = 21 + (-15)$

   (c) $(-5 + 12) + (-9) = -5 + [12 + (-9)]$

   (d) $-2 + [(-4) + 1] = (-2 + 4) + 1$

3. Compute.

   (a) $0 + (-11)$

   (b) $-7 + 7$

   (c) $3 + (+3)$

   (d) $4 + (-4)$

   (e) $5 + (-0)$

blank

Homework/Class Work/Quiz 11

1. Write the following numbers from the smallest to the largest.

$$-(+5), +(-1), -240, 0, -(-50), +5$$

2. Compute.

    (a) $-9+0$

    (b) $-1+(-1)$

    (c) $-0+(-0)$

    (d) $-4+(-3)$

    (e) $-6+6$

    (f) $-5+4$

    (g) $5+(-4)$

    (h) $-3+2$

    (i) $-2+(-2)$

    (j) $2+(-2)$

blank

Homework/Class Work/Quiz 12

Compute and simplify.

1. $(-3)^2$

2. -3^2

3. $-(-2)^4$

4. $-(-2)^3$

5. $-1(-3)$

6. $3 \times (-3)$

7. $-2(2)$

8. -1×4

9. $(-10)^2$

10. $(-0)^7$

11. $(-1)^3 1$

12. $2 \times (-1) \times 3$

blank

Homework/Class Work/Quiz 13

Compute and simplify.

1. $(-1)^{42}$

2. -1^{42}

3. $-(-100)^2$

4. $-1 + (-2)$

5. $-1(-2)$

6. $31 \times (-10)$

7. $-5 \times 2 \times (-15)$

8. $28 \times (-45) \times 0$

9. $-5 + 6$

10. -5×6

11. $-5 + 4$

12. $-5(+4)$

blank

Homework/Class Work/Quiz 14

Compute and simplify.

1. $-1+0$

2. $-1(+0)$

3. $-6+(+6)$

4. $-3\times(-3)\times(-1)$

5. 0^5

6. $(-1)^{13}$

7. $-9+5$

8. $-9+(-5)$

9. $-9(+5)$

10. $-9(-5)$

11. $-5+(-5)$

12. $+5+(-5)$

13. -7^2

14. $(-6)^2$

15. $-1+3$

16. $-(-1)^{12}$

blank

Homework/Class Work/Quiz 15

Compute if possible or write 'undefined'.

1. $-1-4$

2. $-1(-4)$

3. $\dfrac{-4}{-1}$

4. $0 \div (-4)$

5. $-2+(-3)$

6. $-2-(-3)$

7. $\dfrac{5}{0}$

8. $-1+3$

9. $-1(+3)$

10. $(-12) \div (-4)$

11. $\dfrac{0}{2}$

12. $-1-1$

13. $-1+(-1)$

14. $2-3$

blank

Homework/Class Work/Quiz 16

Compute if possible or write 'undefined'.

1. $(-1)^5$

2. $(-1)5$

3. $-1+5$

4. $5+(-5)$

5. $5 \div (-5)$

6. $5(-5)$

7. $-9-9$

8. $-9+8$

9. $-9+10$

10. $35 \times (-10)$

11. $(-10)^0$

12. $\frac{9}{0}$

13. $8 \div (-2)$

14. $-2 \times (-2)$

blank

Homework/Class Work/Quiz 17

Compute if possible or write 'undefined'.

1. $-3 + (+7)$

2. $-3(+7)$

3. $-3 - (+7)$

4. $-3 - (-7)$

5. $\frac{10}{-5}$

6. $(-5) \div 0$

7. $0 \div (-5)$

8. $(-1) \times (-1) \times (-3)$

9. $-(-1)^{38}$

10. $(-5)^3$

11. $-2 - 2$

12. $4 - 5$

13. $-4 - 5$

14. -3^0

blank

Homework/Class Work/Quiz 18

Compute if possible or write 'undefined'.

1. $-3 + (+3)$
2. $-7 - (-2)$
3. $9 + (-8)$
4. $-5 - 5$
5. $(-1)^9$
6. $-(-1)^{10}$
7. $-(-10)^3$
8. $(85)^0$
9. $-9 \times (-10) \times (-1)$
10. $8 \times (-8) \times 2$
11. $(-49) \div 7$
12. $0 \div 51$
13. $-8 \div 0$
14. $1 \div (-1)$

blank

LESSON 10 Order of Operations.

Let's talk again about the order of operations if we include subtraction and division. Operations should be performed in the following order:

1. Operations inside grouping symbols. If there are grouping symbols inside grouping symbols, then we need to work from inside out.

   WARNING: If the result coming from a grouping symbol is a negative number, keep the number in a grouping symbol.

2. Exponentials.

3. Multiplications and divisions are associated LEFT TO RIGHT.

   WARNING: There is not priority of multiplication over division or the other way around. You perform the one that comes first from left to right, regardless of whether it is multiplication or division.

4. Additions and subtractions are associated LEFT TO RIGHT.

Be aware that if within grouping symbols there is more than one operation then, inside the grouping symbol we should also follow the order, starting with exponentials.

9.1) Examples.

1. $(-3+2-2)\times 10 = (-1-2)\times 10 = (-3)\times 10 = -30$
2. $(2-5)^3 = (-3)^3 = -3\times(-3)\times(-3) = -27$
3. $2-5^3 = 2-125 = -123$
4. $-3-3+2\times 4 = -3-3+8 = -6+8 = 2$
5. $(-2)^2 - 36\div(-6)\times 2 = 4-36\div(-6)\times 2 = 4+6\times 2 = 4+12 = 16$
6. $-3^2+9\times 6\div 3-20 = -9+9\times 6\div 3-20 = -9+54\div 3-20 = -9+18-20 = 9-20 = -11$
7. $-2-(-3+1)\div(-2)\times 2 = -2-(-2)\div(-2)\times 2 = -2+2\div(-2)\times 2 = -2-1\times 2 = -2-2 = -4$
8. $3+2(5-7) = 3+2(-2) = 3-4 = -1$

9. $(3+2)(5-7) = 5(-2) = -10$

   WARNING: Notice that if you do not use a parentheses for the result of the second parentheses, you will be writing $5-2$ which is actually a subtraction and not a multiplication which is the right operation.

10. $-(3-2\times 2)-(8\div 2\div 2) = -(3-4)-(4\div 2) = -(-1)-2 = 1-2 = -1$

11. $2[4-3(1-2\times 2)+1] = 2[4-3(1-4)+1] = 2[4-3(-3)+1] = 2[4+9+1] = 2\times 14 = 28$

Exercises

Compute if possible, or write that the expression is undefined. When computing, show one step for each operation. Make sure that you use the '=' sign correctly.

1. $-5-3$

2. $(-5)(-3)$

3. $-1-1(-1)$

4. $(4-2\times 2)^0$

5. $[5+5(-1)]\div[4-5]$

6. $(4-5)\div[3+3(-1)]$

7. $-1^2-10\div(-5)\times 2$

8. $7-3\times 2$

9. $(7-3)2$

10. $-2+4\times(-1)+3$

11. $-3-3(4-5)$

12. $(-3-3)(4-5)$

13. $5-2^2+3(-2)$

14. $(5-2)^3+3(-2)$

15. $-[5-2(1+1\times 3)]-[2-(2^2-8\div 2)]$

16. $(-1)(-2)(-5)$

17. $(-10)10^3\times 10^0$

18. $-(-3)[-(-7)]$

19. $(-8^2-3^4-2^5)(-6-4)$

20. $-12-(-4)+3-(4+3)$

21. $-125\div 25+[(-6-3)-6-3]$

22. $-[(-7-5)(-7+5)(7-5)(7+5)]$

23. $-876\times(-954)^0-3^1$

LESSON 11
REVIEW

Homework/Class Work/Quiz 19

Compute if possible or write 'undefined'. Show one step of each operation. Make sure that you use the '=' sign correctly.

1. $(1-3)^3$

2. $1-3^3$

3. $(-1-1+1)\times 10$

4. $4-3\times 2-1$

5. $(-1)^2-9\div(-3)$

6. $1+1(-2-2)$

7. $(2+1)(-1+1)(8-3)$

8. $7-3^2+2(-1)$

blank

Homework/Class Work/Quiz 20

Compute if possible or write 'undefined'. Show one step of each operation. Make sure that you use the '=' sign correctly.

1. $2-(2-3)^{20}$

2. $-5+9^0$

3. $[2+2(-2)]\div(3-5)$

4. $-2^3+0\div 2$

5. $3+2(-1-1)$

6. $9-10\div 5\times 2$

7. $-[4-3(-1-1\times 2)]+2-(-1)^3$

8. $(-10)^3\div 10^2\times 10$

blank

Homework/Class Work/Quiz 21

Compute if possible or write 'undefined'. Show one step of each operation. Make sure that you use the '=' sign correctly.

1. $-(-2)[-(-5)]$

2. $(-3^2 - 2^3 - 5^0)(-1 - 2)$

3. $9 - 3 - 3 - (-5 + 1)$

4. $36 \div (-2) + [-(-7 - 3) \times 2]$

5. $-535 \times (536)^0 + 2^1$

6. $-3 + 5 \times (-2) \div 2 \times 5$

7. $-2[3 - 3(2 - 1 \times 3)] - [4 - (1^3 - 9 \div (-9))]$

blank

LESSON 12

We already mentioned that sometimes division of two integers is not equal to an integer. This means that we need more numbers (those representing the result of a division when the result is not an integer). We will write those numbers as the **fraction** representing the division.

A fraction is an expression of the form

$$\frac{a}{b} \quad \text{where } a \text{ and } b \text{ are integers and } b \neq 0$$

a is the numerator of the fraction and b the denominator of the fraction.

Since we know that a fraction also indicates a division, then some fractions are actually integers.

For example:

1. $\dfrac{-6}{3} = -2$
2. $\dfrac{10}{1} = 10$

   Any fraction with denominator 1 is actually an integer. In fact, it is equal to the numerator of the fraction. This also means that any integer can be written as a fraction. Any integer is equal to the fraction with numerator equal to the same integer and denominator equal to 1.

   For example:

   (a) $7 = \dfrac{7}{1}$

   (b) $-14 = \dfrac{-14}{1}$

3. $\dfrac{-5}{-5} = 1$

   When the numerator and denominator are equal (and both are different from zero!), then the fraction is equal to 1.

4. $\dfrac{0}{3} = 0$

   When the numerator is zero (and the denominator is different from zero) then the fraction is equal to zero.

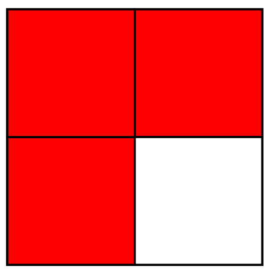
5. $\dfrac{4}{0}$

   An expression like this is considered **undefined**.

Now let's try to understand the meaning of a fraction that is not an integer. Let's start with the case when the numerator and denominator are positive integers: If m and n are positive integers, we can use the part-whole interpretation for the fraction $\frac{m}{n}$. That is, we can think that units of something are divided into n equal pieces and then we take m of those pieces. Notice that if $m > n$, in order to be able to take m pieces, we will need more than one unit. If $m < n$ one unit will be enough to cover the m pieces.

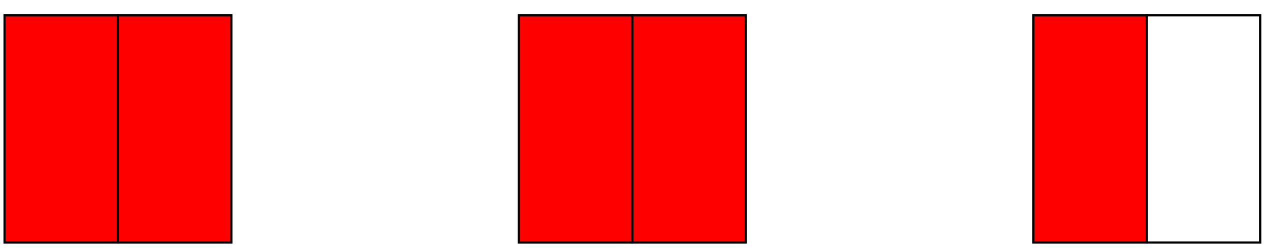
12.1)Examples

1. To understand the meaning of the fraction $\frac{3}{4}$ we consider a "unit" (for example, a square) divided into 4 equal pieces. Then we take 3 pieces.

   If we think of the "unit" as the number 1 then we can conclude that $\frac{3}{4}$ is a number between 0 and 1, closer to 1 than to 0. Notice that the numerator is less than the denominator.

2. If we want to understand the fraction $\frac{5}{2}$ now we need to divide units into 2 equal pieces and then take 5 pieces. Therefore, not only we need more than one "unit", we actually need more than 2 "units".



   This interpretation is telling us that $\frac{5}{2}$ is a number between 2 and 3 and in fact, it is right in the middle between 2 and 3. Now, when we say "middle", we are basically thinking of the location of the fraction as a number on a number line. If the fraction is not an integer, then the fraction corresponds to a point on the line that happens to be between two integers.

Location of a fraction on a number line.

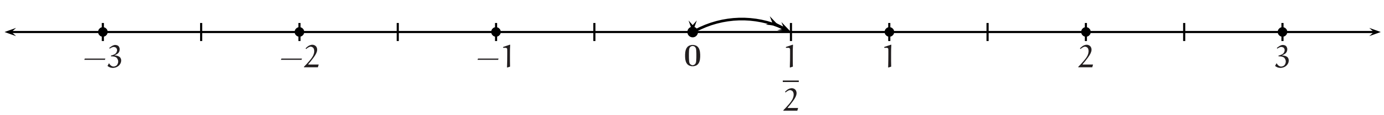
Based on the rules for division of signs, we can conclude that when a fraction is not an integer, it can be written as either $\frac{m}{n}$ (positive fraction) or $-\frac{m}{n}$ (negative fraction) where m and n are positive integers. One way to understand the fraction as a number is by finding a location on a number line. To do that, we will consider the segment between two consecutive integers as a "unit". For example, the segment between -3 and -2 is a unit, the segment between -2 and -1 is a unit, the segment between -1 and 0 is a unit, the segment between 0 and 1 is a unit, etc.

We will understand that each unit is divided into n equal pieces. To find the location of the fraction on a number line we start on 0. If the fraction is positive, we will move to the right till we cover m pieces. If the fraction is negative, then we move to the left till we cover m pieces.

12.2) Examples.

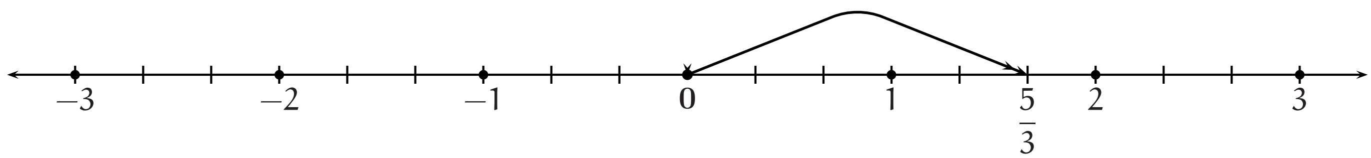
Locate the fractions on a number line.

1. $\frac{1}{2}$



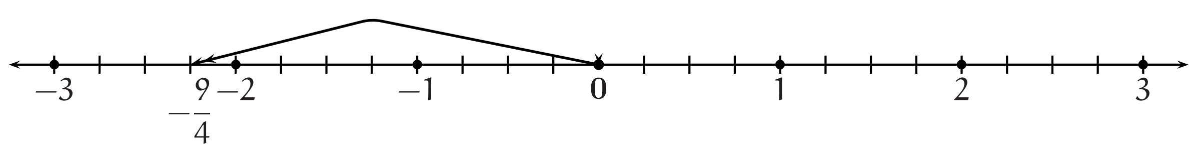
In this case, each unit is divided into two equal pieces (because the denominator is 2). Since the fraction is positive, we start on "0" and move to the right until we cover one piece (because the numerator is 1). Notice that this location is showing that $\frac{1}{2}$ is the number in the "middle" between 0 and 1.

2. $\frac{5}{3}$



To locate this fraction, each unit is divided into three equal pieces (because the denominator is 3). Since the fraction is positive, we start on 0 and move to the right until we cover five pieces (because the denominator is 5).This location shows that $\frac{5}{3}$ is a number between 1 and 2 closer to 2 than to 1.

3. $-\frac{9}{4}$



Now we need to divide each unit into four equal pieces (the denominator is equal to 4). Since this fraction is negative, we start on 0 and move to the left until we cover nine pieces (the numerator is equal to 9). This location shows that $-\frac{9}{4}$ is a number between -3 and -2, closer to -2 than to -3.

The set of numbers that can be written as fractions (and that includes the integers!) is called the set of **Rational Numbers**.We will use $\mathbb{Q}$ to denote this new set.

Rational numbers can be located on a number line. Therefore, we have an "order" in $\mathbb{Q}$ that we will express using the symbols '$<$'(is less than) or '$>$' (is greater than).

12.3) Examples.

Based on the examples in 12.2, now we can say that

1. $\frac{1}{2} > 0$
2. $\frac{1}{2} < 1$
3. $0 < \frac{1}{2} < 1$
4. $\frac{5}{3} > 1$
5. $\frac{5}{3} < 2$
6. $1 < \frac{5}{3} < 2$
7. $-\frac{9}{4} > -3$
8. $-\frac{9}{4} < -2$
9. $-3 < -\frac{9}{4} < -2$

It is also important to notice that if a and b are integers, with $b \neq 0$, then:

1. $\frac{-a}{b} = -\frac{a}{b}$

   For example, $\frac{-3}{7} = -\frac{3}{7}$
2. $\frac{a}{-b} = -\frac{a}{b}$

   For example, $\frac{3}{-7} = -\frac{3}{7}$
3. Based on the two previous items, we can write:

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

   For example,

$$\frac{-3}{7} = \frac{3}{-7} = -\frac{3}{7}$$

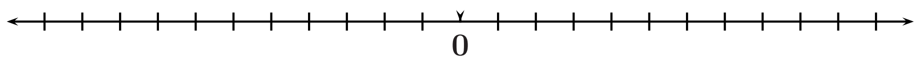
4. $\dfrac{-a}{-b} = \dfrac{a}{b}$

For example, $\dfrac{-3}{-7} = \dfrac{3}{7}$

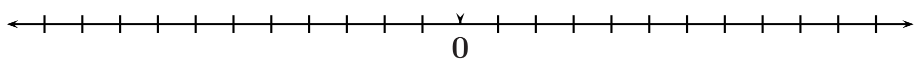
Exercises

1. Locate the number on a number line.

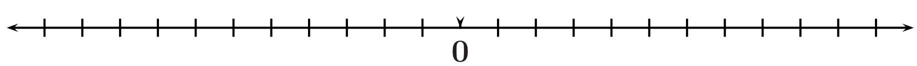
(a) $\dfrac{7}{6}$



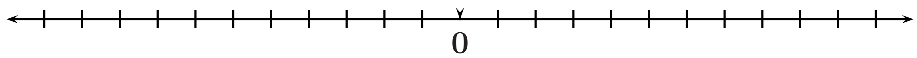
(b) $-\dfrac{11}{3}$



(c) $\dfrac{4}{5}$



(d) $-\dfrac{1}{3}$



2. Fill in the blank using either '=', '<', or '>' as appropriate.

   (a) $-\dfrac{24}{5} \qquad 0$

   (b) $0 \qquad \dfrac{1}{3}$

   (c) $\dfrac{6}{-3} \qquad 2$

   (d) $\dfrac{-6}{3} \qquad -2$

   (e) $\dfrac{-3}{-5} \qquad \dfrac{5}{-3}$

   (f) $\dfrac{9}{8} \qquad \dfrac{10}{10}$

   (g) $1 \qquad \dfrac{9}{10}$

   (h) $\dfrac{1}{3} \qquad \dfrac{5}{15}$

   (i) $-1 \qquad -\dfrac{9}{10}$

   (j) $\dfrac{0}{3} \qquad \dfrac{-1}{100}$.

