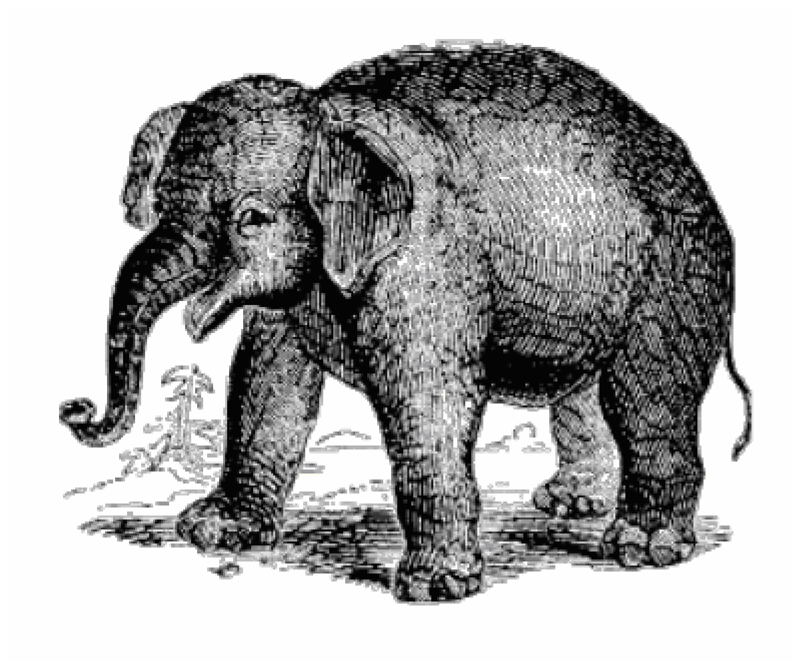
Math 016



Materials With Exercises

December 2011

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<u>Help available</u>: You can find help in the **Learning Lab for math in room B2-36** weekdays and in room B1-28 Monday- Thursday evenings and Saturdays on Main Campus. During fall and spring semesters, free, peer tutoring is available beginning with the second week of classes for all current CCP students and free, weekly workshops begin in the third week of the semester. The peer tutors are experienced CCP students who have taken many of the courses in which they tutor. Math specialists also tutor as well as lead workshops. Check at Regional Centers for days and times of services. Also, during summer sessions, offerings may vary.

Lesson 1

Topics: Natural numbers; Operations on natural numbers. Multiplication by powers of 10; Opposite operations; Commutative Property of Addition and Multiplication; Exponential notation; Order of operations; The meaning of the "=" sign.

Arithmetic is a branch of mathematics in which ***numbers, relations among numbers,*** and ***operations on numbers*** (like addition, subtraction, multiplication and division) are studied.

We will start with

<u>*Natural numbers*</u>

Numbers

0, 1, 2, 3, …

are called natural numbers.

(the dots ' … ' indicate that the list goes forever)

<u>*Basic operations on natural numbers*</u>

The basic operations on natural numbers are addition, subtraction, multiplication and division.

Addition: $6+2=8$

The result of addition is called the sum. For example, the sum of 6 and 2 is 8. Numbers that are being added are called **terms of addition.**

$$\underbrace{\mathbf{6} \quad + \quad \mathbf{2}}_{\textit{terms of addition}} \quad = \quad \underbrace{\mathbf{8}}_{\textit{sum}}$$

Subtraction: $6-2=4$

The result of subtraction is called the difference. For example, the difference of 6 and 2 is 4. Numbers that are being subtracted are called **terms of subtraction.**

$$\underbrace{\mathbf{6} \quad - \quad \mathbf{2}}_{\textit{terms of subtraction}} \quad = \quad \underbrace{\mathbf{4}}_{\textit{difference}}$$

Multiplication: $6 \times 2 = 12$

Other notations are also used: $6 \cdot 2$, $6(2)$, $(6)2$. **If no sign is explicitly displayed we assume multiplication (**for example, $6(2)$ means 6×2). **The result of multiplication is called the product**. The product of 6 and 2 is 12. Numbers that are being multiplied are called **factors**.

$$\underbrace{\mathbf{6} \quad \times \quad \mathbf{2}}_{\textit{factors}} \quad = \quad \underbrace{\mathbf{8}}_{\textit{product}}$$

Division: $6 \div 2 = 3$

Equivalently, we can write $\frac{6}{2} = 3$ (it is called fraction notation).