LESSON 13

Multiplication of Fractions.

We would like the multiplication of fractions to extend the multiplication of integers. In particular, if we write integers as fractions and we multiply them as fractions, the result should be the same as when we use the multiplication of integers.

For example $3 \times 2 = 6$, as fractions $\frac{3}{1} \times \frac{2}{1} = \frac{6}{1}$. This would make us think that the way to multiply positive fractions is by multiplying the numerators and multiplying the denominators. That is in fact the way to do it. When multiplying fractions that could be either positive or negative, we will use the rules of multiplication of signs, along with multiplying numerators and denominators.

In general, we can set the following rules regarding multiplication of fractions:

1. $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \qquad b \neq 0; d \neq 0$
2. Notice that as a consequence of the previous point,
$$\frac{a}{b} \times 0 = 0 \times \frac{a}{b} = 0 \qquad b \neq 0$$
Let's check it
$$\frac{a}{b} \times 0 = \frac{a}{b} \times \frac{0}{1} = \frac{a \times 0}{b \times 1} = \frac{0}{b} = 0$$
3. Also as a consequence of the first point we have
$$\frac{a}{b} \times 1 = 1 \times \frac{a}{b} = \frac{a}{b} \qquad b \neq 0$$
$$\frac{a}{b} \times 1 = \frac{a}{b} \times \frac{1}{1} = \frac{a \times 1}{b \times 1} = \frac{a}{b}$$

13.1) Examples.

1. $\frac{3}{4} \times \frac{7}{2} = \frac{3 \times 7}{4 \times 2} = \frac{21}{8}$
2. $\frac{5}{2} \times 9 = \frac{5}{2} \times \frac{9}{1} = \frac{5 \times 9}{2 \times 1} = \frac{45}{2}$

WARNING: When multiplying a fraction times an integer we need to write the integer as a fraction first, and then apply what we know about multiplying fractions. In particular
$$\frac{5}{2} \times 9 \neq \frac{45}{18}$$

We are not supposed to invent our own rules!!!

We should not multiply both numerator and denominator by 9!

3. $11 \times \frac{10}{7} = \frac{11}{1} \times \frac{10}{7} = \frac{11 \times 10}{1 \times 7} = \frac{110}{7}$

4. $\frac{8}{5} \times 1 = \frac{8}{5}$

Also notice the properties:

1. Commutative property.
$$\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$$

2. Associative property.
$$\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right)$$

13.1)***Exercises.***

Compute. If the resulting fraction is an integer, indicate it.

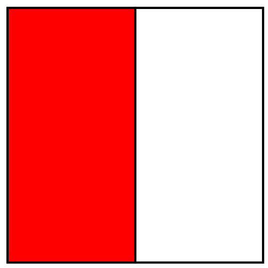
1. $\frac{7}{3} \times \frac{2}{5}$
2. $3 \times \frac{1}{5}$
3. $\frac{2}{7} \times 5$
4. $\frac{11}{12} \times \frac{0}{5}$
5. $\frac{9}{2} \times 1$
6. $1 \times \frac{3}{5}$
7. $\frac{1}{4} \times \frac{7}{4}$
8. $\frac{1}{4} \times 4$
9. $\frac{3}{2} \times \frac{4}{3}$
10. $0 \times \frac{2}{3}$

Equivalent Fractions.

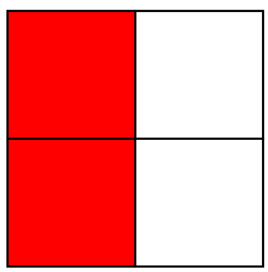
Two fractions may look different and still represent the same number. In that case, we say that they are equivalent, and we can state that they are equal.

For example, $\frac{1}{2} = \frac{2}{4}$. Let's see it from the part-whole interpretation point of view:

$\frac{1}{2}$



$\frac{2}{4}$



From the arithmetic point of view, we can take advantage of the rules for multiplication of fraction. That is:

$$\frac{1}{2} = \frac{1}{2} \times 1$$

Since

$$1 = \frac{2}{2}$$

and **equals can be substituted for equals**, then:

$$\frac{1}{2} = \frac{1}{2} \times 1 = \frac{1}{2} \times \frac{2}{2} = \frac{1 \times 2}{2 \times 2} = \frac{2}{4}$$

In general for any fraction, we have:

$$\frac{a}{b} = \frac{a}{b} \times 1$$

For any integer $n \neq 0$, we have that

$$1 = \frac{n}{n}$$

and since **equals can be substituted by equals**, then:

$$\frac{a}{b} = \frac{a}{b} \times 1 = \frac{a}{b} \times \frac{n}{n} = \frac{a \times n}{b \times n}$$

This shows that if we multiply both numerator and denominator by the same integer (different from zero!), the "new" fraction is equivalent to the original one.

13.2) Examples.

1. Find a fraction equivalent to $\frac{7}{3}$

   $$\frac{7}{3} = \frac{7}{3} \times 1 = \frac{7}{3} \times \frac{5}{5} = \frac{7 \times 5}{3 \times 5} = \frac{35}{15}$$

   Therefore, one possible fraction equivalent to $\frac{7}{3}$ is $\frac{35}{15}$.

   Notice that replacing 1 with $\frac{5}{5}$ was just a choice. We could have picked any other fraction equal to 1.

2. Find a fraction equivalent to $\frac{7}{3}$ with denominator 18.

   In this case we are looking for a particular new denominator. It is possible to make it be 18 because $18 = 3 \times 6$. If would not be possible, for example, to have 19 for the new denominator.

   $$\frac{7}{3} = \frac{7}{3} \times 1 = \frac{7}{3} \times \frac{6}{6} = \frac{7 \times 6}{3 \times 6} = \frac{42}{18}$$

   Therefore, the fraction we are looking for is $\frac{42}{18}$.

3. Find a fraction equivalent to $\frac{5}{2}$ with denominator 25.

   This is not possible because 25 can not be the result of multiplying 2 by another integer!

4. Find a fraction equivalent to $\frac{11}{6}$ with denominator 24.

   $$\frac{11}{6} = \frac{11}{6} \times 1 = \frac{11}{6} \times \frac{4}{4} = \frac{11 \times 4}{6 \times 4} = \frac{44}{24}$$

   So, the fraction that we are looking for is $\frac{44}{24}$.

13.2) ***Exercises***

1. Find 2 fractions equivalent to $\frac{5}{2}$. Show all the steps.

2. Find a fraction equivalent to $\frac{3}{5}$ with denominator 100. Show all the steps.

3. Find a fraction equivalent to $\frac{7}{10}$ with denominator 50. Show all the steps.

4. Determine which of the following pairs of fractions are equivalent. Justify your answer.

   (a) $\frac{3}{7}, \frac{12}{28}$

   (b) $\frac{2}{3}, \frac{5}{6}$

   (c) $\frac{0}{5}, \frac{0}{3}$

   (d) $\frac{7}{7}, \frac{345}{345}$

   (e) $\frac{14}{2}, \frac{21}{3}$

   (f) $\frac{1}{16}, \frac{1}{32}$

   (g) $\frac{1}{4}, \frac{2}{8}$

LESSON 14

Simplification of Fractions.

Notice that the process that we explained in the previous lesson to find an equivalent fraction, will make the "new" fraction have a bigger numerator and denominator.

Now, let's consider what happens if we do this process backwards. In other words, given a fraction, we want to find an equivalent fraction with a smaller numerator and denominator (if possible). This process is called " simplifying the fraction" or "reducing the fraction to lowest terms".

Take for instance the last example we discussed. In this case, the original fraction is $\dfrac{44}{24}$ and we want to show that $\dfrac{11}{6}$ is equivalent. We can just copy the steps we did before but backwards; however, we need to understand the meaning of the steps.

Simplify $\dfrac{44}{24}$.

FIRST STEP

$$\frac{44}{24} = \frac{11 \times 4}{6 \times 4}$$

Notice that we expressed both numerator and denominator as a multiplication. These multiplications have a factor in commom (in this case it is 4). We will call this factor a **common factor**. In this example, we factored 44 and 24 using the common factor 4.

SECOND STEP

$$\frac{44}{24} = \frac{11 \times 4}{6 \times 4} = \frac{11}{6} \times \frac{4}{4}$$

Here we are using the rule for multiplication of fractions "backwards". We certainly know that

$$\frac{11}{6} \times \frac{4}{4} = \frac{11 \times 4}{6 \times 4}$$

so we can say

$$\frac{11 \times 4}{6 \times 4} = \frac{11}{6} \times \frac{4}{4}$$

Also, we are making sure to use the fraction with numerator and denominator given by the common factor. Such fraction will be equal to 1.

THIRD STEP

$$\frac{44}{24} = \frac{11 \times 4}{6 \times 4} = \frac{11}{6} \times \frac{4}{4} = \frac{11}{6} \times 1$$

We know that

$$\frac{4}{4} = 1$$

and since **equals can be substituted for equals**,then:

$$\frac{11}{6} \times \frac{4}{4} = \frac{11}{6} \times 1$$

FOURTH STEP

$$\frac{44}{24} = \frac{11 \times 4}{6 \times 4} = \frac{11}{6} \times \frac{4}{4} = \frac{11}{6} \times 1 = \frac{11}{6}$$

Let's comment on some things regarding this example.

Based on the last step we could write:

$$\frac{11 \times 4}{6 \times 4} = \frac{11}{6}$$

We can say that the effect was of "cancelling" the common factor 4. Once we advance some more in this class, we will show simplification without providing all the steps. We will go straight from the fraction that shows the common factor (or factors) to the simplified fraction, thinking about "cancelling". However, it is very important that we understand what allows us to do the "cancelling". That is, the fact that we have a multiplication in the numerator and denominator showing a common factor.

WARNING:

$$\frac{11+4}{6+4} \neq \frac{11}{6} \quad \text{4 can not be cancelled!!!}$$

In order to cancel, we must have **multiplication** in the numerator and denominator showing a common factor.

14.1)Examples.

Simplify the fractions as much as possible. Show all the steps of the simplification.

1. $\dfrac{18}{27}$

$$\frac{18}{27} = \frac{2 \times 9}{3 \times 9} = \frac{2}{3} \times \frac{9}{9} = \frac{2}{3} \times 1 = \frac{2}{3}$$

Notice that we could have done:

$$\frac{18}{27} = \frac{6 \times 3}{9 \times 3} = \frac{2 \times 3 \times 3}{3 \times 3 \times 3} = \frac{2}{3} \times \frac{3}{3} \times \frac{3}{3} = \frac{2}{3} \times 1 \times 1 = \frac{2}{3} \times 1 = \frac{2}{3}$$

2. $\dfrac{84}{132}$

$$\frac{84}{132} = \frac{7 \times 12}{11 \times 12} = \frac{7}{11} \times \frac{12}{12} = \frac{7}{11} \times 1 = \frac{7}{11}$$

Or

$$\frac{84}{132} = \frac{42 \times 2}{66 \times 2} = \frac{6 \times 7 \times 2}{11 \times 6 \times 2} = \frac{6}{6} \times \frac{7}{11} \times \frac{2}{2} = 1 \times \frac{7}{11} \times 1 = \frac{7}{11} \times 1 = \frac{7}{11}$$

Notice that since multiplication of rational numbers is commutative and associative, we don't need the factors to be written in any particular order and when having more than two numbers in a multiplication we can multiply left to right or right to left.

Or

$$\frac{84}{132} = \frac{42 \times 2}{66 \times 2} = \frac{21 \times 2 \times 2}{33 \times 2 \times 2} = \frac{7 \times 3 \times 2 \times 2}{11 \times 3 \times 2 \times 2} = \frac{7}{11} \times \frac{3}{3} \times \frac{2}{2} \times \frac{2}{2} = \frac{7}{11} \times 1 \times 1 \times 1 = \frac{7}{11}$$

3. $\dfrac{63}{8}$

This fraction can not be simplified because 63 and 8 do not have any common factors.

Exercises

Simplify the fractions. Show all the steps of the simplification.

1. $\dfrac{9}{21}$
2. $\dfrac{81}{54}$
3. $\dfrac{0}{5}$
4. $\dfrac{88}{4}$

5. $\frac{1230}{1230}$

6. $\frac{9240}{220}$

7. $\frac{5 \times 45}{45 \times 3}$

8. $\frac{7 \times 100}{14 \times 10}$

9. $\frac{28}{5 \times 28}$

10. $\frac{9 \times 10 \times 25}{25 \times 9 \times 10}$

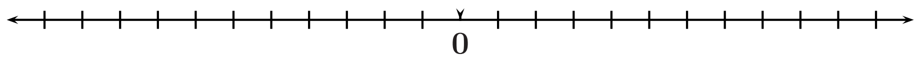
LESSON 15

REVIEW

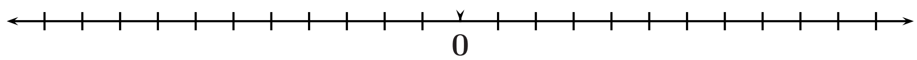
Homework/Class Work/Quiz 22

1. Locate the number on a number line.

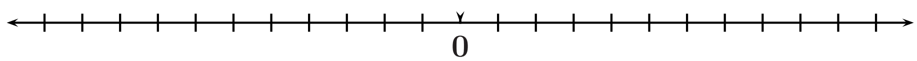
   (a) $\frac{7}{4}$

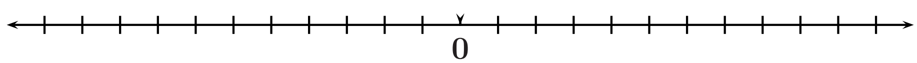
   (b) $-\frac{3}{4}$

   (c) $\frac{2}{3}$

   (d) $-\frac{7}{3}$

2. Fill in the blank using either '=', '<', or '>' as appropriate.

   (a) $0 \qquad -\frac{2}{3}$

   (b) $\frac{-3}{10} \qquad \frac{3}{-10}$

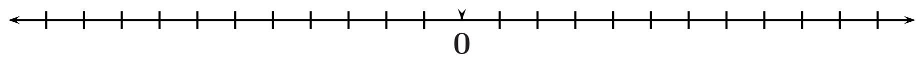
   (c) $\frac{5}{7} \qquad 0$

   (d) $\frac{-4}{5} \qquad \frac{-7}{-8}$

blank

Homework/Class Work/Quiz 23

1. Locate $\frac{4}{3}$ and $-\frac{5}{3}$ on a number line.



2. Fill in the blank using either ’=’, ’<’, or ’>’ as appropriate.

   (a) $0 \qquad \frac{0}{-8}$

   (b) $\frac{7}{8} \qquad 1$

   (c) $\frac{-7}{-3} \qquad \frac{7}{3}$

   (d) $\frac{1000}{-1000} \qquad \frac{1}{1000}$

3. Compute. If the resulting fraction is an integer, indicate it.

   (a) $\frac{5}{4} \times \frac{1}{3}$

   (b) $7 \times \frac{7}{8}$

   (c) $\frac{1}{3} \times \frac{1}{3}$

   (d) $\frac{5}{6} \times 5$

blank

Homework/Class Work/Quiz 24

1. Compute. If the resulting fraction is an integer, indicate it.
   (a) $\frac{2}{5} \times \frac{7}{3}$

   (b) $1 \times \frac{8}{7}$

   (c) $\frac{0}{3} \times \frac{1}{10}$

   (d) $\frac{3}{8} \times 7$

2. Find two fractions equivalent to $\frac{2}{7}$

3. Find a fraction equivalent to $\frac{1}{10}$ with denominator 800.

4. Determine which of the following pair of fractions are equivalent.
   (a) $\frac{3}{13}; \frac{7}{17}$

   (b) $\frac{1}{5}; \frac{5}{25}$

   (c) $\frac{0}{7}; \frac{0}{8}$

   (d) $\frac{1}{15}; \frac{1}{30}$

blank

Homework/Class Work/Quiz 25

1. Find a fraction equivalent to $\frac{2}{15}$ with denominator 60.

2. Determine which of the following pair of fractions are equivalent.

   (a) $\frac{54}{2}; \frac{81}{3}$

   (b) $\frac{6}{7}; \frac{12}{21}$

   (c) $\frac{8}{8}; \frac{1000}{1000}$

   (d) $\frac{3}{8}; \frac{12}{32}$

3. Simplify the fractions. Show all the steps of the simplification.

   (a) $\frac{45}{5}$

   (b) $\frac{31}{93}$

blank

Homework/Class Work/Quiz 26

Simplify the fractions. Show all the steps of the simplification.

1. $\dfrac{90}{12}$

2. $\dfrac{0}{7}$

3. $\dfrac{232}{4}$

4. $\dfrac{820}{220}$

5. $\dfrac{9 \times 37}{37 \times 6}$

6. $\dfrac{14 \times 22}{33 \times 7}$

7. $\dfrac{9 \times 6}{36}$

8. $\dfrac{8 \times 17 \times 11}{11 \times 16 \times 17}$

9. $\dfrac{19 \times 36 \times 42}{21 \times 5 \times 12}$

blank

LESSON 16

Multiplication of Fractions and Simplification.

We explained that simplifying a fraction is possible when we can express, both the numerator and denominator, as a product with common factors. On the other hand, multiplying fractions will result in having the numerators and denominators multiplied. This means that the numerators of the fractions will be factors of the resulting fraction, and the denominators will be factors of the denominator of the resulting fraction. Therefore, we will be able to simplify before actually performing the multiplication.

16.1) Examples.

1.
$$\frac{7}{3} \times \frac{3}{8} = \frac{7 \times 3}{3 \times 8} = \frac{7}{8} \times \frac{3}{3} = \frac{7}{8} \times 1 = \frac{7}{8}$$

We can think that we "cancelled" the 3 in the denominator of the first fraction with the 3 in the numerator of the second fraction. We understand that $3 = 3 \times 1$, so when cancelling, what we have left at the denominator of the first fraction and numerator of the second, is 1.

$$\frac{7}{3} \times \frac{3}{8} = \frac{7}{1 \times 3} \times \frac{3 \times 1}{8} = \frac{7 \times 3 \times 1}{1 \times 3 \times 8} = \frac{7}{1} \times \frac{1}{8} \times \frac{3}{3} = \frac{7}{1} \times \frac{1}{8} \times 1 = \frac{7}{1} \times \frac{1}{8} = \frac{7}{8}$$

Most of the time we go straight from the multiplication to what is left after cancelling the common factor. In this case, it would be:

$$\frac{7}{3} \times \frac{3}{8} = \frac{7}{1} \times \frac{1}{8} = \frac{7}{8}$$

2.
$$\frac{7}{3} \times \frac{9}{8} = \frac{7}{1 \times 3} \times \frac{3 \times 3}{8} = \frac{7}{1} \times \frac{3}{8} = \frac{21}{8}$$

As a result of the associative property, we can multiply more than two fractions, grouping in any way we want. We will end up multiplying all numerators and all denominators. Also, keep in mind what it means to have a number followed by an expression in parentheses, or the other way around. It means that the operation in between is multiplication.

If we include negative fractions in the multiplication, we should follow the rules for multiplication of signs and again multiply numerators and denominators.

Another important point is that when multiplying fractions, it is convenient to try to simplify before multiplying. **Multiplication is the only operation with fractions that allows this procedure.**

16.2) Examples.

Compute and simplify. Show all the factors that can be cancelled.

1. $\frac{6}{5} \times \frac{9}{4} \times \frac{15}{18} = \frac{3 \times 2}{5 \times 1} \times \frac{9 \times 1}{2 \times 2} \times \frac{3 \times 5}{2 \times 9} = \frac{3}{1} \times \frac{1}{2} \times \frac{3}{2} = \frac{3 \times 1 \times 3}{1 \times 2 \times 2} = \frac{9}{4}$
2. $\frac{4}{5}\left(-\frac{10}{3}\right) = -\left(\frac{4}{5} \times \frac{10}{3}\right) = -\left(\frac{4}{1 \times 5} \times \frac{5 \times 2}{3}\right) = -\left(\frac{4}{1} \times \frac{2}{3}\right) = -\frac{8}{3}$
3. $-\frac{7}{4} \times \left(-\frac{1}{14}\right) = \frac{7 \times 1}{4} \times \frac{1}{2 \times 7} = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$
4. $-\frac{9}{2}\left(-\frac{3}{7}\right)\left(-\frac{14}{30}\right) = -\left(\frac{9}{1 \times 2} \times \frac{3 \times 1}{7 \times 1} \times \frac{7 \times 2}{3 \times 10}\right) = -\left(\frac{9}{1} \times \frac{1}{1} \times \frac{1}{10}\right) = -\frac{9}{10}$
5. $-\frac{121}{90}\left(\frac{45}{10}\right)\left(-\frac{2}{121}\right) = \frac{121}{45 \times 2 \times 1} \times \frac{45 \times 1}{10} \times \frac{2 \times 1}{121 \times 1} = \frac{1}{1} \times \frac{1}{10} \times \frac{1}{1} = \frac{1}{10}$
6. $3\left(-\frac{4}{9}\right)\frac{7}{2} = -\left(\frac{3}{1} \times \frac{4}{9} \times \frac{7}{2}\right) = -\left(\frac{3 \times 1}{1} \times \frac{2 \times 2}{3 \times 3} \times \frac{7}{2 \times 1}\right) = -\left(\frac{1}{1} \times \frac{2}{3} \times \frac{7}{1}\right) = -\frac{14}{3}$
7. $\frac{1}{2}(-5) = -\left(\frac{1}{2} \times \frac{5}{1}\right) = -\frac{5}{2}$

Exponential Notation.

Let's talk again about exponential notation. This time we will allow the base to be a fraction. In that case, we need to use a parentheses for the base and the exponent should be written outside the parentheses. The reason for this is that, if the parentheses is not there, it would look like if the exponent is only for the numerator and not for the whole fraction. Also, if a negative sign is inside the parentheses (and in front of the fraction), then the negative sign is part of the base. If we see a negative sign outside the parentheses that is used for the base, then it would be "external" to the exponential.

16.3) Examples.

Compute.

1. $\left(\frac{1}{2}\right)^3 = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

2. $\left(\frac{3}{4}\right)^2 = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$

3. $\left(-\frac{2}{3}\right)^4 = \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) = \frac{16}{81}$

4. $-\left(\frac{1}{3}\right)^4 = -\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{1}{81}$

5. $\left(-\frac{4}{3}\right)^3 = \left(-\frac{4}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{4}{3}\right) = -\frac{64}{27}$

6. $\left(-\frac{2}{5}\right)^2 = \left(-\frac{2}{5}\right) \times \left(-\frac{2}{5}\right) = \frac{2}{5} \times \frac{2}{5} = \frac{2 \times 2}{5 \times 5} = \frac{4}{25}$

7. $\left(-\frac{1}{3}\right)^4 = \left(-\frac{1}{3}\right) \times \left(-\frac{1}{3}\right) \times \left(-\frac{1}{3}\right) \times \left(-\frac{1}{3}\right) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1 \times 1 \times 1 \times 1}{3 \times 3 \times 3 \times 3} = \frac{1}{81}$

8. $\left(-\frac{2}{3}\right)^3 = \left(-\frac{2}{3}\right) \times \left(-\frac{2}{3}\right) \times \left(-\frac{2}{3}\right) = -\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = -\frac{2 \times 2 \times 2}{3 \times 3 \times 3} = -\frac{8}{27}$

9. $-\left(\frac{1}{2}\right)^2 = -\left(\frac{1}{2} \times \frac{1}{2}\right) = -\frac{1 \times 1}{2 \times 2} = -\frac{1}{4}$

10. $-\left(-\frac{1}{2}\right)^4 = -\left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) = -\frac{1}{16}$

11. $-\left(-\frac{1}{2}\right)^3 = -\left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) = \frac{1}{8}$

Exercises

Compute and simplify. Show all the factors that can be cancelled.

1. $\frac{10}{3} \times \frac{9}{2}$

2. $\frac{16}{5} \times \frac{9}{7} \times \frac{70}{12}$

3. $\frac{100}{9}\left(-\frac{6}{20}\right)$

4. $-\frac{2}{7}\left(-\frac{21}{10}\right)\left(-\frac{15}{4}\right)$

5. $-\frac{36}{25} \times \frac{90}{7} \times \left(-\frac{35}{120}\right)$

6. $\frac{5}{8}(-16)$

7. $9 \times \left(-\frac{4}{15}\right) \times \frac{1}{2}$

8. $-\left(-\frac{1}{3}\right)^3$

9. $\left(\frac{2}{5}\right)^2$

10. $-\left(-\frac{1}{2}\right)^4$

11. $\left(\frac{3}{5}\right)^2$

12. $-\left(\frac{1}{5}\right)^2$

13. $\left(-\frac{1}{3}\right)^2$

14. $-\left(-\frac{1}{9}\right)^2$

15. $\left(\frac{2}{3}\right)^3$

16. $\left(-\frac{3}{4}\right)^3$

17. $-\left(-\frac{2}{3}\right)^3$

18. $\left(-\frac{1}{2}\right)^4$

LESSON 17

In order to figure out what would be the "right" rule for division of fractions, let's notice some facts.

1. Division of integers can be written as **multiplication of an integer and a fraction**. For example

   (a)
   $$10 \div 2 = \frac{10}{2} = 10 \times \frac{1}{2}$$

   (b)
   $$3 \div 5 = \frac{3}{5} = 3 \times \frac{1}{5}$$

2. Division of integers can be written as **multiplication of fractions**. For example

   (a)
   $$10 \div 2 = \frac{10}{2} = \frac{10}{1} \times \frac{1}{2}$$

   (b)
   $$3 \div 5 = \frac{3}{5} = \frac{3}{1} \times \frac{1}{5}$$

3. Division of integers can be written as **division of fractions**. For example

   (a)
   $$10 \div 2 = \frac{10}{1} \div \frac{2}{1}$$

   The previous examples showed that
   $$10 \div 2 = \frac{10}{1} \times \frac{1}{2}$$

   And, since **equals can be substituted for equals**, we can write
   $$\frac{10}{1} \div \frac{2}{1} = \frac{10}{1} \times \frac{1}{2}$$

   (b) Similarly we can also write
   $$\frac{3}{1} \div \frac{5}{1} = \frac{3}{1} \times \frac{1}{5}$$

4. If a and b are integers different from zero then

$$\frac{a}{b} \times \frac{b}{a} = \frac{a \times b}{b \times a} = 1$$

When the product of two numbers is equal to 1, we say that the numbers are **reciprocals**. Therefore,

$$\textbf{the reciprocal of } \frac{a}{b} \textbf{ is } \frac{b}{a}\textbf{, and the reciprocal of } \frac{b}{a} \textbf{ is } \frac{a}{b}.$$

Also notice that

(a) The reciprocal of a positive number is also positive.

(b) The reciprocal of a negative number is also negative.

(c) Zero is the only number that does not have a reciprocal.

17.1)Examples

1. Write the division of integers as a fraction, then as product of an integer and a fraction, and then as product of fractions.

   (a) $81 \div 3 = \frac{81}{3} = 81 \times \frac{1}{3} = \frac{81}{1} \times \frac{1}{3}$

   (b) $-7 \div 5 = -\frac{7}{5} = -7 \times \frac{1}{5} = -\frac{7}{1} \times \frac{1}{5}$

   (c) $10 \div (-3) = -\frac{10}{3} = -10 \times \frac{1}{3} = -\frac{10}{1} \times \frac{1}{3}$

2. Determine the reciprocal of the number.

   (a) 5

   Since $5 = \frac{5}{1}$, then its reciprocal is $\frac{1}{5}$

   (b) $-\frac{3}{5}$

   The reciprocal is $-\frac{5}{3}$

   (c) 1

   The reciprocal is 1

(d) -1

The reciprocal is -1

(e) $\frac{1}{9}$

The reciprocal is 9

Everything we just discussed leads us to the following rule for division of fractions:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

In words, to divide two fractions we have to multiply the first one times the reciprocal of the second one.

Once again, fraction notation can be used to indicate division of fractions:

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}}$$

17.2) Examples.

Divide and simplify. When simplifying show all the factors that are cancelled.

1. $\frac{2}{9} \div \frac{4}{3} = \frac{2}{9} \times \frac{3}{4} = \frac{2 \times 1}{3 \times 3} \times \frac{3 \times 1}{2 \times 2} = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

2. $\frac{10}{3} \div 5 = \frac{10}{3} \div \frac{5}{1} = \frac{10}{3} \times \frac{1}{5} = \frac{2 \times 5}{3} \times \frac{1}{5 \times 1} = \frac{2}{3} \times \frac{1}{1} = \frac{2}{3}$

3. $9 \div \frac{6}{5} = \frac{9}{1} \div \frac{6}{5} = \frac{9}{1} \times \frac{5}{6} = \frac{3 \times 3}{1} \times \frac{5}{2 \times 3} = \frac{3}{1} \times \frac{5}{2} = \frac{15}{2}$

4. $\frac{-\frac{3}{10}}{\frac{6}{25}} = -\frac{3}{10} \div \frac{6}{25} = -\frac{3}{10} \times \frac{25}{6} = -\frac{3 \times 1}{2 \times 5} \times \frac{5 \times 5}{2 \times 3} = -\frac{1}{2} \times \frac{5}{2} = -\frac{5}{4}$

5. $\frac{11}{5} \div \left(-\frac{3}{10}\right) = \frac{11}{5} \times \left(-\frac{10}{3}\right) = -\frac{11}{5 \times 1} \times \frac{5 \times 2}{3} = -\frac{11}{1} \times \frac{2}{3} = -\frac{22}{3}$

6. $\frac{-7}{-\frac{7}{9}} = \frac{7}{1} \div \frac{7}{9} = \frac{7}{1} \times \frac{9}{7} = \frac{7 \times 1}{1} \times \frac{9}{7 \times 1} = \frac{1}{1} \times \frac{9}{1} = \frac{9}{1} = 9$

Exercises

Divide and simplify. When simplifying show all the factors that are cancelled.

1. $\dfrac{1}{5} \div \dfrac{5}{4}$

2. $\dfrac{-\dfrac{8}{25}}{\dfrac{2}{5}}$

3. $-\dfrac{16}{9} \div \left(-\dfrac{3}{2}\right)$

4. $\dfrac{-\dfrac{49}{64}}{-\dfrac{7}{8}}$

5. $-\dfrac{36}{25} \div 12$

6. $\dfrac{3}{-\dfrac{27}{2}}$

7. $-\dfrac{1}{3} \div (-3)$

8. $\dfrac{\frac{4}{9}}{8}$

LESSON 18

Comparing Fractions.

If we consider two fractions that are not equivalent, then, one of them is less than the other one. How do we know which one? Let's first mention some situations that are predictable.

1. Any negative fraction is less than zero. For example
$$-\frac{5}{2} < 0$$
2. Zero is less than any positive fraction. For example
$$0 < \frac{1}{1000}$$
3. Any negative fraction is less than any positive fraction. For example
$$-\frac{101}{5} < \frac{1}{2}$$
4. A positive fraction with the numerator less than the denominator is less than 1. For example
$$\frac{99}{100} < 1$$
5. A positive fraction with the numerator greater than the denominator is greater than 1. For example
$$\frac{1001}{1000} > 1$$
6. Any positive fraction less than one (numerator less than denominator) is less than any fraction greater than 1 (numerator greater than denominator). For example
$$\frac{235}{236} < \frac{8}{7}$$

If we were dealing with, for example, two positive fractions that are both less than 1 or both greater than 1, then we need to look at the situation more carefully.

Based on the part-whole interpretation, a positive fraction is related to equal pieces of a "unit". The denominator determines what "kind of pieces" we are using. Most of the time, in real life, we compare only objects of the same kind. We compare two chairs, maybe in terms of which one is more comfortable, but there is no point in comparing say a chair and an apple!.

We can think that fractions are of the "same kind" when they have the same denominator. So, in order to understand how to compare two fractions that are not equivalent, we would like them to have the same denominator.

Consider, for example $\frac{2}{5}$ and $\frac{3}{5}$, we can say immediately that $\frac{2}{5} < \frac{3}{5}$.

However, it would take some thinking to compare $\frac{3}{5}$ and $\frac{4}{7}$ (notice that both fractions are less than 1). We would like them to be the "same kind" of fractions, meaning, having the same denominator.

Remember that we can change the denominator of a fraction without changing the number represented by the fraction. We can do that by multiplying the numerator and the denominator by the same integer different from zero. In this case we have denominators 5 and 7, and we want to change them to the same number (we want a common denominator). One possibility is to change them to be 35 ($35 = 5 \times 7$), in the following way:

$$\frac{3}{5} = \frac{3 \times 7}{5 \times 7} = \frac{21}{35}$$

also

$$\frac{4}{7} = \frac{4 \times 5}{7 \times 5} = \frac{20}{35}$$

Since $\frac{21}{35} > \frac{20}{35}$, then we can conclude that $\frac{3}{5} > \frac{4}{7}$.

Let's now compare the fractions $\frac{7}{12}$ and $\frac{11}{18}$.

First, since they are both positive fractions less than one, they require some thinking. We are going to make them have the same denominator. In other words, we need a common denominator. One possibility for a common denominator is 12×18. This number would work, but it is fairly large, so we might prefer to work with a smaller number. Keep in mind that any common denominator would result of multiplying each denominator by an integer greater or equal to one. Therefore a common denominator can not be less than the largest denominator.

WARNING: In this example we should not think that 2 could be a common denominator. In fact it can not be any number less than 18.

To try to find a smaller common denominator, we should start considering the largest denominator. For this particular example it is 18. If it were possible to multiply the other denominator (12) by an integer and get 18 as the result, that would mean that 18 is a possible common denominator. In fact it would be the smallest possibility. However, since it is not possible, then we need to consider a number greater than 18. If we notice that

$$36 = 12 \times 3 \quad \text{and} \quad 36 = 18 \times 2$$

we can conclude that 36 is a possible common denominator.

Now, let's compare the fractions:

$$\frac{7}{12} = \frac{7 \times 3}{12 \times 3} = \frac{21}{36}$$

$$\frac{11}{18} = \frac{11 \times 2}{18 \times 2} = \frac{22}{36}$$

Since $\frac{21}{36} < \frac{22}{36}$ then $\frac{7}{12} < \frac{11}{18}$.

18.1)Examples.

1. For each pair of fractions, find equivalent fractions having the same denominator.

   (a) $\frac{7}{6}; \frac{5}{2}$

   We should start by considering the largest denominator: 6. Since the other denominator is 2 and $2 \times 3 = 6$, then 6 is not only a possible common denominator, but in fact it is the smallest possibility.

   $$\frac{5}{2} = \frac{5 \times 3}{2 \times 3} = \frac{15}{6}$$

   $\frac{7}{6}$ and $\frac{15}{6}$ have the same denominator.

   (b) $\frac{9}{7}$; $\frac{11}{6}$

   The largest denominator, in this case 7, is not a possible common denominator. However, since the other denominator is 6 and $6 \times 7 = 42$, then we can use 42.

   $$\frac{9}{7} = \frac{9 \times 6}{7 \times 6} = \frac{54}{42}$$

$$\frac{11}{6} = \frac{11 \times 7}{6 \times 7} = \frac{77}{42}$$

$\frac{54}{42}$ and $\frac{77}{42}$ have the same denominator.

(c) $\frac{7}{12}$; $\frac{9}{16}$

The largest denominator is 16. We can not use it as a common denominator. We could use the result of 12×18. However, if we realize that

$$16 \times 3 = 48 \quad \text{and} \quad 12 \times 4 = 48$$

then we can conclude that 48 can be used as a common denominator.

$$\frac{7}{12} = \frac{7 \times 4}{12 \times 4} = \frac{28}{48}$$
$$\frac{9}{16} = \frac{9 \times 3}{16 \times 3} = \frac{27}{48}$$

$\frac{28}{48}$ and $\frac{27}{48}$ have the same denominator.