The result of division is called the quotient. The quotient of 6 and 2 is 3. The number we divide by is called a **divisor;** 2 is the divisor in this division (*).

$$\mathbf{6} \quad \div \quad \underbrace{\mathbf{2}}_{\textit{divisor}} \quad = \quad \underbrace{\mathbf{3}}_{\textit{quotient}}$$

An important fact about division.

The operation of division by 0 is not defined

For example, $\frac{7}{0}$, $\frac{5}{0}$, $0 \div 0$ are not defined (**but** $\frac{0}{5} = 0$ **is defined! We can divide 0 by any number different from 0**).

Numbers on which addition, subtraction, multiplication or division are performed are called operands. So, in all of the above examples, i.e. $6 + 2 = 8$, $6 - 2 = 4$, $6 \times 2 = 12$, $6 \div 2 = 3$, we can refer to the numbers 6 and 2 as to operands.

Example 1.1 Rewrite the following numerical expressions inserting a multiplication sign '$\times$' whenever multiplication is implied.

a) $3(7)$ b) $4(5+6)$

Solution:

a) $3(7) = 3 \times 7$

b) $4(5+6) = 4 \times (5+6)$

(*) Number that is being divided is called dividend (6 is the dividend in the division $6 \div 2 = 3$). The name is not used that often, so if you do not remember it, it is OK.

Example 1.2 Name the operation that is to be performed and also the name of its result, then evaluate the expression.

a) $71+2$ b) 35×0 c) $24\div 6$

Solution:

a) addition; the result of addition is called the sum; $71+2=73$
b) multiplication; the result of multiplication is called the product; $35\times 0=0$
c) division; the result of division is called quotient; $24\div 6=4$

Example 1.3 Which of the following operations are not defined? If the operation is defined, perform it, otherwise write "not defined".

a) $243\div 0$ b) $0\div 243$ c) $\frac{243}{243}$

Solution:

a) $243\div 0$ cannot be performed since operation of division by 0 is not defined. We must write "not defined".

b) $0\div 243=0$ (0 divided by any number different from 0 is equal to 0).

c) $\frac{243}{243}=1$

Example 1.4 Write the following statements as numerical expressions, and then evaluate them.

a) The product of 6 and 7
b) 8 subtracted from 12
c) The sum of 4 and 16
d) The quotient of 36 and 12

Solution:

a) $6\times 7=42$
b) $12-8=4$ (Please, notice the order of the numbers 12 and 8)
c) $4+16=20$
d) $36\div 12=3$

The opposite operations

Addition and subtraction are opposite operations. This means that one can be used to 'undo' the other. If we start with any number and add any number and then subtract the same number we added, we return to the number we started with. The same will happen if we first subtract and then add the same number.

$$5+2-2=5 \quad \text{also} \quad 5-2+2=5$$

Multiplication and division are opposite operations. Multiplying any number by another number and then dividing the result by the same number we multiplied by, will give us as a result the number we started with. Similarly, if we first divide and then multiply by the same number, we will get the number we started with (we need to remember that division by zero is not defined).

$$12\times 6\div 6=12 \quad \text{also} \quad 12\div 6\times 6=12$$

Example 1.5 Identify the operation that is performed in the following examples, and then find the opposite operation together with the appropriate operand to "undo it".

a) $14-6$ b) $27 \div 9$

Solution:

a) The opposite operation to subtraction is addition. We need to add 6 to "undo" the operation $14-6+6=14$.

a) The opposite operation to division is multiplication. We need to multiply by 9 to "undo" the operation $27 \div 9 \times 9 = 27$.

Commutative property of multiplication and addition

We know that whether we add 7 to 3, or 3 to 7, the result, in both cases, will be the same. Multiplication has the same property, for example 6 times 5 is equal to 5 times 6. In other words, we can change the order in which we add or multiply numbers. Such property is called the commutative property and it means that it does not matter in which order we do things (*).

The commutative property of addition (multiplication) says it does not matter in which order we add (multiply) numbers.

For example,
$$4+2=2+4$$
$$4 \times 2 = 2 \times 4$$

It is important to note that this property does not apply to all the operations. It does apply to addition and multiplication but does not apply to subtraction or division.

Subtraction and division are NOT commutative. We cannot change the order in which we perform subtraction or division.