(d) $\frac{43}{50}$; $\frac{97}{105}$

The largest denominator is 105. It can not be used as a common denominator. Let's use the result of 50×105

$50 \times 105 = 5250$

$$\frac{43}{50} = \frac{43 \times 105}{50 \times 105} = \frac{4515}{5250}$$

$$\frac{97}{105} = \frac{97 \times 50}{105 \times 50} = \frac{4850}{5250}$$

$\frac{4515}{5250}$ and $\frac{4850}{5250}$ have the same denominator.

2. Fill in the blank using either '$<$' or '$>$' as appropriate. Explain your answer.

(a) $\frac{9}{10} \qquad \frac{13}{12}$

$\frac{9}{10} < 1$ and $1 < \frac{13}{12}$

Then

$$\frac{9}{10} < \frac{13}{12}$$

(b) $-\frac{1}{10} \qquad \frac{1}{1000}$

$-\frac{1}{10} < 0$ and $0 < \frac{1}{1000}$

Then

$$-\frac{1}{10} < \frac{1}{1000}$$

(c) $\frac{9}{10} \qquad \frac{91}{100}$

$\frac{9}{10} = \frac{9 \times 10}{10 \times 10} = \frac{90}{100}$

Then

$$\frac{9}{10} < \frac{91}{100}$$

(d) $\frac{7}{12} \qquad \frac{9}{16}$

$\frac{7}{12} = \frac{7 \times 4}{12 \times 4} = \frac{28}{48}$

$\frac{9}{16} = \frac{9 \times 3}{16 \times 3} = \frac{27}{48}$

Then

$$\frac{7}{12} > \frac{9}{16}$$

(e) $\dfrac{43}{50} \qquad \dfrac{89}{105}$

$\dfrac{43}{50} = \dfrac{43 \times 105}{50 \times 105} = \dfrac{4515}{5250}$

$\dfrac{89}{105} = \dfrac{89 \times 50}{105 \times 50} = \dfrac{4450}{5250}$

Then

$$\frac{43}{50} > \frac{89}{105}$$

Exercises

1. For each pair of fractions find equivalent fractions having the same denominator.

   (a) $\dfrac{2}{3}; \dfrac{5}{9}$

   (b) $\dfrac{7}{12}; \dfrac{3}{5}$

   (c) $\dfrac{7}{24}; \dfrac{11}{30}$

   (d) $\dfrac{33}{20}; \dfrac{65}{56}$

2. Fill in the blank using either '<' or '>' as appropriate. Explain your answer.

   (a) $\dfrac{235}{233} \qquad \dfrac{498}{501}$

   (b) $-\dfrac{143}{4} \qquad 0$

   (c) $-1 \qquad -\dfrac{7}{8}$

   (d) $-1230 \qquad \dfrac{1}{10}$

   (e) $\dfrac{7}{12} \qquad \dfrac{3}{5}$

   (f) $\dfrac{9}{24} \qquad \dfrac{11}{30}$

LESSON 19

REVIEW

Homework/Class Work/Quiz 27

Compute and simplify. When simplifying show all the factors that can be cancelled.

1. $\frac{3}{8} \times \frac{5}{7}$

2. $\frac{14}{5} \times \frac{10}{7}$

3. $\frac{11}{3} \times 5$

4. $\frac{10}{9}\left(-\frac{12}{25}\right)$

5. $8 \times \frac{7}{16}$

6. $\left(-\frac{25}{18}\right)\left(-\frac{27}{10}\right)$

7. $\left(\frac{3}{2}\right)^3$

8. $\frac{7}{3}(-8)$

blank

Homework/Class Work/Quiz 28

Compute and simplify. When simplifying show all the factors that can be cancelled.

1. $\frac{16}{9} \times \frac{1}{5} \times \frac{2}{5}$

2. $\frac{32}{14} \times \frac{2}{25} \times \frac{35}{16}$

3. $-\frac{81}{8} \times \frac{10}{9} \times \left(-\frac{10}{3}\right)$

4. $-\frac{1}{3}\left(-\frac{2}{7}\right)\left(-\frac{5}{9}\right)$

5. $-\left(\frac{2}{3}\right)^4$

6. $\left(-\frac{2}{3}\right)^3$

7. $-\left(-\frac{1}{4}\right)^2$

8. $-\left(-\frac{1}{4}\right)^3$

blank

Homework/Class Work/Quiz 29

Compute and simplify. When simplifying show all the factors that can be cancelled.

1. $\frac{3}{14} \div \frac{2}{3}$

2. $\frac{12}{5} \div \frac{9}{10}$

3. $\frac{15}{2} \div 3$

4. $7 \div \frac{21}{4}$

5. $\frac{\frac{21}{8}}{\frac{9}{4}}$

6. $\frac{\frac{12}{5}}{16}$

7. $\frac{100}{\frac{20}{3}}$

blank

Homework/Class Work/Quiz 30

Compute and simplify. When simplifying show all the factors that can be cancelled.

1. $\frac{9}{7} \div \left(-\frac{7}{3}\right)$

2. $-\frac{3}{4} \div \frac{9}{10}$

3. $-\frac{14}{15} \div \left(-\frac{21}{10}\right)$

4. $\frac{7}{9} \div (-9)$

5. $(-6) \div \left(-\frac{9}{7}\right)$

6. $\frac{-\frac{36}{25}}{\frac{6}{100}}$

7. $\frac{\frac{11}{13}}{-\frac{22}{39}}$

8. $\frac{-\frac{8}{7}}{-14}$

blank

Homework/Class Work/Quiz 31

Compute and simplify. When simplifying show all the factors that can be cancelled.

1. $\frac{18}{35} \times \left(-\frac{14}{9}\right)$

2. $\frac{18}{35} \div \left(-\frac{14}{9}\right)$

3. $8\left(\frac{10}{12}\right)$

4. $8 \div \frac{10}{12}$

5. $\left(-\frac{1}{5}\right)^2$

6. $-\left(-\frac{2}{5}\right)^2$

7. $-\left(-\frac{1}{6}\right)^3$

8. $\frac{-\frac{9}{10}}{2}$

9. $\frac{-\frac{5}{8}}{-\frac{25}{12}}$

blank

Homework/Class Work/Quiz 32

1. For each pair of fractions find equivalent fractions having the same denominator.

   (a) $\frac{7}{27}\ ;\ \frac{5}{3}$

   (b) $\frac{2}{7}\ ;\ \frac{4}{5}$

   (c) $\frac{21}{100}\ ;\ \frac{12}{150}$

2. Fill in the blank using either ’$<$’ or ’$>$’ as appropriate. Explain your answer.

   (a) $\frac{532}{535} \qquad \frac{451}{448}$

   (b) $\frac{9}{8} \qquad \frac{25}{26}$

   (c) $\frac{0}{25} \qquad \frac{38}{-7}$

   (d) $\frac{13}{16} \qquad \frac{19}{24}$

blank

Homework/Class Work/Quiz 33

1. For each pair of fractions find equivalent fractions having the same denominator.

   (a) $\frac{8}{21}$; $\frac{7}{6}$

   (b) $\frac{15}{39}$; $\frac{25}{26}$

   (c) $\frac{5}{80}$; $\frac{3}{40}$

2. Fill in the blank using either '<' or '>' as appropriate. Explain your answer.

   (a) $\frac{-19}{-18} \qquad \frac{-21}{-22}$

   (b) $\frac{10}{-10} \qquad -\frac{10}{9}$

   (c) $\frac{5}{3} \qquad -\frac{10}{6}$

   (d) $\frac{4}{21} \qquad \frac{3}{14}$

blank

LESSON 20

Addition and Subtraction of Fractions.

In our day to day life we are constantly adding and subtracting. If we have two pencils, and then we buy three more pencils, then we say that we have five pencils ($5 = 2 + 3$). If we have $100 and we spend $35 in groceries, then we say that we have $ 65 left ($65 = 100 - 35$).

However, if we have two pencils and then we buy three erasers, we will still say that we have two pencils and three erasers!. In this case, we don't add because we are not talking about the same kind of object.

In general we add or subtract quantities related to the same kind of object. That same thing happens when we want to add or subtract fractions. We need them to be the same "kind" of fractions, In other words we need them to have the same denominator. If that is the case, the denominator indicates the "kind of object" and the numerator says "how many " of them. The resulting fraction should be the same "kind of fraction", meaning a fraction with the same denominator as the original ones. The numerator of the resulting fraction will be the result of adding or subtracting the original numerators.

20.1) Examples.

1. $$\frac{5}{3} + \frac{4}{3} = \frac{\overbrace{5+4}^{\text{add numerators}}}{\underbrace{3}_{\text{keep denominator}}} = \frac{9}{3} = 3$$

2. $$\frac{10}{14} - \frac{3}{14} = \frac{\overbrace{10-3}^{\text{subtract numerators}}}{\underbrace{14}_{\text{keep denominator}}} = \frac{7}{14} = \frac{7 \times 1}{7 \times 2} = \frac{1}{2}$$

These examples show the general rule, that is:

when adding or subtracting fractions with the same denominator, we keep the denominator and add or subtract the numerators.

If the fractions do not have the same denominator, we need to "make" them have a common denominator, in other words, we need to replace the fractions in the sum or difference by equivalent fractions with a **common denominator**.

20.2) Examples.

1. $\dfrac{7}{6}-\dfrac{13}{12}=\dfrac{7\times 2}{6\times 2}-\dfrac{13}{12}=\dfrac{14}{12}-\dfrac{13}{12}=\dfrac{14-13}{12}=\dfrac{1}{12}$

2. $\dfrac{11}{2}+\dfrac{1}{6}=\dfrac{11\times 3}{2\times 3}+\dfrac{1}{6}=\dfrac{33}{6}+\dfrac{1}{6}=\dfrac{33+1}{6}=\dfrac{34}{6}=\dfrac{17\times 2}{3\times 2}=\dfrac{17}{3}$

3. $\dfrac{7}{12}+\dfrac{7}{16}=\dfrac{7\times 4}{12\times 4}+\dfrac{7\times 3}{16\times 3}=\dfrac{28}{48}+\dfrac{21}{48}=\dfrac{28+21}{48}=\dfrac{49}{48}$

4. $\dfrac{7}{12}-\dfrac{5}{18}=\dfrac{7\times 3}{12\times 3}-\dfrac{5\times 2}{18\times 2}=\dfrac{21}{36}-\dfrac{10}{36}=\dfrac{21-10}{36}=\dfrac{11}{36}$

5. $\dfrac{11}{10}+\dfrac{1}{15}=\dfrac{11\times 3}{10\times 3}+\dfrac{1\times 2}{15\times 2}=\dfrac{33}{30}+\dfrac{2}{30}=\dfrac{33+2}{30}=\dfrac{35}{30}=\dfrac{7\times 5}{6\times 5}=\dfrac{7}{6}$

6. $\dfrac{7}{5}+2=\dfrac{7}{5}+\dfrac{2}{1}=\dfrac{7}{5}+\dfrac{2\times 5}{1\times 5}=\dfrac{7}{5}+\dfrac{10}{5}=\dfrac{7+10}{5}=\dfrac{17}{5}$

7. $9-\dfrac{5}{3}=\dfrac{9}{1}-\dfrac{5}{3}=\dfrac{9\times 3}{1\times 3}-\dfrac{5}{3}=\dfrac{27}{3}-\dfrac{5}{3}=\dfrac{27-5}{3}=\dfrac{22}{3}$

Addition of fractions is also commutative and associative. When adding more than two fractions, we can group the fractions in any way. In particular we can add left to right or right to left. However, since we know that when we add two fractions we need them to have the same denominator, when adding more than two, it is convenient to start by making sure that all fractions have the same denominator. It can be shown than we can now add by keeping the common denominator and adding all numerators.

20.3)Examples.

1. $\dfrac{2}{5}+\dfrac{1}{10}+\dfrac{1}{6}=\dfrac{2\times 6}{5\times 6}+\dfrac{1\times 3}{10\times 3}+\dfrac{1\times 5}{6\times 5}=\dfrac{12}{30}+\dfrac{3}{30}+\dfrac{5}{30}=\dfrac{12+3+5}{30}=\dfrac{20}{30}=\dfrac{2\times 10}{3\times 10}=\dfrac{2}{3}$

2. $\dfrac{3}{2}+\dfrac{1}{4}+1=\dfrac{3}{2}+\dfrac{1}{4}+\dfrac{1}{1}=\dfrac{3\times 2}{2\times 2}+\dfrac{1}{4}+\dfrac{1\times 4}{1\times 4}=\dfrac{6}{4}+\dfrac{1}{4}+\dfrac{4}{4}=\dfrac{6+1+4}{4}=\dfrac{11}{4}$

Now let's include negative fractions in additions or subtractions. We will take advantage of the fact that

$$-\frac{a}{b} = \frac{-a}{b}$$

20.4) Examples.

1. $-\frac{3}{5} + \frac{1}{3} = -\frac{3\times 3}{5\times 3} + \frac{1\times 5}{3\times 5} = -\frac{9}{15} + \frac{5}{15} = \frac{-9}{15} + \frac{5}{15} = \frac{-9+5}{15} = \frac{-4}{15} = -\frac{4}{15}$

2. $-\frac{7}{5} - \frac{1}{10} = -\frac{7\times 2}{5\times 2} - \frac{1}{10} = -\frac{14}{10} - \frac{1}{10} = \frac{-14}{10} - \frac{1}{10} = \frac{-14-1}{10} = \frac{-15}{10} = -\frac{3\times 5}{2\times 5} = -\frac{3}{2}$

3. $\frac{1}{3} - \frac{1}{2} = \frac{1\times 2}{3\times 2} - \frac{1\times 3}{2\times 3} = \frac{2}{6} - \frac{3}{6} = \frac{2-3}{6} = \frac{-1}{6} = -\frac{1}{6}$

4. $\frac{5}{9} + \left(-\frac{1}{2}\right) = \frac{5\times 2}{9\times 2} + \left(-\frac{1\times 9}{2\times 9}\right) = \frac{10}{18} + \left(-\frac{9}{18}\right) = \frac{10}{18} + \left(\frac{-9}{18}\right) = \frac{10+(-9)}{18} = \frac{1}{18}$

5. $-\frac{12}{7} + \left(-\frac{2}{3}\right) = -\frac{12\times 3}{7\times 3} + \left(-\frac{2\times 7}{3\times 7}\right) = -\frac{36}{21} + \left(-\frac{14}{21}\right) = \frac{-36}{21} + \left(\frac{-14}{21}\right) = \frac{-36+(-14)}{21} =$

$= \frac{-50}{21} = -\frac{50}{21}$

6. $-5 + \frac{6}{5} = \frac{-5}{1} + \frac{6}{5} = \frac{-5\times 5}{1\times 5} + \frac{6}{5} = \frac{-25}{5} + \frac{6}{5} = \frac{-25+6}{5} = \frac{-19}{5} = -\frac{19}{5}$

7. $\frac{2}{3} - \left(-\frac{5}{6}\right) = \frac{2}{3} + \frac{5}{6} = \frac{2\times 2}{3\times 2} + \frac{5}{6} = \frac{4}{6} + \frac{5}{6} = \frac{4+5}{6} = \frac{9}{6} = \frac{3\times 3}{3\times 2} = \frac{3}{2}$

Exercises.

Compute and simplify. Show all the steps of the computation. When simplifying show the factors that are cancelled.

1. $\frac{5}{9} + \frac{4}{9}$

2. $\frac{5}{9} - \frac{2}{9}$

3. $\frac{2}{3} + 4$

4. $\frac{12}{5} - 1$

5. $\frac{9}{5} + \frac{3}{10}$

6. $\frac{7}{9} - \frac{11}{6}$

7. $\frac{1}{2} + \frac{5}{3} + \frac{7}{6}$

8. $\frac{2}{5} + 3 + \frac{5}{6}$

9. $-\frac{5}{9} + \frac{4}{9}$

10. $-\frac{5}{9} - \frac{2}{9}$

11. $\frac{2}{3} - 4$

12. $\frac{12}{5} - 3$

13. $\frac{9}{5} + \left(-\frac{3}{10}\right)$

14. $\frac{7}{9} - \frac{5}{6}$

LESSON 21

Now we will combine addition and subtraction of fractions. One way to proceed is to make all fractions have the same denominator. Then, keep the denominator and perform the additions or subtractions of the numerators from left to right.

21.1)Examples.

1. $-\frac{2}{3}+\frac{4}{9}-\frac{1}{2}=-\frac{2\times 6}{3\times 6}+\frac{4\times 2}{9\times 2}-\frac{1\times 9}{2\times 9}=-\frac{12}{18}+\frac{8}{18}-\frac{9}{18}=\frac{\overbrace{-12+8}^{-4}-9}{18}=\frac{-13}{18}=-\frac{13}{18}$

2. $\frac{5}{4}+\frac{3}{5}-\frac{5}{6}=\frac{5\times 15}{4\times 15}+\frac{3\times 12}{5\times 12}-\frac{5\times 10}{6\times 10}=\frac{75}{60}+\frac{36}{60}-\frac{50}{60}=\frac{\overbrace{75+36}^{111}-50}{60}=\frac{61}{60}$

3. $\frac{1}{8}-\frac{1}{6}-\frac{1}{9}=\frac{1\times 9}{8\times 9}-\frac{1\times 12}{6\times 12}-\frac{1\times 8}{9\times 8}=\frac{9}{72}-\frac{12}{72}-\frac{8}{72}=\frac{\overbrace{9-12}^{-3}-8}{72}=\frac{-11}{72}=-\frac{11}{72}$

4. $\frac{1}{4}-\frac{2}{3}+\frac{3}{2}+3=\frac{1\times 3}{4\times 3}-\frac{2\times 4}{3\times 4}+\frac{3\times 6}{2\times 6}+\frac{3\times 12}{1\times 12}=\frac{3}{12}-\frac{8}{12}+\frac{18}{12}+\frac{36}{12}=\frac{\overbrace{\overbrace{3-8}^{-5}+18}^{13}+36}{12}=\frac{49}{12}$

5. $-2-\frac{1}{6}-\frac{1}{2}=-\frac{2\times 6}{1\times 6}-\frac{1}{6}-\frac{1\times 3}{2\times 3}=-\frac{12}{6}-\frac{1}{6}-\frac{3}{6}=\frac{\overbrace{-12-1}^{-13}-3}{6}=\frac{-16}{6}=-\frac{8\times 2}{3\times 2}=-\frac{8}{3}$

Now let's combine all operations, keeping in mind the order of operations.
For example,

$$\frac{1}{2}+\frac{1}{2}\times\frac{1}{3}=\frac{1}{2}+\frac{1}{6}=\frac{1\times 3}{2\times 3}+\frac{1}{6}=\frac{3}{6}+\frac{1}{6}=\frac{3+1}{6}=\frac{4}{6}=\frac{2\times 2}{2\times 3}=\frac{2}{3}$$

Also notice that when we have a fraction with operations in the numerator and in the denominator, it means the same as having the expression at the numerator in parentheses, divided by the expression in the denominator also in parentheses. That is:

$$\frac{A}{B} = (A) \div (B)$$

Therefore, this indicates that the first thing to do is the operations in the numerator and in the denominator and then work with the resulting fraction.

For example,

$$\frac{\frac{3}{5} - \frac{1}{5}}{1 + \frac{2}{5}} = \frac{\frac{2}{5}}{\frac{7}{5}} = \frac{2}{5} \times \frac{5}{7} = \frac{2}{7}$$

However, in the case where the operations at the numerator and at the denominator are ONLY multiplications, then we are in the situation that allows us to simplify before performing the multiplications.

For example,

$$\frac{2 \times 5 \times 7}{6 \times 2 \times 14} = \frac{5 \times 7}{6 \times 14} = \frac{5 \times 7}{6 \times 7 \times 2} = \frac{5}{6 \times 2} = \frac{5}{12}$$

21.2)Examples.

1. $\left(\frac{1}{2} + \frac{1}{2}\right) \times \frac{1}{3} = \frac{1+1}{2} \times \frac{1}{3} = \frac{2}{2} \times \frac{1}{3} = 1 \times \frac{1}{3} = \frac{1}{3}$

2. $1 + \left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right) = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} = \frac{1 \times 2}{1 \times 2} - \frac{1}{2} + \frac{1}{2} = \frac{2}{2} - \frac{1}{2} + \frac{1}{2} = \frac{2-1+1}{2} = \frac{2}{2} = 1$

3. $1 - \frac{1}{2} + \left(-\frac{1}{2} + 1\right) = 1 - \frac{1}{2} + \left(-\frac{1}{2} + \frac{2}{2}\right) = 1 - \frac{1}{2} + \left(\frac{-1+2}{2}\right) = 1 - \frac{1}{2} + \frac{1}{2} =$

   $= 1 + \left(-\frac{1}{2}\right) + \frac{1}{2} = 1 + \left(\underbrace{\left(-\frac{1}{2}\right) + \frac{1}{2}}_{0}\right) = 1 + 0 = 1$ Notice that here we used the associative property for addition!!

4. $\frac{2}{5} - \left(\frac{3}{2}\right)^2 = \frac{2}{5} - \frac{9}{4} = \frac{2 \times 4}{5 \times 4} - \frac{9 \times 5}{4 \times 5} = \frac{8}{20} - \frac{45}{20} = \frac{8-45}{20} = \frac{-37}{20} = -\frac{37}{20}$

5. $2+\frac{3}{5}\div\frac{9}{5}\times 5=2+\frac{1}{3}\times 5=2+\frac{5}{3}=\frac{6}{3}+\frac{5}{3}=\frac{6+5}{3}=\frac{11}{3}$

6. $\frac{1}{1+1+2}+\frac{3}{9-6-7}=\frac{1}{4}+\frac{3}{-4}=\frac{1}{4}+\left(-\frac{3}{4}\right)=\frac{1+(-3)}{4}=\frac{-2}{4}=-\frac{2\times 1}{2\times 2}=-\frac{1}{2}$

7. $\frac{125\times 125}{25\times 10\times 25}=\frac{25\times 5\times 25\times 5}{25\times 2\times 5\times 25}=\frac{5}{2}$

8. $\left(\frac{\frac{1}{2}}{\frac{3}{2}}\right)^5\times\left(\frac{3}{2}\right)^1=\left(\frac{1}{2}\times\frac{2}{3}\right)^5\times\left(\frac{3}{2}\right)^1=\left(\frac{1}{3}\right)^5\times\left(\frac{3}{2}\right)^1=\frac{1\times 1\times 1\times 1\times 1}{3\times 3\times 3\times 3\times 3}\times\frac{3}{2}=\frac{1}{3\times 3\times 3\times 3}\times\frac{1}{2}=$

9. $\frac{1-3\left(\frac{1+2}{4-1}\right)}{5(7-6)-\frac{10}{2}}=\frac{1-3\left(\frac{3}{3}\right)}{5\times 1-\frac{10}{2}}=\frac{1-3\times 1}{5-\frac{10}{2}}=\frac{1-3}{5-5}=\frac{-2}{0}$ This is undefined!

Exercises.

Compute and simplify. Show all the steps of the computation and when simplifying show the factors that are cancelled.

1. $\frac{1}{2}-\frac{5}{3}-\left(-\frac{7}{6}\right)$

2. $-2+\frac{5}{7}-\frac{1}{3}$

3. $-\frac{2}{9}-\frac{1}{3}-\frac{3}{2}$

4. $1+\frac{1}{2}-\frac{3}{2}-\frac{5}{2}$

5. $\frac{2}{3}-\frac{3}{5}+\frac{1}{2}$

6. $\frac{1}{3}+2-\frac{1}{2}-\frac{1}{4}$

7. $\frac{1}{3} - \frac{1}{3} \times 3$

8. $\left(\frac{1}{3} - \frac{1}{3}\right) \times 3$

9. $\frac{1}{3}\left(1 + \frac{2}{3}\right) + \frac{1}{3}$

10. $-\frac{2}{5} + \left(-\frac{2}{5}\right) - \left(-\frac{2}{5}\right)$

11. $1 - \left(\frac{1}{3}\right)^2 + \frac{1}{3}$

12. $\left(\frac{1}{2}\right)^2 + \frac{1}{4}$

13. $1 - \frac{2}{7} \div 7 \times 7$

14. $\left(1 - \frac{2}{3}\right) \times 10 - 5$

15. $-\frac{1}{2} - \left(-\frac{1}{2}\right)^3 + \frac{1}{3^2}$

16. $\dfrac{\frac{1}{5} - \frac{5}{3}}{-\left(-\frac{1}{3}\right) - \frac{1}{3}}$

17. $\frac{1}{2} - \frac{1}{2} \times \frac{5}{3}$

18. $\left(\frac{1}{2} - \frac{1}{2}\right) \times \frac{5}{3}$

19. $\dfrac{2}{3}+\dfrac{1}{6}\left(2+\dfrac{2}{3}\right)$

20. $-\dfrac{1}{2}+\left(-\dfrac{1}{2}\right)-\left(-\dfrac{1}{2}\right)$

21. $1+\left(\dfrac{1}{2}\right)^{3}+\dfrac{1}{3}$

22. $\left(-\dfrac{1}{2}\right)^{2}-\dfrac{1}{4}$

23. $3-\dfrac{2}{5}\div 5\times 2$

24. $\left(1-\dfrac{2}{5}\right)\times 10-10$

25. $-\left(-\dfrac{1}{3}\right)^{3}+\dfrac{1}{3^2}-\dfrac{1}{3}$

26. $\dfrac{\frac{1}{3}-\frac{3}{2}}{-\left(-\frac{1}{2}\right)-\frac{1}{2}}$

27. $\dfrac{1200\times 625}{100\times 25\times 50}$

28. $\left(\dfrac{\frac{2}{3}}{\frac{4}{9}}\right)^{4}\times\left(\dfrac{2}{3}\right)^{3}$

29. $\dfrac{1}{1-3+1}+\dfrac{5}{3-4+4}$

30. $\dfrac{2-3\left(\frac{3-2}{1-3}\right)}{2\left(\frac{1}{3}-\frac{1}{2}\right)-\frac{1}{2}}$

31. $1-\left(\dfrac{\frac{1}{2}}{\frac{1}{2}-\frac{1}{4}}\right)\times\dfrac{1}{2}$

32. $\dfrac{1-\left(\frac{1}{2}-\frac{1}{3}\right)+\frac{1}{6}}{(1+1)\left(1+\frac{1}{2}\right)}$

LESSON 22

REVIEW

Homework/Class Work/Quiz 34

Compute and simplify. Show all the steps of the computation and when simplifying show the factors that are cancelled.

1. $\dfrac{7}{6}+\dfrac{8}{6}$

2. $\dfrac{5}{3}-\dfrac{2}{3}$

3. $\dfrac{3}{5}+2$

4. $\dfrac{9}{4}-1$

5. $\dfrac{4}{5}+\dfrac{2}{15}$

6. $\dfrac{10}{7}-\dfrac{5}{21}$

7. $\dfrac{2}{5}+\dfrac{1}{3}+\dfrac{19}{15}$

8. $\dfrac{5}{2}+2+\dfrac{1}{3}$

blank

Homework/Class Work/Quiz 35

Compute and simplify. Show all the steps of the computation and when simplifying show the factors that are cancelled.

1. $5 + \frac{4}{5} + \frac{3}{2}$

2. $\frac{1}{2} + \frac{1}{3} + \frac{1}{5}$

3. $\frac{5}{9} + \frac{4}{3} + \frac{8}{6}$

4. $-\frac{3}{7} + \frac{17}{7}$

5. $-\frac{8}{10} - \frac{4}{10}$

6. $\frac{9}{5} - \frac{11}{5}$

7. $3 - \frac{15}{7}$

8. $\frac{16}{7} - 3$

blank

Homework/Class Work/Quiz 36

Compute and simplify. Show all the steps of the computation and when simplifying show the factors that are cancelled.

1. $\frac{3}{4} - \frac{7}{3} - \left(-\frac{5}{6}\right)$

2. $\frac{5}{3} - 4 - \frac{3}{2}$

3. $-\frac{1}{10} - \frac{1}{5} - \frac{1}{5}$

4. $\frac{5}{3}(-4)\left(-\frac{3}{2}\right)$

5. $\frac{1}{2} + \frac{1}{3} - 3 - \frac{5}{6}$

6. $-\frac{2}{5} + \frac{1}{3} - \frac{2}{15}$

7. $\frac{3}{7} - \left(-\frac{2}{3}\right) - 2$

blank

Homework/Class Work/Quiz 37

Compute and simplify. Show all the steps of the computation and when simplifying show the factors that are cancelled.

1. $\frac{3}{4}\left(-\frac{12}{5}\right)$

2. $\frac{3}{4}-\frac{12}{5}$

3. $\left(-\frac{5}{9}\right)\div\left(-\frac{2}{15}\right)$

4. $\frac{1}{2}-\frac{1}{4}+\frac{5}{2}$

5. $-\left(-\frac{1}{10}\right)^2$

6. $3\left(-\frac{7}{15}\right)$

7. $3-\frac{7}{15}$

8. $\frac{7}{3}+\frac{1}{2}-\frac{1}{9}-\frac{1}{18}$

blank

Homework/Class Work/Quiz 38

Compute and simplify. Show all the steps of the computation and when simplifying show the factors that are cancelled.

1. $\dfrac{2}{3} + \dfrac{2}{3} \times \dfrac{6}{5}$

2. $\left(\dfrac{2}{3} + \dfrac{2}{3}\right) \times \dfrac{6}{5}$

3. $-1 - \dfrac{2}{5} \div \dfrac{3}{10}$

4. $-\dfrac{1}{3} - \left(-\dfrac{1}{3}\right)^2$

5. $\dfrac{2-3}{\frac{1}{2} - \frac{1}{3}}$

6. $-2\left(-\dfrac{1}{3}\right)^3$

7. $-2 - \dfrac{3}{5} \div 5$

blank

Homework/Class Work/Quiz 39

Compute and simplify. Show all the steps of the computation and when simplifying show the factors that are cancelled.

1. $\dfrac{1}{9} - \left(\dfrac{1}{5} - \dfrac{1}{5}\right)$

2. $\dfrac{1}{9} - \dfrac{1}{5} - \dfrac{1}{5}$

3. $\left(\dfrac{\frac{1}{10}}{\frac{1}{5}}\right)^2 \times \left(-\dfrac{8}{7}\right)$

4. $\dfrac{15 - (-5)}{5(-5)}$

5. $\dfrac{1}{5} - \dfrac{2}{3}\left(2 - \dfrac{3}{4}\right)$

6. $\left(\dfrac{1}{5} - \dfrac{2}{3}\right)\left(2 - \dfrac{3}{4}\right)$

7. $\dfrac{25 \times 18}{81 \times 75} - \dfrac{1}{9}$

blank

Homework/Class Work/Quiz 40

Compute and simplify. Show all the steps of the computation and when simplifying show the factors that are cancelled.

1. $-3\left(-\frac{1}{3}\right)^3 + \frac{1}{2} \times \frac{4}{3}$

2. $-2 - \left(\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)$

3. $\dfrac{\frac{1}{3} - \left(-\frac{1}{3}\right)^2}{\frac{1}{3}\left(-\frac{1}{3}\right) + 1}$

4. $\dfrac{4 + 2 \times 5 - 2^3}{1 + 32 \div (-16) \div (-4)}$

5. $-\left(\dfrac{-\frac{2}{3}}{3}\right)^2 \times \left(\dfrac{-3}{\frac{5}{2}}\right)^2$

blank

Homework/Class Work/Quiz 41

Compute and simplify. Show all the steps of the computation and when simplifying show the factors that are cancelled.

1. $\dfrac{2}{9} + \dfrac{10}{9} \div \dfrac{20}{9} - \dfrac{5}{3}$

2. $\left(-\dfrac{1}{5} + \dfrac{2}{10}\right) \times \dfrac{2}{3} - 2$

3. $-2\left(\dfrac{\frac{3}{2}}{\frac{1}{3} - \frac{1}{9}}\right) \times \dfrac{2}{15}$

4. $\dfrac{-2 - \left(\frac{1}{3} - \frac{1}{9}\right) + \frac{1}{9}}{(2+3)(-1+1)}$

5. $\dfrac{-2 - \left(\frac{1}{3} - \frac{1}{9}\right) + \frac{1}{9}}{(2+3)(-1-1)}$

blank

LESSON 23

Mixed Numbers.

We already discussed that a positive fraction with the numerator bigger than the denominator represents a number bigger than one. If that is the case, then we may wonder if it is bigger than two or three, and so on. More precisely we want to know if the number is between one and two or between two and three, etc. One way to find this out, is by writing the fraction as a mixed number. A mixed number is a number written as a positive integer, followed by a positive fraction with numerator less than denominator (the fraction represents a number between zero and one).

For example: $2\frac{1}{3}$, $1\frac{4}{5}$, $3\frac{10}{11}$.

Now let's talk about the meaning of the mixed number. Take for example $2\frac{1}{3}$. This is a number between 2 and 3, and in fact:

$$2\frac{1}{3} = 2 + \frac{1}{3}$$

therefore, written as a fraction

$$2\frac{1}{3} = 2 + \frac{1}{3} = \frac{7}{3}$$

Any mixed number can be written as a fraction, the result of adding the positive integer and the fraction. Notice that the resulting fraction will represent a number greater than one. Also, any positive fraction **bigger than one** can be written as a mixed number. To do so, we can take advantage of the rule for addition of fractions "backwards".

23.1)Examples.

1. Write the fraction as a mixed number, if it applies.

   (a) $\frac{7}{3} = \frac{6+1}{3} = \frac{6}{3} + \frac{1}{3} = 2 + \frac{1}{3} = 2\frac{1}{3}$

   Notice that we decide to "break" 7 into $6+1$ because 6 is the largest number that is less than 7 and for which 3 is a factor.

   (b) $\frac{13}{15}$

   This fraction can not be written as a mixed number because it is actually a number between 0 and 1.

(c) $\frac{15}{4} = \frac{12+3}{4} = \frac{12}{4} + \frac{3}{4} = 3 + \frac{3}{4} = 3\frac{3}{4}$

(d) $\frac{19}{17} = \frac{17+2}{17} = \frac{17}{17} + \frac{2}{17} = 1 + \frac{2}{17} = 1\frac{2}{17}$

(e) $\frac{19}{5} = \frac{15+4}{5} = \frac{15}{5} + \frac{4}{5} = 3 + \frac{4}{5} = 3\frac{4}{5}$

2. Write the mixed number as a fraction.

(a) $1\frac{2}{3} = 1 + \frac{2}{3} = \frac{3}{3} + \frac{2}{3} = \frac{5}{3}$

(b) $5\frac{3}{5} = 5 + \frac{3}{5} = \frac{25}{5} + \frac{3}{5} = \frac{28}{5}$

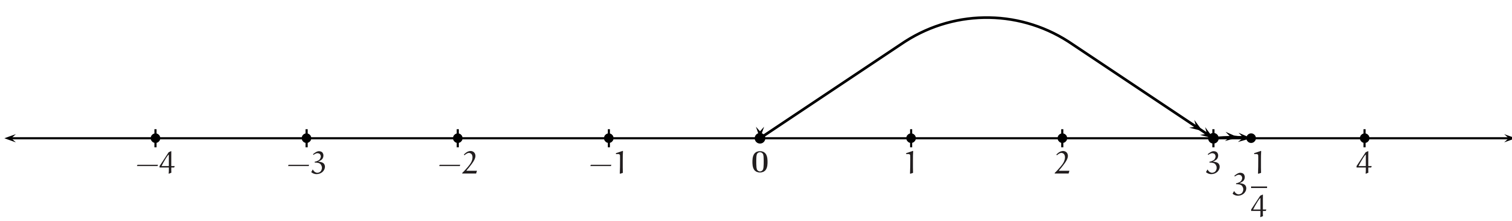
(c) $4\frac{1}{7} = 4 + \frac{1}{7} = \frac{28}{7} + \frac{1}{7} = \frac{29}{7}$

3. For each fraction, write as a mixed number (if it applies) and then locate it on a number line.

(a) $\frac{13}{4}$

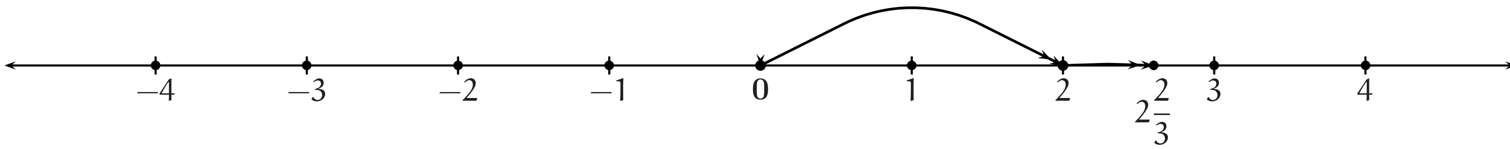
$\frac{13}{4} = \frac{12+1}{4} = \frac{12}{4} + \frac{1}{4} = 3 + \frac{1}{4} = 3\frac{1}{4}$

Now let's locate it on a number line.