For example,
$$8-5 \neq 5-8$$
$$20 \div 5 \neq 5 \div 20$$
(the sign '$\neq$' means that the objects are not equal).

Multiplication by powers of 10

Let us recall the following fact about multiplying any number by 10, 100, 1000, … (numbers whose first digit left to right is one, followed by zeros). To multiply a natural number by 10, we rewrite the number and add one zero to the right of the number.

(*) There is another important property of multiplication and addition: the associative property. It is needed because addition and multiplication are binary operations which means that we can only do multiplication or addition with two numbers at one time. So what does 1+2+3 (or $4 \times 5 \times 6$) mean? We can either say (1+2)+3 or 1+(2+3) where the parentheses indicate which operation is to be performed first. It turns out that we get the same answer both ways. The associative property of multiplication (addition) tells us that when we multiply (add) three numbers, it does not make any difference which numbers we "group together", we will always get the same answer. For example, $(4 \times 5) \times 6 = 4 \times (5 \times 6)$ or $(1+2)+3=1+(2+3)$. Subtraction and division do not have the associative property.

$3 \times 10 = 30$

$10 \times 47 = 470$

If we multiply by 100, we add two zeros to the right of the number.

$100 \times 45 = 4500$

If by 1000, three zeros are needed.

$347 \times 1000 = 347000$

In general, we add as many zeros to the right as appears in the natural number by which we multiply.

$52 \times 100000 = 5200000$ (five zeros are added since there are five zeros in 100000)

Example 1.6 Perform the following operations. Please, make sure that you display your answer in a correct way, using the '=' sign.

a) 2×100
b) 33×10
c) 1000×540

Solution:

a) $2 \times 100 = 200$
b) $33 \times 10 = 330$
c) $1000 \times 540 = 540000$

Exponential notation as a shorthand for repeated multiplication

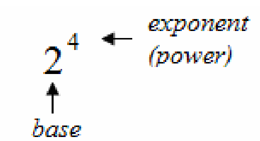
Suppose that we want to perform the following operation 3+3+3+3+3+3. Instead of writing such a long expression, we can use the multiplication to denote the above operation. Adding 3 six times to each other means 3 times 6.

$$3 + 3 + 3 + 3 + 3 + 3 = 3 \times 6$$

We can view multiplication as a shorthand for repeated addition. There is also a shorthand for repeated multiplication. It is called exponentiation. Instead of writing $3 \times 3 \times 3 \times 3 \times 3 \times 3$, we can write 3^6.

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6$$

The expressions of the form $3^6, 2^4, 5^{24}$ are called **exponential expressions**. In the exponential expression 2^4, the number 2 is called the **base,** and 4 is called the **exponent (or power)**.



Notice the relative position of the base and the exponent. The exponent is not in the same line as the base, it is written slightly higher.

The expression 2^4 means "2 multiplied by itself as many times as the exponent indicates".

Thus, for example,

$$2^1 = 2$$
$$2^2 = 2 \times 2$$
$$\dots$$
$$2^5 = \underbrace{2 \times 2 \times 2 \times 2 \times 2}_{5\ times}$$
$$2^6 = \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2}_{6\ times}$$
$$\dots$$

The expression 2^6 is read "two raised to the sixth power". In case of expressions raised to the second power, like 6^2, 3^2, they can also be read 'six squared' or 'three squared', respectively. Expressions raised to the third power 4^3 are often read 'four cubed'.

Example 1.7 Expand 9^3, meaning write without exponential notation.

Solution:
$9^3 = 9 \times 9 \times 9$

Example 1.8 Write the following expressions using exponential notation whenever possible.

a) $7(7)(7)(7)(7)$
b) $6 \times 6 \times 6 \times 8 \times 8$
c) $2 + 2 + 2 + 2 \times 2 \times 2$

Solution:

a) 7^5
b) $6^3 \times 8^2$
c) Exponentiation is used only if the same number is multiplied several times. When 2 is added, we cannot use the exponential notation. So,
$2 + 2 + 2 + \mathbf{2} \times \mathbf{2} \times \mathbf{2} = 2 + 2 + 2 + 2^3$

Example 1.9 Perform the following operations.

a) 3^2
b) 2^3
c) 24^1
d) 10^5

Solution:

a) $3^2 = 3 \times 3 = 9$
b) $2^3 = 2 \times 2 \times 2 = 8$
c) $24^1 = 24$
d) $10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100000$

Suppose that we are asked to perform the operation $2+3\times 4$. Notice that more than one operation needs to be performed (addition and multiplication). The question is which operation should be performed first. The following convention has been commonly adopted.