(b) $\frac{8}{3}$

$\frac{8}{3} = \frac{6+2}{3} = \frac{6}{3} + \frac{2}{3} = 2 + \frac{2}{3} = 2\frac{2}{3}$



Exercises.

1. Write the mixed number as a fraction.

   (a) $3\frac{4}{7}$

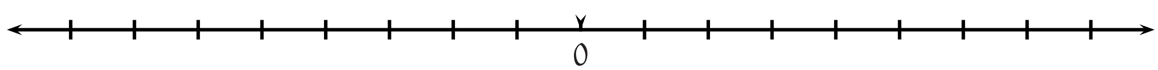
   (b) $1\frac{5}{6}$

   (c) $2\frac{3}{5}$

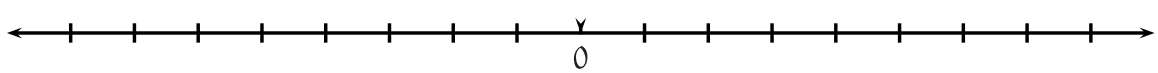
   (d) $5\frac{1}{6}$

2. Write the fraction as a mixed number if possible. Then, locate it on a number line.

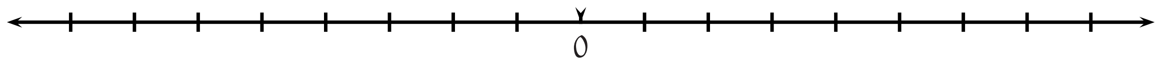
   (a) $\frac{19}{3}$



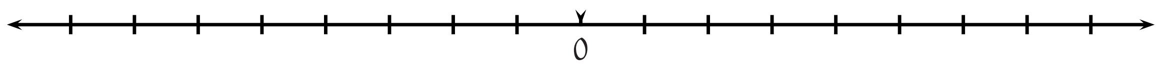
   (b) $\frac{11}{12}$



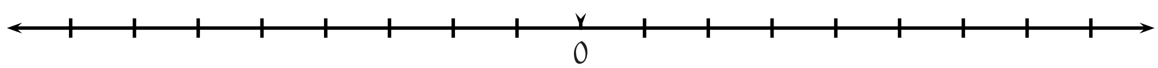
(c) $\frac{14}{5}$



(d) $\frac{28}{4}$



(e) $\frac{40}{7}$



LESSON 24

Operations on Mixed Numbers.

To perform operations on mixed number we can always write them as fractions and then perform the operations. If we want the answer as a mixed number, we just have to write the resulting fraction again as a mixed number, if possible.

There are some short cuts for addition or subtraction.

For example, for additions we can proceed as follows:

24.1)Examples.

1. $2\frac{2}{3}+1\frac{3}{4}=2+\frac{2}{3}+1+\frac{3}{4}=(2+1)+\left(\frac{2}{3}+\frac{3}{4}\right)=3+\frac{8+9}{12}=3+\frac{17}{12}=3+\frac{12+5}{12}==3+\frac{12}{12}+\frac{5}{12}=$ $4\frac{5}{12}$

2. $3\frac{1}{5}+4\frac{1}{3}=3+\frac{1}{5}+4+\frac{1}{3}=(3+4)+\left(\frac{1}{5}+\frac{1}{3}\right)=7+\frac{8}{15}=7\frac{8}{15}$

   It is important to understand that here we are applying both the commutative and associative property for addition, even though we are not showing the formal details.

24.2)Examples.

Compute. Give your answer both as a fraction and as a mixed number, if it applies.

1. $3\frac{1}{3}-2\frac{4}{5}=\left(3+\frac{1}{3}\right)-\left(2+\frac{4}{5}\right)=\left(\frac{9+1}{3}\right)-\left(\frac{10+4}{5}\right)=\frac{10}{3}-\frac{14}{5}=\frac{50-42}{15}=\frac{8}{15}$

   The answer as a fraction is $\frac{8}{15}$.

   Since this is a number between 0 and 1, it can not be written as a mixed number.

2. $2\frac{2}{5}\times 3\frac{1}{2}=\left(2+\frac{2}{5}\right)\times\left(3+\frac{1}{2}\right)=\left(\frac{12}{5}\right)\times\left(\frac{7}{2}\right)=\frac{6\times 2}{5}\times\frac{7}{2}=\frac{6}{5}\times\frac{7}{1}=\frac{42}{5}$

   The answer as a fraction is $\frac{42}{5}$

   As a mixed number,

$$\frac{42}{5} = \frac{40+2}{5} = \frac{40}{5} + \frac{2}{5} = 8 + \frac{2}{5} = 8\frac{2}{5}$$

3. $4\frac{2}{7} \div 2\frac{2}{3} = \left(4 + \frac{2}{7}\right) \div \left(2 + \frac{2}{3}\right) = \frac{30}{7} \div \frac{8}{3} = \frac{30}{7} \times \frac{3}{8} = \frac{2 \times 15}{7} \times \frac{3}{2 \times 4} = \frac{15}{7} \times \frac{3}{4} = \frac{45}{28}$

   The answer as a fraction is $\frac{45}{28}$

   As a mixed number,
   $\frac{45}{28} = \frac{28+17}{28} = \frac{28}{28} + \frac{17}{28} = 1 + \frac{17}{28} = 1\frac{17}{28}$

4. $1\tfrac{1}{5} + 2\tfrac{4}{5} = 1 + \frac{1}{5} + 2 + \frac{4}{5} = (1+2) + \left(\frac{1}{5} + \frac{4}{5}\right) = 3 + 1 = 4$

5. $\frac{4}{5} \times 2\frac{1}{4} = \frac{4}{5} \times \left(2 + \frac{1}{4}\right) = \frac{4}{5} \times \frac{9}{4} = \frac{9}{5}$

   The answer as a fraction is $\frac{9}{5}$

   As a mixed number,
   $\frac{9}{5} = \frac{5+4}{5} = \frac{5}{5} + \frac{4}{5} = 1 + \frac{4}{5} = 1\frac{4}{5}$

6. $1\frac{2}{5} \div 5 = \left(1 + \frac{2}{5}\right) \div 5 = \frac{7}{5} \div \frac{5}{1} = \frac{7}{5} \times \frac{1}{5} = \frac{7}{25}$

7. $2\frac{7}{8} + \frac{3}{4} = \left(2 + \frac{7}{8}\right) + \frac{3}{4} = 2 + \left(\frac{7}{8} + \frac{3}{4}\right) = 2 + \frac{13}{8} = \frac{29}{8}$

   As a mixed number,
   $\frac{29}{8} = \frac{24+5}{8} = \frac{24}{8} + \frac{5}{8} = 3 + \frac{5}{8} = 3\frac{5}{8}$

8. $4\frac{3}{5} - \frac{9}{10} = \left(4 + \frac{3}{5}\right) - \frac{9}{10} = \frac{23}{5} - \frac{9}{10} = \frac{46}{10} - \frac{9}{10} = \frac{37}{10}$

   As a mixed number,
   $\frac{37}{10} = \frac{30+7}{10} = \frac{30}{10} + \frac{7}{10} = 3 + \frac{7}{10} = 3\frac{7}{10}$

Exercises

Compute. Write your answer both as a fraction (or integer) and a mixed number (if it applies).

1. $3\frac{2}{5} + 7\frac{1}{3}$

2. $1\frac{4}{5} + 2\frac{1}{2}$

3. $5\frac{1}{3} - 2\frac{8}{9}$

4. $4\frac{2}{7} - 3$

5. $10 \div 2\frac{1}{2}$

6. $4\frac{4}{7} + 3$

7. $3\frac{2}{3} + \frac{3}{7}$

8. $2\frac{1}{4} \times 3\frac{5}{9}$

9. $5\frac{1}{5} \div 1\frac{3}{10}$

10. $10 \times 7\frac{3}{5}$

LESSON 25

REVIEW

Homework/Class Work/Quiz 42

1. Write the mixed number as a fraction.

   (a) $1\frac{4}{5}$

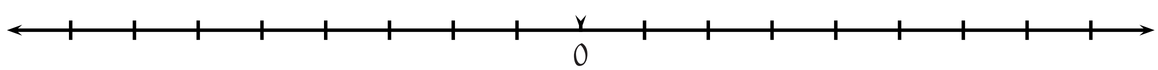
   (b) $2\frac{3}{7}$

   (c) $4\frac{1}{2}$

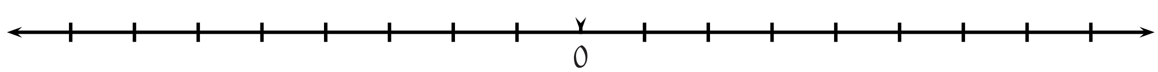
   (d) $6\frac{3}{5}$

2. Write the fraction as a mixed number if possible. Then, locate it on a number line.

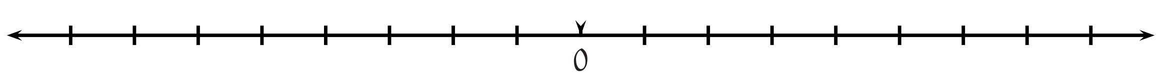
   (a) $\frac{9}{5}$



   (b) $\frac{9}{10}$



   (c) $\frac{31}{4}$



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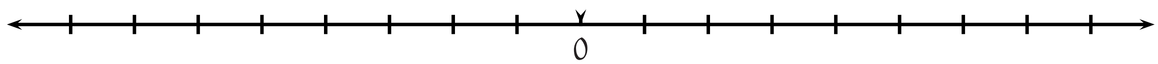
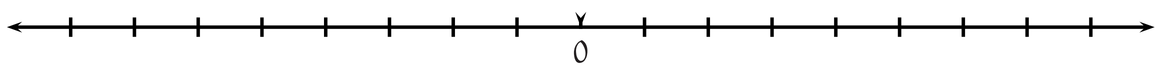
Homework/Class Work/Quiz 43

1. Write the fraction as a mixed number if possible. Then, locate it on a number line.

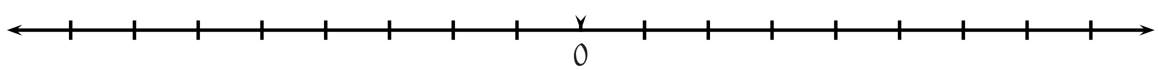
   (a) $\frac{9}{8}$



   (b) $\frac{10}{4}$



   (c) $\frac{4}{5}$



2. Compute. Give your answer both as a fraction and as a mixed number, if it applies.

   (a) $5\frac{2}{7} + 3\frac{1}{4}$

   (b) $1\frac{7}{8} + 4\frac{2}{3}$

   (c) $2\frac{1}{4} - 1\frac{4}{5}$

   (d) $5\frac{1}{2} - 2\frac{1}{3}$

blank

Homework/Class Work/Quiz 44

Compute. Give your answer both as a fraction (or an integer) and as a mixed number, if it applies.

1. $3\frac{1}{5} \times 4\frac{2}{3}$

2. $3\frac{1}{5} + 4\frac{2}{3}$

3. $6\frac{3}{7} \div 2\frac{5}{6}$

4. $2\frac{3}{5} + 1\frac{5}{6}$

5. $2\frac{2}{3} - 1\frac{5}{6}$

6. $9 \times 5\frac{2}{3}$

7. $9 \div 5\frac{2}{3}$

8. $5\frac{5}{7} - \frac{3}{5}$

blank

Homework/Class Work/Quiz 45

Compute. Give your answer both as a fraction (or an integer) and as a mixed number, if it applies.

1. $7\frac{3}{5} \div \frac{5}{2}$

2. $-\frac{3}{2} + 2\frac{2}{5}$

3. $\frac{4}{5} \times 5\frac{1}{6}$

4. $3\frac{3}{5} \div 2$

5. $2\frac{4}{5} \times 10$

6. $5\frac{1}{3} + \frac{2}{5}$

7. $4\frac{2}{3} - \frac{1}{9}$

blank

LESSON 26

Decimal Notation.

Decimal notation refers to writing a number using a horizontal arrangement of the ten digits $0,1,2,3,4,5,6,7,8$ and 9. The position of the digit will determine its value based on powers of ten. For example:

$$232 = 2 \times 100 + 3 \times 10 + 2 = 2 \times 10^2 + 3 \times 10^1 + 2 \times 10^0$$

How do we write decimal notation for a fraction that is not an integer? Let's concentrate only on positive fractions.

First we need to understand decimal; notation for a particular kind of fractions, that is, those with numerator 1 and denominator given by a positive power of 10 (10, 100, 1000 and so on).

For such fractions, decimal notation will include a point called "decimal point". The first digit (left to right) after the decimal point is related to the fraction $\frac{1}{10}$, the second digit is related to the fraction $\frac{1}{100}$, the third digit is related to the fraction $\frac{1}{1000}$, etc.

In particular:

1. $\frac{1}{10} = 0.1$ (or $.1$)

   This is showing that

$$0.1 = 0 + 1 \times \frac{1}{10}$$

2. $\frac{1}{100} = 0.01$ (or $.01$)

   This is showing that

$$0.01 = 0 + 0 \times \frac{1}{10} + 1 \times \frac{1}{100}$$

3. $\frac{1}{1000} = 0.001$ (or $.001$)

   This is showing that

$$0.001 = 0 + 0 \times \frac{1}{10} + 0 \times \frac{1}{100} + 1 \times \frac{1}{1000}$$

Notice that these three fractions have a numerator less than the denominator. Therefore, they are numbers **between zero and one**. Also, notice that the decimal expression has the natural number zero before the decimal point.

26.1)Examples

1. $\frac{3}{10} = 3 \times \frac{1}{10} = 0.3$

   This is a number between zero and one.

2. $\frac{5}{1000} = 5 \times \frac{1}{1000} = 0.005$

   This is a number between zero and one.

3. $\frac{3201}{1000} = \frac{3000 + 200 + 1}{1000} = \frac{3000}{1000} + \frac{200}{1000} + \frac{1}{1000} = 3 + \frac{2}{10} + \frac{1}{1000} =$
   $= 3 + 2 \times \frac{1}{10} + 0 \times \frac{1}{100} + 1 \times \frac{1}{1000} = 3.201$

   Two things to mention are that, first, this is a number between 3 and 4. Second, the decimal notation has the same digits as the number in the numerator of the original fraction. However, the decimal point is located such that there are three digits after the point. The denominator of the fraction is a power of 10, and in fact it is 10^3 ($10^3 = 1000$). In other words, **the exponent for 10 (or the numbers of zeros in the denominator) is the number of digits after the point in the decimal notation**.

4. $\frac{1}{2} = \frac{1 \times 5}{2 \times 5} = \frac{5}{10} = 5 \times \frac{1}{10} = 0.5$

5. $\frac{1}{4} = \frac{1 \times 25}{4 \times 25} = \frac{25}{100} = \frac{20 + 5}{100} = \frac{20}{100} + \frac{5}{100} = \frac{2}{10} + \frac{5}{100} = 2 \times 10 + 5 \times \frac{1}{100} = 0.25$

6. $\frac{13}{5} = \frac{10 + 3}{5} = \frac{10}{5} + \frac{3}{5} = \underbrace{2 + \frac{3}{5}}_{\text{number between 2 and 3}} = 2 + \frac{3 \times 2}{5 \times 2} = 2 + \frac{6}{10} = 2.6$

These examples will make the following facts plausible (even though we are not giving a formal proof!):

1. For a positive fraction with the denominator a positive power of 10 (10, 100, 1000, and so on), the decimal notation will have the same digits as the natural number in the numerator. The decimal point is located such that there will be as many digits after the point as zeros in the denominator. If the digits of the natural number in the numerator do not provide the necessary number of digits after the point in the decimal notation, we must put zeros to the left of the natural number in the numerator. For example:
   (a) $\frac{31}{10} = 3.1$
   (b) $\frac{31}{100} = 0.31$
   (c) $\frac{31}{1000} = 0.031$
   (d) $\frac{31}{10000} = 0.0031$
2. If a fraction does not have the denominator written as a power of 10, then, if possible, we could try using an equivalent fraction with denominator that is a power of 10. We can then determine the decimal notation as we did above. For example:
   (a) $\frac{1}{2} = \frac{5}{10} = 0.5$
   (b) $\frac{1}{4} = \frac{25}{100} = 0.25$
   (c) $\frac{1322}{500} = \frac{2644}{1000} = 2.644$
3. At this point, it is important to be aware of the following:

   We know that for natural numbers (actually also for integers!) zeros on the left hand side do not have any value, for example

$$4 = 04 = 004 = 0000000004$$

   Something similar happens for decimals regarding zeros on the right hand side, after the decimal point. For example:

$$2 = \frac{2}{1} = \frac{20}{10} = \frac{200}{100} = \frac{20000000}{10000000}$$

On the other hand

$$\frac{20}{10} = 2.0$$
$$\frac{200}{100} = 2.00$$
$$\frac{20000000}{10000000} = 2.0000000$$

therefore:

$$2 = 2.0 = 2.00 = 2.0000000$$

In general, any integer is equal to the decimal number that we get when adding a decimal point and zeros after it.

4. A positive number that is not an integer and is written in decimal notation, will be a number between the natural number before the decimal point and the same natural number plus one. For example:

   (a) 0.003 is a number between 0 and 1. Therefore we can write:

   $$0 < 0.003 < 1$$

   (b) 2.999 is a number between 2 and 3. Then we can write:

   $$2 < 2.999 < 3$$

   (c) 36.12111 is a number between 36 and 37. Therefore:

   $$36 < 36.12111 < 37$$

26.2) Examples

1. Find decimal notation for each of the following fractions.

   (a) $\frac{351}{100}$

   $$\frac{351}{100} = 3.51$$

(b) $\frac{12}{5}$

$$\frac{12}{5} = \frac{24}{10} = 2.4$$

(c) $\frac{11}{4}$

$$\frac{11}{4} = \frac{11 \times 25}{4 \times 25} = \frac{275}{100} = 2.75$$

2. Fill in the blank using either '=', '<', or '>' as appropriate:

(a) $3 \quad 3.01$

Since 3.01 is between 3 and 4, then

$3 < 3.01$

(b) $4.0 \quad 3.9999$

Since $4.0 = 4$ and 3.9999 is between 3 and 4, then

$4.0 > 3.9999$

(c) $\frac{86}{10} \quad 8.6$

Since the decimal notation for $\frac{86}{10}$ is 8.6, then

$\frac{86}{10} = 8.6$

Exercises

1. Find decimal notation for each of the following fractions.

(a) $\frac{2391}{100}$

(b) $\frac{91}{1000}$

(c) $\frac{9}{5}$

(d) $\frac{13}{4}$

(e) $\frac{7}{2}$

(f) $\frac{7}{100}$

(g) $\frac{2}{25}$

(h) $\frac{501}{500}$

2. Fill in the blank using either '=', '<', or '>' as appropriate:

   (a) $\frac{501}{500}$ ___ 2

   (b) 5.0000 ___ 5

   (c) 5.000001 ___ 5

   (d) 9 ___ 8.9999999999

LESSON 27

Sometimes trying to change the denominator of a fraction into a power of ten could involve big numbers or it could be impossible! For example, if the denominator of the fraction is 3, we can not change the denominator into a power of 10 because 3 is not a factor for any power of 10. Therefore, we would like to count with an algorithm that will provide us with the decimal notation, without having to rewrite the fraction with denominator which is a power of 10. That algorithm is the "extended" division algorithm. We are using the word "extended" because we will go beyond the point of division with natural numbers, until we can make the remainder 0 or we conclude that we can never make it be 0. In the last case we need to understand the meaning of this situation.

11.2 Examples.

1. $\dfrac{1}{10}$

$$\frac{1}{10} = 1 \div 10$$

Let's see the division step by step.

STEP ONE

$$\begin{array}{r} 0 \\ 10\overline{)1} \\ -\underline{\ 0} \\ 1 \end{array}$$

In the first step the remainder is not zero but we do not have any more digits to bring down. We will write the dividend 1 as 1.0 and when bringing down the first digit after the point (in this case 0), we will also put a decimal point in the quotient and then proceed with the division.

STEP TWO

$$
\begin{array}{rl}
 & 0.1 \\
10 & \overline{)1.0} \\
- & \underline{0} \\
 & 1\,0 \\
- & \underline{1\,0} \\
 & 0
\end{array}
$$

So the division is showing that

$$\frac{1}{10} = 0.1$$

2. $\dfrac{13}{8}$

$$\frac{13}{8} = 13 \div 8$$

Let's start the division:

$$
\begin{array}{rl}
 & 1 \\
18 & \overline{)13} \\
- & \underline{8} \\
 & 5
\end{array}
$$

In this case, we need to write 13 as 13.000 so we have enough digits to bring down until the remainder is zero. Again keep in mind that when bringing down the first digit after the point (in this case it is zero), we have to put a decimal point in the quotient before performing the division.

$$
\begin{array}{rl}
 & 1.625 \\
8 & \overline{)13.000} \\
- & \underline{8} \\
 & 5\,0 \\
- & \underline{4\,8} \\
 & 20 \\
 & \underline{-16} \\
 & 40 \\
 & \underline{-40} \\
 & 0
\end{array}
$$

Therefore we have that

$$\frac{13}{8} = 1.625$$

3. $\frac{1}{3}$

$\frac{1}{3} = 1 \div 3$

Let's start the division:

$$\begin{array}{r} 0 \\ 3\overline{)1} \\ -\ \underline{0} \\ 1 \end{array}$$

Now let's write 1 as 1.0 and go on with the division. Now we get:

$$\begin{array}{r} 0.3 \\ 3\overline{)1.0} \\ -\ \underline{0} \\ 1\ 0 \\ -\ \underline{\ 9} \\ 1 \end{array}$$

Since the remainder is not zero now, we need to write 1 as 1.00 so we can go on with the division. Now we get:

$$\begin{array}{r} 0.33 \\ 3\overline{)1.00} \\ -\ \underline{0} \\ 1\ 0 \\ -\ \underline{\ 9} \\ 10 \\ -\ \underline{9} \\ 1 \end{array}$$

It is not going to take long to arrive at the conclusion that the remainder will always be 1, meaning that there is no way to end the division with a remainder of zero! The meaning of this is that the decimal notation that we are looking for will **not** be **terminating**. In other words, the digits after the decimal point will never stop. However, there is a pattern that repeats infinitely (in this case the digit 3). For that reason, we say that the decimal expression is **repeating**. In order to indicate the pattern, we will use a horizontal bar on top of it.

So, in this case we write:

$$\frac{1}{3} = 0.\bar{3}$$

Examples 1 and 2 show decimal expressions that are terminating. Example 3 shows a decimal expression that is not terminating and repeating.

All fractions have decimal representations that are either **terminating** or **not terminating and repeating**.

27.1) Examples

Use division to find decimal notation for each of the following fractions.

1. $\dfrac{15}{16}$

```
      0.9375
  16)15.0000
  -   0
     15 0
  -  14 4
        60
      - 48
       120
       112
         80
       - 80
          0
```

Therefore
$\dfrac{15}{16} = 0.9375$

2. $\frac{5}{3}$

$$
\begin{array}{r}
1.66 \\
3\overline{)5.00} \\
-\ 3 \\
\hline
2\ 0 \\
-\ 1\ 8 \\
\hline
20 \\
-18 \\
\hline
2
\end{array}
$$

Since the remainder will always be 2, then what we have is a decimal notation which is not terminating and repeating, where the pattern that repeats is 6.

In this case

$$\frac{5}{3} = 1.\bar{6}$$

From Decimal Notation to Fraction Notation.

Using the facts that we pointed out regarding decimal notation for a fraction with denominator given by a positive power of 10, we can write a terminating decimal as a fraction.

27.2) Examples.

1. $1.9 = \frac{19}{10}$

2. $0.037 = \frac{37}{1000}$

3. $5.3111 = \frac{53111}{10000}$

4. $3.15 = \frac{315}{100}$

It is important to notice that for decimals, zeros on the right hand side **after the decimal point** do not have any value.

For example:

$$5.3 = \frac{53}{10} = \frac{530}{100} = \frac{5300}{1000} = \frac{53000}{10000}$$

therefore

$$5.3 = 5.30 = 5.300 = 5.3000$$

WARNING: It is incorrect to think that,for example, 2.50 is greater than 2.5 because 50 is greater than 5! According with what we just noticed, actually

$$2.50 = 2.5$$

Comparing Positive Decimals.

If we want to compare two different positive decimals that are not integers, it is convenient to first notice the consecutive positive integers having the decimals in between. In other words, we first look at the natural number before the point for each decimal. If they are different, then it should be clear how to compare the decimals. For example:

1. $3.312 > 1.5234$

   Since 3.312 is a number between 3 and 4 and 1.5234 is a number between 1 and 2, then it is easy to determine that 3.312 is greater than 1.5234

2. $4.0001 < 5.3$

   Since 4.0001 is a number between 4 and 5 and 5.3 is a number between 5 and 6, then it is easy to determine that 4.0001 is less than 5.3

However, if the natural number to the left of the point is the same, then we need to look at the digits after the point in order to compare the numbers. Let's take for example the numbers 3.12 and 3.2.

WARNING: it is **incorrect** to think that 3.12 is bigger than 3.2 because after the point in 3.12 there is a 12, and for 3.2, there is a 2 after the point. Although $12 > 2$, 3.12 is not bigger than 3.2.

In fact,

$$3.12 = \frac{312}{100}$$

$$3.2 = \frac{32}{10} = \frac{320}{100}$$

Since

$$\frac{312}{100} < \frac{320}{100}$$

then we can conclude that

$$3.12 < 3.2$$

Comparing two positive decimals with the same natural number before the point requires comparing the digits after the point "digit by digit". Do not try to interpret the digits after the point as whole numbers. The bigger number will be the one with the bigger digit in the same position. In the case of 3.12 and 3.2, the first digit after the point for 3.12 is 1 and the first digit after the point for 3.2 is 2, so, since $1 < 2$, then $3.12 < 3.2$.

27.3) Examples.

Fill in the blank using either '=', '<' or '>' as appropriate.

1. $42.235 \quad\quad 42.24$

   Since these are positive decimals with the same natural number before the point (42), then we need to look at the digits after the point. The first digit after the point is the same, but the second digit of the first number is less than the second digit of the second number. Therefore,

$$42.235 < 42.24$$

2. $-1.4567 \qquad -1.5$

Let's see first what happen with the positive decimals 1.4567 and 1.5. Again, the natural number before the point is the same (1), so, we need to look at the digits after the point. Since the first digit after the point for the first number is less than the first digit after the point for the second number, it follows that

$$1.4567 < 1.5$$

This means that

$$-1.4567 > -1.5$$

3. $\frac{1}{3} \qquad 0.333333$

Since $\frac{1}{3} = 0.\overline{3}$, then

$$\frac{1}{3} > 0.333333$$

4. $\frac{101}{100} \qquad 1.0001$

Since $\frac{101}{100} = 1.01$, then

$$\frac{101}{100} > 1.0001$$

5. $4.5000 \qquad 4.5$

Based on our previous discussions

$$4.5000 = 4.5$$

Exercises

1. Find decimal notation for each of the following numbers

   (a) $\frac{139}{8}$

   (b) $-\frac{32}{25}$

   (c) $\frac{31}{12}$

   (d) $2\frac{103}{330}$

   (e) $-\frac{5}{16}$

2. Write each of the following numbers as a fraction.

   (a) 25.6655

   (b) 0.0043

   (c) -12.01

   (d) $(1.3)^2$

3. Fill in the blank using either '=', '<' or '>' as appropriate. Explain your answer.

   (a) $\frac{2}{3} \qquad 0.\overline{6}$

   (b) $2.999 \qquad 3.00$

   (c) $9.330 \qquad 9.6$

   (d) $-34.2230 \qquad -34.24$

   (e) $0.0001 \qquad -234$

   (f) $3.25 \qquad \frac{3250}{1000}$

LESSON 28

Addition and Subtraction of Decimals.

We know than when arranging natural numbers vertically to perform an addition, we need to make sure that digits in a particular place go under those in the same place: ones under ones, tens under tens and so on.

When adding decimals, we need to use the same idea. That means that when arranging them vertically, the decimal points need to be lined up so ones go under ones, tens under tens and so on. Tenths (the first digit after the decimal point) must also be lined up under tenths, hundredths (the second digit after the decimal point) under hundredths, and so on.

The same principle applies when subtracting a positive decimal number from a bigger positive decimal number.

Keep in mind that all natural numbers can be written as decimals by putting the decimal point at the right hand side, and then writing zeros after the decimal point (for example, $5 = 5.0 = 5.00$, etc).

28.1) Examples

Compute

1. $23.25 + 7.9$

$$\begin{array}{rr} & 23.25 \\ + & 7.9 \\ \hline & 31.15 \end{array}$$

or

$$\begin{array}{rr} & 23.25 \\ + & 7.90 \\ \hline & 31.15 \end{array}$$

Then

$$23.25 + 7.9 = 31.15$$

2. $15 - 8.7$

$$\begin{array}{rr} & 15.0 \\ - & 8.7 \\ \hline & 6.3 \end{array}$$

Then

$$15 - 8.7 = 6.3$$

3. $9.456 + 11 + 5.9$

$$\begin{array}{rr} & 9.456 \\ & 11 \\ + & 5.9 \\ \hline & 26.356 \end{array}$$

or

$$\begin{array}{rr} & 9.456 \\ & 11.000 \\ + & 5.900 \\ \hline & 26.356 \end{array}$$

Therefore

$$9.456 + 11 + 5.9 = 26.356$$

4. $132.52 - 14.7$

$$\begin{array}{rr} & 132.52 \\ - & 14.7 \\ \hline & 117.82 \end{array}$$

or

$$\begin{array}{rr} & 132.52 \\ - & 14.70 \\ \hline & 117.82 \end{array}$$

Therefore

$$132.52 - 14.7 = 117.82$$

Now, let's do some additions and subtractions including negative numbers written in decimal notation. The rules regarding the signs are the same as in the case of integers.

28.2)Examples.

1. $-3.15 + 2.7 = -(3.15 - 2.7) = -0.45$

$$\begin{array}{rr} & 3.15 \\ - & 2.7 \\ \hline & 0.45 \end{array}$$

2. $-11.3 - 134.47 = -(11.3 + 134.47) = -145.77$

$$\begin{array}{rr} & 134.47 \\ + & 11.3 \\ \hline & 145.77 \end{array}$$

3. $9.55 + (-10.1) = -(10.1 - 9.55) = -0.55$

$$\begin{array}{rr} & 10.10 \\ - & 9.55 \\ \hline & 0.55 \end{array}$$

4. $0.91 - 1 = -(1 - 0.91) = -0.09$

$$\begin{array}{rr} & 1.00 \\ - & 0.91 \\ \hline & 0.09 \end{array}$$

5. $-3.3-(-2.22) = -3.3+2.22 = -(3.3-2.22) = -1.08$

$$\begin{array}{rr} & 3.30 \\ - & 2.22 \\ \hline & 1.08 \end{array}$$

6. $9.151-(-11.5) = 9.151+11.5 = 20.651$

$$\begin{array}{rr} & 9.151 \\ + & 11.5 \\ \hline & 20.651 \end{array}$$

Also if we combine additions and subtractions, they are associated from left to right.

28.3)Examples.

1. $\underbrace{9.73-11.7}_{-1.97}-3.59+5 = \underbrace{-1.97-3.59}_{-5.56}+5 = -5.56+5 = -0.56$

2. $\underbrace{-1.1-1.1}_{-2.2}+1+1 = \underbrace{-2.2+1}_{-1.1}+1 = -1.1+1 = -0.1$

3. $\underbrace{-0.3+1.12}_{0.82}-(-2.3) = 0.82+2.3 = 3.12$

4. $1.2-1.2-3.12+0.352 = 0-3.12+0.352 = -3.12+0.352 = -(3.12-0.352) = -2.768$

Exercises

Compute

1. $17.3+9.77$

2. $12.12+7+9.9$

3. $221.321+45.9+99+5.35$

4. $35.1-12.532$

5. $9 - 7.351$

6. $14.12 - 8$

7. $-2.2 - 3.31$

8. $-1.15 + 5.321$

9. $0.16 - 1.76$

10. $3.41 - (-1)$

11. $1.93 - 3$

12. $-5 - (-6.713)$

13. $8.2 - 9.17 - 3 + 1.1$

14. $-(-1.1) - (-1.1) - 1.1$

15. $0.351 - 1.23 + 2$

16. $-5.37 - (-5.37) - 4.65 + 7 - 3.3339$

17. $8 - 8.001 + 1 - (-0.3) + 1$

LESSON 29

Multiplication of Decimals.

If we want to multiply two decimals, one thing that we could do is to write them as fractions and then multiply them.

For example:

$$2.5 \times 3.71 = \frac{25}{10} \times \frac{371}{100} = \frac{25 \times 371}{10 \times 100} = \frac{9275}{1000} = 9.275$$

The example shows that the digits of the result (9.275) are exactly the same ones we would get if we "eliminate" the decimal points from the original numbers and simply multiply the natural numbers ($25 \times 371 = 9275$). Also the result (9.275) has three digits after the point. The first number in the multiplication (2.5) has one digit after the point, and the second number in the multiplication (3.71) has two digits after the point. In other words, the result of the multiplication has as many digits after the point as the total of digits after the point among the numbers multiplied.

In conclusion, we note that we could have done the multiplication by multiplying the original numbers just as we multiply natural numbers. Then, locate the decimal point, so that the number of digits after it is the same as the total number of digits after the decimal point for the numbers that we are multiplying.

$$\begin{array}{rr} & 2.5 \\ \times & 3.71 \\ \hline & 25 \\ & 1750 \\ + & 7500 \\ \hline & 9.275 \end{array}$$

Notice that when arranging the numbers vertically to perform the multiplication, **we do not need to line up the decimal points**.

Also, it is important to realize that if we write the decimals as fractions and multiply, we would get the same result but written as a fraction. Therefore, all the properties that are true for operations with fractions are also true for operations with decimals. In particular, keep in mind commutative and associative properties.

Besides, we can also consider negative decimals. We will use the same rules for multiplication of signs that we discussed before.

29.1)Examples.

1. $4 \times (-5.31) = -(4 \times 5.31) = -21.24$

$$\begin{array}{rr} & 5.31 \\ \times & 4 \\ \hline & 21.24 \end{array}$$

2. $-4.11 \times (-3.102) = 4.11 \times 3.102 == 12.74922$

$$\begin{array}{rr} & 3.102 \\ \times & 4.11 \\ \hline & 3102 \\ & 31020 \\ + & 1240800 \\ \hline & 12.74922 \end{array}$$

3. $3.25 \times 10 = \dfrac{325}{100} \times 10 = \dfrac{325 \times 10}{10 \times 10} = \dfrac{325}{10} = 32.5$

   Notice that in this example one of the numbers is 10. Also, the result has the same digits as the other number in the multiplication (3.25), but the decimal point has been "moved" one place to the right.

   In general, if we multiply a decimal by a positive power of 10 ($10 = 10^1$,$100 = 10^2$,$1000 = 10^3$, and so on), the result is what we get by "moving" the decimal point of the decimal number to the right, as many places as we have zeros in the power of 10. If we run out of places to continue "moving" the point to the right, then we start adding zeros, as it shows in the next example.

4. $-35.3 \times 100 = -\dfrac{353}{10} \times 100 = -353 \times 10 = -3530$

5. $4.2 \times 0.1 = \dfrac{42}{10} \times \dfrac{1}{10} = \dfrac{42}{100} = 0.42$

   Notice that in this last example the decimal point of the first number in the multiplication "moved" one place to the left!

It happened because the other number in the multiplication is 0.1. Something similar will happen when multiplying a decimal by 0.01 or 0.001, and so on. In other words, when the other number in the multiplication is a negative integer power of 10 (0.1, 0.01, 0.001, etc), we must "move" the decimal point to the left.

6. $-53.1 \times (-0.001) = 53.1 \times 0.0001 = \dfrac{531}{10} \times \dfrac{1}{1000} = \dfrac{531}{10000} = 0.0531$

7. $2.11 \times (-1.2) \times 3 = -2.532 \times 3 = 7.596$

8. $(-1.3) \times (-2.56) \times (-10) = 3.328 \times (-10) = -33.28$

9. $4.51 \times 0.1 \times (-3.3) \times 10 = 4.51 \times (-3.3) \times \underbrace{0.1 \times 10}_{1} = 4.51 \times (-3.3) \times 1 = -14.883 \times 1 - -14.883$

   Notice that in this example we used the commutative and associative properties of multiplication.

Now we can also compute exponentials that have a decimal number for the base.

29.2)Examples.