When evaluating arithmetic expressions, the order of operations is

1) **Perform all operations in parentheses first.**
2) **Next, do any work with exponents.**
3) **Perform all multiplications and divisions, working from left to right.**
4) **Perform all additions and subtractions, working from left to right.**

If a numerical expression includes a fraction bar, perform all calculations above and below the fraction bar before dividing the top by the bottom number.

Sometimes, instead of parentheses ( ), we use brackets [ ], or braces { }. There is no difference in meaning among those three. The reason why they are used sometimes is when the expression is so complicated and contains so many parentheses that introducing different types will make it easier to read the expression.

Example 1.10 Name the first operation that should be performed in accordance to the order of operations. Then evaluate the expressions performing *one operation at a time.* Please, make sure that you display your answer in a correct way, using the '=' sign.

a) $2+3\times 4$ b) $(2+3)\times 4$ c) 3×4^2

Solution:

a) The first operation that has to be performed is multiplication.
$2+3\times 4 = 2+12 = 14$

b) The first operation that has to be performed is addition (because of parentheses around addition). $(2+3)\times 4 = 5\times 4 = 20$

c) The first operation that has to be performed is exponentiation.
$3\times 4^2 = 3\times 16 = 48$

Example 1.11 Perform the following operations, if possible. If not possible, explain why it is not possible. Please, make sure that you display your answer in a correct way, using the '=' sign.

a) $\frac{4}{3-3}$ b) $(4-3)\times 5$ c) $7+10\div 5$ d) $12\div 4\times 3$

Solution:

a) $\frac{4}{3-3} = \frac{4}{0}$ Since division by 0 is not defined it is not possible to perform this operation.

b) $(4-3)\times 5 = 1\times 5 = 5$

c) $7+10\div 5 = 7+2 = 9$

d) $12 \div 4 \times 3 = 3 \times 3 = 9$

Example 1.12 Write the following statements using as one numerical expression, and then evaluate.
a) First find the product of 2 and 5, and then raise the result to the fourth power
b) First subtract 3 from 10, and then find the sum of the result and 12

Solution:
a) $(2 \times 5)^4 = 10^4 = 10 \times 10 \times 10 \times 10 = 10000$
b) $10 - 3 + 12 = 7 + 12 = 19$

Use of the correct mathematical language and the meaning of equal sign

Any time we evaluated expressions we used '=' sign. The equal sign " =" is placed between two symbols or expressions to indicate that they have the same value. Let us consider the following evaluation $5(3 \times 2) = 5 \times 6 = 30$ Notice, that what we did was substituting 6 for 3×2.

$$5 \boxed{(3 \times 2)} = 5 \times \boxed{6} = 30$$

We replace 3×2 by 6

We could do that because 6 is equal to 3×2. If two things are equal, then one can be put in place of the other and nothing will change (see also Mistake 1.4 at the end of this Lesson 1). The following fundamental principle underlies the process of evaluation.

If two quantities are equal, you can always substitute one for the other. **"Equals can be substituted for equals"**

Example 1.13 Apply the principle 'equals can be substituted for equals' to evaluate
$(1234 \times 5678) \times 1000$
if you know that $1234 \times 5678 = 7006652$.

Solution:
Since 'equals can be substituted for equals', we can substitute 7006652 for 1234×5678 to get

$$\boxed{(1234 \times 5678)} \times 1000 = \boxed{7006652} \times 1000 = 7006652000$$

We replace these two

Example 1.14 If you know that $46 \times 23 = 1053$, what is the result of $1053 \div 23$?

Solution:

Since ‘Equals can be substituted for equals’, we can replace 1053 with 46×23. This substitution gives us

$$\boxed{1053} \div 23 = \boxed{46\times 23} \div 23 = 46$$

We replace these two

Division by 23 is the opposite operation to multiplication by 23 and one can be used to “undo” the other. Thus the final result is 46.

Finally, let us recall the following property of “=” sign. It is symmetrical. If 2=1+1, we equivalently can write it as 1+1=2. Both statements have exactly the same meaning.