Compute

1. $(1.2)^2 = 1.2 \times 1.2 = 1.44$

2. $(-2.12)^2 = (-2.12) \times (-2.12) = 2.12 \times 2.12 = 4.4944$

3. $(1.1)^3 = 1.1 \times 1.1 \times 1.1 = 1.21 \times 1.1 = 1.331$

4. $-(2.3)^4 = -(2.3 \times 2.3 \times 2.3 \times 2.3) = -(5.29 \times 2.3 \times 2.3) = -(12.167 \times 2.3) = -27.9841$

Division of Decimals.

Let's start with the case where the dividend is a positive decimal but the divisor is a positive integer.

We can always write the decimal as a fraction and perform the division. For example:

$$3.99 \div 3 = \frac{399}{100} \div \frac{3}{1} = \frac{399}{100} \times \frac{1}{3} = \frac{399}{100 \times 3} = \frac{3 \times 133}{100 \times 3} = \frac{133}{100} = 1.33$$

However, sometimes the numbers may not be "nice" enough to be handled in this way. Therefore, we would like to know how to perform the long division algorithm.

The first thing to do is to divide the natural number before the decimal point by the divisor (in the usual way). This basically means that in the first step of the division we can not go beyond the point. We go on with the usual process. When bringing down the first digit after the decimal point, before dividing, we need to put a decimal point in the quotient and then proceed with the division.

Keep in mind that if necessary, we can always add zeros at the end of the decimal number (it would not change its value), so we can go on with the division until either the remainder is zero, or we determine that the quotient will not be terminating but a repeating decimal.

29.3)Examples.

1. $1.21 \div 2 = 0.605$

   Let's show the long division step by step.

   FIRST STEP

   We divide the natural number before the decimal point by the divisor.

$$\begin{array}{rl} & 0 \\ 2) & \overline{1.21} \\ - & \underline{0} \\ & 1 \end{array}$$

   SECOND STEP

   Now we bring down the first digit after the decimal point (2), put a point at the quotient and then divide.

$$\begin{array}{rl} & 0.6 \\ 2) & \overline{1.21} \\ - & \underline{0} \\ & 1\,2 \\ - & \underline{1\,2} \\ & \;\;0 \end{array}$$

THIRD STEP

Continue with the division bringing down the next digit (1).

```
   0.60
 2)1.21
 - 0
   1 2
 - 1 2
     01
    - 0
      1
```

FOURTH STEP

Since the remainder is not zero and there are no more digits to bring down, we may add a zero at the end of the dividend and continue with the division

```
   0.605
 2)1.210
 - 0
   1 2
 - 1 2
     01
    - 0
      10
     -10
        0
```

2. $24.382 \div 12 = 2.0318\bar{3}$

$$\begin{array}{l}
\quad\;\; 2.031833 \\
12\overline{)24.382000} \\
-\ \underline{24} \\
\quad\;\; 0\ 3 \\
\quad -\ \underline{\;\;\, 0} \\
\qquad\;\; 38 \\
\qquad -\underline{36} \\
\qquad\quad 22 \\
\qquad\; -\ \underline{12} \\
\qquad\quad 100 \\
\qquad\;\; -\ \underline{\;96} \\
\qquad\qquad 40 \\
\qquad\quad -\ \underline{36} \\
\qquad\qquad\; 40 \\
\qquad\qquad -\ \underline{36} \\
\qquad\qquad\quad 4
\end{array}$$

We can see that from this point forward, the remainder will always be 4. That means that the decimal expression of the quotient is not terminating and repeating. In this case the pattern that repeats is the digit 3.

3. $135.36 \div 3 = 45.12$

$$\begin{array}{l}
\quad\; 45.12 \\
3\overline{)135.36} \\
-\ \underline{12} \\
\quad\;\; 15 \\
-\ \underline{\;\;15} \\
\qquad 0\ 3 \\
\qquad -\ \underline{3} \\
\qquad\;\; 06 \\
\qquad\; -\ \underline{6} \\
\qquad\quad 0
\end{array}$$

4. $23.5 \div 10 = 2.35$

$$23.10 \div 10 = \frac{235}{10} \div \frac{10}{1} = \frac{235}{10} \times \frac{1}{10} = \frac{235}{100} = 2.35$$

Notice that the result has the same digits as the dividend with the decimal point "moved" to the left one place. This is what happens when the divisor is 10.

In general, when dividing a decimal by a positive integer power of 10 (10, 100, 1000, and so on), the result will be the one corresponding to "moving" the decimal point of the dividend to the left, as many spaces as zeros we have in the power of 10. We understand that if we run out of places to continue "moving" the decimal point, then we can add zeros to the left hand side of the dividend as necessary.

5. $23.5 \div 1000 = 0.0235$

Now let's also consider negative decimals. In this case, all we need to do is to use the same rules for division of signs that we learned when working with integers.

29.4)Examples.

1. $31.5972 \div (-12) = -(31.5972 \div 12) = -2.6331$

```
      2.6331
   ----------
12)31.5972
 - 24
   ---
    7 5
 -  7 2
   ----
      39
     -36
      ---
       37
      -36
      ---
        12
       -12
      ----
         0
```

2. $(-65.32) \div 1000 = -0.06532$

3. $(-4.328) \div (-0.01) = 432.8$

Exercises

Compute.

1. $3.52 \times (-4.111)$

2. $(-2.8) \times (-9.7)$

3. 53.41×1000

4. 9×0.01

5. $3.25 \times 5.5 \times (-2.1)$

6. $(-0.1) \times (-0.1) \times (-0.01)$

7. $(-6.63) \times 0.01 \times (-2.2) \times 1000$

8. $(-5.5)^3$

9. $-(-2.11)^3$

10. $-(-33.1)^2$

11. $(0.1)^5$

12. $40.1951 \div (-11)$

13. $-5.932 \div (-12)$

14. $-893.16 \div 200$

15. $(-4.78999) \div 1000$

16. $35.8722 \div (-0.001)$

17. $0.1 \div 200$

18. $-69.797 \div 13$

19. $0.35888 \div (-9)$

LESSON 30

Now let's consider the case where the divisor is a positive decimal. We can "change" the division into one where the divisor is a positive integer. For example:

$$2.351 \div 1.2 = \frac{2351}{1000} \div \frac{12}{10} = \frac{2351}{1000} \times \frac{10}{12} = \frac{2351}{100} \times \frac{1}{12} = \frac{2351}{100} \div \frac{12}{1} = 23.51 \div 12$$

We notice that in this example, the original division is equal to the division where the new numbers are the result of "moving" the decimal point of the original numbers one place to the right.

It could be checked that all rules that are valid for operations with fractions, are also valid when using fraction notation. In other words, even if the numerator and denominator are not integers, the rules remain valid.

In particular we can write:

$$2.351 \div 1.2 = \frac{2.351}{1.2} = \frac{2.351 \times 10}{1.2 \times 10} = \frac{23.51}{12}$$

Now we only need to perform the last division.

30.1)Examples.

1. $2.351 \div 1.2$

   $2.351 \div 1.2 = 23.51 \div 12 = 1.9591\bar{6}$

```
      1.95916
12)23.51000
 -  12
    11 5
 -  10 8
       71
      -60
       110
      -108
         00
        - 12
          80
        -  72
            8
```

2. $59.7432 \div 1.1$

$$59.7432 \div 1.1 = \frac{59.7432}{1.1} = \frac{59.7432 \times 10}{1.1 \times 10} = \frac{597.432}{11} = 54.312$$

```
      54.312
  11)597.432
   -  55
      47
   -  44
       3 4
      -3 3
         13
        -11
          22
        - 22
           0
```

3. $2.81373 \div (-2.13)$

$$2.81373 \div (-2.13) = -(2.81373 \div 2.13) = -\left(\frac{2.81373}{2.13}\right) = -\left(\frac{2.81373 \times 100}{2.13 \times 100}\right) = -\left(\frac{281.373}{213}\right) = -1.321$$

```
        1.321
  213)281.373
   -  213
       68 3
   -   63 9
        4 47
   -    4 26
          213
        - 213
            0
```

4. $(-16.12) \div (0.13)$

$$(-16.12) \div (0.13) = -(16.12 \div 0.13) = -\left(\frac{16.12}{0.13}\right) = -\left(\frac{16.12 \times 100}{0.13 \times 100}\right) = -\left(\frac{1612}{13}\right) = -124$$

5. $(-1.246) \div (-0.02) = \dfrac{1.246 \times 100}{0.02 \times 100} = \dfrac{124.6}{2} = 62.3$

6. $5.1 \div 0.1$

$$5.1 \div 0.1 = \frac{5.1}{0.1} = \frac{5.1 \times 10}{0.1 \times 10} = \frac{51}{1} = 51$$

   Now the decimal point "moved" to the right! This occurred because the divisor was 0.1, which is a negative integer power of 10.

7. $5.1 \div 0.01$

$$5.1 \div 0.01 = \frac{5.1}{0.01} = \frac{5.1 \times 100}{0.01 \times 100} = \frac{510}{1} = 510$$

Combining Operations with Decimals.

Once again let's combine all operations, this time using decimals and integers. Keep in mind the order of operations.

1. Operations inside grouping symbols. If there are grouping symbols inside grouping symbols, then we need to work from inside out.

2. Exponentials.

3. Multiplications and divisions are associated LEFT TO RIGHT.

30.2)Examples.

1. $1 + 2 \times 0.2 = 1 + 0.4 = 1.4$

2. $(1 + 2) \times 0.2 = 3 \times 0.2 = 0.6$

3. $3.12 \times (0.1)^3 = 3.12 \times 0.001 = 0.00312$

4. $-2.43 + 5 \times 1.3 + (-10)^3(0.1)^2 = -2.43 + 5 \times 1.3 + (-1000)(0.01) = -2.43 + 6.5 - 10 = 4.07 - 10 = -5.93$

5. $(-2.43+5) \times 1.3 - (-10)^3 \times (-0.1)^2 = 2.57 \times 1.3 - (-10)^3 \times (-0.1)^2 = 2.57 \times 1.3 - (-1000) \times (0.01) = 3.341 + 10 = 13.341$

6. $-1.15 + 10(7-8.3)^2 + 5 \div 0.1 \times 10 = -1.15 + 10(-1.3)^2 + 5 \div 0.1 \times 10 = -1.15 + 10 \times 1.69 + 5 \div 0.1 \times 10 = -1.15 + 16.9 + 5 \div 0.1 \times 10 = -1.15 + 16.9 + 50 \times 10 = -1.15 + 16.9 + 500 = 15.75 + 500 = 515.75$

7.
$$\frac{(0.2)^2 \times (0.1)^3 \times 10^2}{0.2 \times 100} = \frac{0.2 \times 0.2 \times (0.1)^3 \times 100}{0.2 \times 100} = 0.2 \times 0.001 = 0.0002$$

8.
$$\frac{(1.2-0.2)(1.1 \times 0.1)}{0.11} = \frac{1 \times 0.11}{0.11} = 1$$

9.
$$\frac{11.1 + 0.1}{1.1 - 0.1} = \frac{11.2}{1} = 11.2$$

10.
$$0.3 - (0.7)(0.2) = 0.3 - 0.14 = 0.16$$

11.
$$\frac{1.21 - 2.3}{2 - 2 \times 6} = \frac{-1.09}{2 - 12} = \frac{-1.09}{-10} = \frac{1.09}{10} = 0.109$$

12.
$$\frac{1.21 - 2.3}{(2-2)6} = \frac{-1.09}{0 \times 6} = \frac{-1.09}{0} \qquad \text{This is undefined!}$$

13.
$$\frac{-3.1 + 0.1 \times 31}{5} = \frac{-3.1 + 3.1}{5} = \frac{0}{5} = 0$$

Percents.

As a simple application of decimals we are going to work with some basic problems involving percent.

Percent means "per hundred". To indicate percent we use the symbol "%". We can identify the percent with the fraction with numerator given by the number used in the percent, and denominator 100. We could also use the decimal number corresponding to the same fraction.

If we want to compute certain percent of a quantity q, all we have to do is multiply q by the fraction or the decimal corresponding to the percent.

For example:

1. To compute 5% we need to use either the fraction $\dfrac{5}{100}$ or the decimal 0.05 $(5 \div 100 = 0.05)$.
2. To compute 65% we need to use either the fraction $\dfrac{65}{100}$ or the decimal 0.65 $(65 \div 100 = 0.65)$.

Notice that:

1. The fraction or the decimal corresponding to 100% is actually the natural number 1. In other words, 100% of a quantity is the same quantity.
2. The fraction corresponding to 50% is $\dfrac{50}{100} = \dfrac{1}{2}$. In other words, 50% of a quantity is half of the quantity.

34.1)Examples.

1. What is 3% of 8?.

   It is 0.24

   $$0.03 \times 8 = 0.24$$

2. What is 50% of 60?

   It is 30

   $$\frac{1}{2} \times 60 = 0.5 \times 60 = 30$$

3. What 62.5% of 72?

   62.5% of 72 is 45.

$$0.625 \times 72 = 45$$

4. If a store offers a 35% discount on an item, and the original price was \$80, what is the new price of the item?.

   First we need to find what is 35% of 80.

$$0.35 \times 80 = 28$$

   Therefore, the new price is the result of subtracting 28 from the original price

$$80 - 28 = 52$$

   Then the new price of the item is \$52.

Exercises

Compute

1. $0.9435 \div 0.3$

2. $28.875 \div (-1.25)$

3. $(-3274.5 \div 1.11$

4. $(-5.392) \div (-0.02)$

5. $85938 \div 1.2$

6. $(-10.25) \div (-2.5)$

7. $63.212 \div (-0.01)$

8. $(-9.21) \div (0.001)$

9. $(-538) \div (-0.1)$

10. $0.02 \div 0.1$

11. $-1 + 3 \times (-0.1)$

12. $(-1 + 3) \times (-0.1)$

13. $1.12 \times (0.3)^2$

14. $10 - 5 \times 2.2 - (0.2)^3 \times 10^3$

15. $(10 - 5) \times 2.2 - (-0.2)^3 \times 10^2$

16. $-3 + 10(6 - 7.28) - 3 \div 0.3 \times 10$

17. $$\frac{(0.4)^3 \times (-0.1)^3 \times 10}{(0.4)^2 \times 10^2}$$

18.

$$\frac{(2.2-3.3)(2.5-2.50)}{2.567}$$

19.

$$\frac{2-0.1\times 3}{4-0.1\times 4}$$

20.

$$\frac{2(-2.11+1.1)\times 4}{2\times 1.01}$$

21. A store offers a 15% discount for an item. The original price was \$32. What is the price after the discount?

22. The price of one gallon of gas was \$2.50 and it was increased 2%. What is the new price?.

Homework/Class Work/Quiz 46

1. Find decimal notation for each of the following fractions.

   (a) $\frac{25}{1000}$

   (b) $\frac{17}{5}$

   (c) $\frac{9}{4}$

   (d) $\frac{435}{100}$

   (e) $\frac{5}{2}$

2. Fill in the blank using either '=', '<', or '>' as appropriate:

   (a) $\frac{468}{467} \quad 1$

   (b) $7.0 \quad \frac{700}{100}$

   (c) $8 \quad 7.99$

blank

Homework/Class Work/Quiz 47

1. Find decimal notation for each of the following fractions.
   (a) $\frac{332}{10000}$

   (b) $\frac{7}{5}$

   (c) $\frac{21}{4}$

   (d) $\frac{11}{8}$

   (e) $\frac{4}{3}$

2. Fill in the blank using either '=', '<', or '>' as appropriate:
   (a) $\frac{2}{3} \qquad 0.66666666$

   (b) $4.56 \qquad 4.123$

   (c) $-8 \qquad 0.99$

   (d) $-0.084 \qquad -0.12$

   (e) $5.01 \qquad \frac{5010}{1000}$

blank

Homework/Class Work/Quiz 48

Compute.

1. $58.3 + 9.28$

2. $27 - 9.91$

3. $13.251 + 35 + 7.3$

4. $-4.19 + 1.3$

5. $4.88 + (-15.9)$

6. $351.73 - 38.9$

7. $-21.8 - 285.987$

8. $0.89 - 1$

blank

Homework/Class Work/Quiz 49

Compute.

1. $-4.5-(-3.33)$

2. $8.295-(-25.3)$

3. $8.75-12.9-2.88+4$

4. $-3.3-3.3-2+4.4$

5. $-0.7+3.35-(-0.189)$

6. $3.35-3.35-4.27+0.991$

7. 3.1×5.78

8. 17.12×10

9. What is 7% of 25?

blank

Homework/Class Work/Quiz 50

Compute.

1. -8.71×7

2. $5.01 \times (-9.321)$

3. $-0.01 \times (-3.2)$

4. -12.1×3.12

5. -0.001×8

6. $0.01 \times (-4.31)$

7. $3.11 \times (-4) \times 1.22$

8. $-0.1 \times (-35.8) \times (-0.001)$

9. A store offers a 10% discount for an item. The original price was $\$68$. What is the price after the discount?

blank

Homework/Class Work/Quiz 51

Compute.

1. $(-8.3)^2$

2. $-(-1.1)^4$

3. $(3.1)^3$

4. $(-1.01)^3$

5. $3.63 \div 2$

6. $48.764 \div 6$

7. $-1.9 \div 20$

8. $48.6 \div (-3)$

blank

Homework/Class Work/Quiz 52

Compute.

1. $4.702 \div 2.4$

2. $3.52 \div 0.1$

3. $3.1 \div (-0.02)$

4. $-24.382 \div (-0.3)$

5. $-5.3 \div 100$

6. $6.57 \div (-0.09)$

7. $-52 \div 0.65$

8. $45 \div 72$

9. A store offers a 20% discount for an item. The original price was $\$125$. What is the price after the discount?

blank

Homework/Class Work/Quiz 53

Compute.

1. $2-4\times 1.51$

2. $(2-4)\times 1.51$

3. $(0.1)^4\times 51.35$

4. $-3.681+4\times 2.1+(-10)^4(0.1)^3$

5. $(-3.681+4)\times 2.1-(-10)^5(0.1)^3$

6. $-3.89+10(6-7.31)^2+6\div(-0.02)\times 10$

blank

Homework/Class Work/Quiz 54

Compute.

1. $0.4 - (0.9)(0.3)$

2. $0.01 - 0.346 \div 0.4$

3. $\dfrac{(0.5)^3 \times (0.1)^4 \times 10^3}{0.5 \times 1000}$

4. $\dfrac{(2.3 - 4.3)(2.38 \times 0.1)}{0.0238 \times 10}$

5. $\dfrac{35.4 + 0.01}{2.3 - 3 \times 0.1}$

6. A store offers a 5% discount for an item. The original price was $\$72$. What is the price after the discount?

blank