Common mistakes and misconceptions

Mistake 1.1

It is NOT true that we always perform addition first and subtraction afterward. Addition and subtraction are of equal priority. We perform these operations in order of their appearance from left to right. For example, in $10-3+2$ we would first subtract 3 from 10, and only then add 2.

$$\underbrace{10 - 3}_{\text{we subtract first}} + 2 = 7 + 2 = 9$$

Adding $3+2$ first would give us the wrong answer.

Similarly, multiplication and division are of equal priority, we perform them in order of their appearance from left to right.

$$\underbrace{12 \div 2}_{\text{we divide first}} \times 3 = 6 \times 3 = 18$$

Since the first operation from the left is division, we divide before multiplying.

Mistake 1.2

Do not confuse exponentiation with multiplication. For example,

$$2^7 \neq 2\cdot 7$$

In 2^7, number 2 must be multiplied by itself 7 times, $2^7 = 2\times 2\times 2\times 2\times 2\times 2\times 2$, not by 7.

Mistake 1.3

There is a difference between 4×3^2 and $(4\times 3)^2$. In the expression 4×3^2 only 3 is squared and then the result is multiplied by 4.

$$4\times 3^2 = 4\times 9 = 36$$

In the expression $(4\times 3)^2$, because of parentheses, we first multiply 4 by 3, and then raise the result to the second power.

$$(4\times 3)^2 = 12^2 = 144$$

Mistake 1.4

When evaluating $2\times10+1$, it is INCORRECT to write

$$2\times10+1=20=21 \qquad (20\neq21)$$

Instead, one should write

$$2\times10+1=20+1=21$$

Numbers or expressions not involved in the operation that is being carried out *must always be rewritten*. Equal sign means that the quantities on either side are equal.

Exercises with Answers (For answers see Appendix A)

Ex.1 Fill in the blanks.

In the product 3×7, 3 and 7 are called _____________ .

Numbers 0, 1, 2, 3, ... are called ___________ numbers.

Operation of division by __________ is not defined.

Zero divided by any number except _____________ is equal to ___________ .

The result of addition is called _________. The result of multiplication is called _________ .

In the expression $15\div5=3$, 5 is called __________, 3 is called ___________.

In the sum $4+5$, 4 and 5 are called ___________ of addition.

Ex.2 Which of the following sentences are true? (There may be more than one true statement.)

a) Operation of multiplication by zero is not defined.
b) Zero divided by any number except zero is zero.
c) Operation of division by zero is not defined.
d) Any number multiplied by zero is zero.

Ex.3 Rewrite the following numerical expressions inserting a multiplication sign "×" whenever multiplication is implied.

a) $4(5)$ b) $(4)(7)$
c) $3(2-5)$ d) $8(9)+4(6)$
e) $34-3(4)$ f) $(10-2)(4\div2)$

Ex.4 Write the following division problems using fraction notation. Do not evaluate then.

a) $456\div33$
b) $56\div3$

Ex.5 First name the operation that is to be performed together with the name of its result, then evaluate each expression.

a) $33\div3$ b) $14-3$
c) $6(8)$ d) $123+1$
e) $\frac{25}{5}$ f) 2×5

Ex.6 Write the following statements as numerical expressions, and then evaluate them.

a) The product of 3 and 8 b) 3 subtracted from 73
c) The sum of 100 and 2 d) The product of 5 and 0

e) The quotient of 10 and 2   f) 15 divided by 3

Ex.7 If the operation can be performed, please, perform it. If not, write "not defined".

a) $\frac{15}{0}$ b) $\frac{0}{15}$

c) $\frac{0}{0}$ d) $\frac{15}{15}$

e) 0×15 f) $\frac{15}{1}$

Ex.8 a) What number should be multiplied by 5768 to give the answer 5768?
$5768\times$ ________ $=5768$
b) By what number can 5768 be divided by to give the answer 5768?
$5768\div$ ________ $=5768$
c) What number can be added to 5768 to give the answer 5768?
$5768+$ ________ $=5768$
d) What number can be subtracted from 5768 to give the answer 5768?
$5768-$ ________ $=5768$

Ex.9 Identify the operation that is performed in the following examples and then find the opposite operation together with the appropriate operand to "undo it" (see Example 1.5 for a sample solution).

a) $15-9$ b) $32\div8$

c) $2+5$ d) 4×12

Ex.10 If $46\times891=40986$ **,** then what is 891×46 equal to ? Explain why.

Ex.11 Perform the following operations. Please, make sure that you display your answer in a correct way, using the '=' sign.

a) 2×100 b) 33×10

c) 1000×5 d) 100×450

e) 100×1000 f) 0×10000000

Ex.12 The same number is to be placed in each of the three boxes below. (There may be more than one true statement.)

$$\square\times\square=\square$$

Which of these numbers would make a true statement?

a) 0
b) 10
c) 5
d) 1

Repeat the same exercise with

$$\frac{\square}{\square}=\square$$

Ex.13 Fill in the blanks.
In the exponential expression 5^{25}, 5 is called _________, 25 is called _________ .

Ex.14 Write the following statement using mathematical symbols (do not perform the operations).
a) five raised to the fourteenth power
b) twelve cubed
c) ten squared
d) exponential expression with base equal to 7 and exponent equal to 13

Ex.15 Expand , meaning write without exponential notation. (do not perform the operations).

a) 12^3 | b) 251^4
c) 7^1 | d) 8^5

Ex.16 Write the following expressions using exponential notation whenever possible. Do not evaluate.

a) $4\times4\times4\times4\times4\times4$
b) $(58)(58)(58)(58)$
c) $3\times3\times3\times3\times8\times8$
d) $7(7)(7)(8)(9)$
e) $33(3)(3)(3)(3)$
f) $4+4+4+4\,(4)$
g) $(2+5)(2+5)(2+5)$
h) $(12+8)(12+8)+12+8$

Ex.17 Perform the following operations. Please, make sure that you display your answer in a correct way, using the '=' sign.

a) 9^2
b) 2^4
c) 10^5
d) 4^3
e) 100^3
f) 10000^2
g) 56^1
h) 1^{56}

Ex.18 A light year is approximately 6×10^{12}. What is another way of writing this number?
a) 6 000 000 000 000
b) 6 000 000 000
c) 600 000 000 000
d) 60 000 000 000 000

Ex.19 To what power do we need to raise the following numbers to get 1 000 000?
a) 10 b) 100 c) 1 000 000

Ex.20 Is exponentiation commutative? Explain your answer.

Ex.21 Name the operation that has to be performed first according to the order of operations. Then evaluate the expressions performing *one operation at a time.* Please, make sure that you display your answer in a correct way, using the '=' sign.

a) $14\div7\times2$
b) $(8+2)6$
c) $10-5+2$
d) $14-2\times3$
e) $(2\times4)^2$
f) 3×10^4

Ex.22 Write four different number sentences following both of these rules.

(i) Each number sentence must show a different way of getting the number 42.
(ii) Each number sentence must contain at least two different arithmetic operations.
For example, $42 = 4\times10+2$

Ex.23 For each pair of the following expression, the second one is obtained from the first one by rewriting it without parentheses. Evaluate each of them to find out if inclusion of parentheses gives a different result.

a) $(3+7)\times8$ $\quad$ $3+7\times8$
b) $(2\times50)^3$ $\quad$ 2×50^3
c) $20\div(5\times2)$ $\quad$ $20\div5\times2$
d) $14-(2+3)$ $\quad$ $14-2+3$

Ex.24 Write each of the following statements as one numerical statement and then evaluate. Remember about the use of parentheses, but use them only when needed.

a) First find the sum of 3 and 2, and then multiply the result by 1000.
b) First find the quotient of 44 and 44, and then multiply the result by 16.
c) First subtract 8 from 10, and then add 5 to the result.
d) First find the product of 3 and 6, and then add it to 7.
e) First subtract 1 from 7, and then divide 18 by the result.
f) First multiply 5 and 2, then raise the result to the seventh power.
g) First cube 2, and then multiply the result by 10.
h) First divide 36 by 9, and then subtract the result from 18.
i) First double 7, and then subtract it from 30.
j) First find the value of the exponential expression with the base 1 and the exponent 5, and then add 2 to it.

Ex.25 Perform the following operations any time they are defined. If they are not defined, please write "not defined". Please, make sure that you display your answer in a correct way, using the '=' sign.

a) $40\div2\times5$ $\quad$ b) $33-3\times5$
c) $(2+1)\times4$ $\quad$ d) $30-6\div2$
e) $(30-6)\div2$ $\quad$ f) $\dfrac{8+6}{7}$
g) 4×2^3 $\quad$ h) $(4\times2)^3$
i) $4+6\div2$ $\quad$ j) $5+2-7$
k) $11-2^3$ $\quad$ l) $5+10\times2$
m) $(5+10)\times2$ $\quad$ n) $100\times2\times7$
o) $(36-36)\div0$ $\quad$ p) $4-3+15$
q) $100\times(981-2)$ $\quad$ r) $10\times2+3\times6$
s) $10\times(2+3)\times6$ $\quad$ t) $(5-2)(4+3)$
u) $22+4\times2$ $\quad$ v) $7+4^2$
w) $398+2+10\div5$ $\quad$ x) $\left(\dfrac{43-1}{7}\right)^2$
y) $15-(2+4)$ $\quad$ z) $12\div3-2$

Ex.26 You know that $2+2=4$. Can you write $4=2+2$ instead? Do both statements have the same meaning?

Ex.27 Compare each of the following two expressions and write '=' between them, if they are equal, or '≠' if the expressions are not equal.

a) 7^5 _____ 5×7

b) 7^5 ______ $7\times 7\times 7\times 7\times 7$

c) 7^5 ______ $5\times 5\times 5\times 5\times 5\times 5\times 5$

d) 4×3^5 _____ $(4\times 3)^5$

e) 65213×678 ___ 678×65213

f) 9×34^3 ______ $34^3\times 9$

g) $12+690\div 345$ ___ $12+(690\div 345)$

h) $6534-356$ ____ $356-6534$

i) 2345×0 ______ 0×45

j) $\dfrac{35}{35}$ ______ 0

Ex.28 a) If $4758+1236=5994$, then what is $5994-1236$ equal to? Explain how you arrived at you answer.

b) If $50076\div 52=963$, then what is 963×52 equal to? Explain how you arrived at you answer.

Ex.29 If you know that $42^3=74008$, apply the principle "Equals can be substituted for equals' to evaluate.

a) 10×42^3 b) $42\times 42\times 42$ c) 42^3+1

d) $42^3\times 1000$ e) $74008-42^3$ f) 1×42^3

Ex.30 If you know that $436\times 78=34008$, apply the principle "Equals can be substituted for equals' to evaluate.

a) 78×436 b) $436\times 78\times 10$ c) $436\times 78+1$

d) $436\times 78-34008$ e) $\dfrac{436\times 78}{1}$ f) $436\times 78-8$

Ex.31 Replace the triangle Δ with a number to make the statement true.

a) $\Delta\times 10=3480$ b) $3^{\Delta}=27$ c) $459+\Delta=781+459$

d) $15\div\Delta=5$ e) $7-2=\Delta+1$ f) $57\times\Delta=5700$

g) $\dfrac{135}{\Delta}=1$ h) $\Delta^2=36$ i) $\dfrac{823}{\Delta}=823$

j) $24\div 6=\Delta+2$ k) $67\times\Delta=835\times 67$ l) $25+\Delta=30$

m) $\Delta\div 2=50$ n) $5\times 7=30+\Delta$ o) $746-\Delta=740$

Lesson 2

Topics: Integers; Meaning of ‘<’, ‘>’; Comparing integers; Number line; Opposite numbers; Adding and subtracting integers.

Suppose that you have 15 dollars and would like to buy a book that costs 14 dollars. You have enough money to buy it. But if the book costs 16 dollars, you no longer can afford it. The only way you can do it is to borrow one dollar. If you do borrow one dollar in order to buy the book, you would then have one dollar less than nothing. In fact, you would be in debt for one dollar or it could be said that you have “minus one dollar”. Numbers like this are called integers.

Integers

Integers are numbers of the form

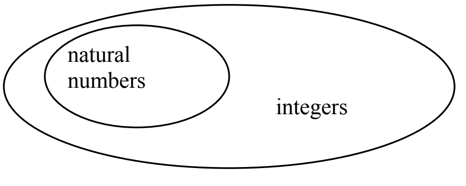
$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

(dots ‘ … ‘, as always, indicate that the list goes forever).

We introduce integers, because sometimes (not always), the operation of subtraction of natural numbers cannot be performed (i.e., the result of subtraction of two natural numbers is not necessarily a natural number, for example $2-5$). With the help of integers, any subtraction can be performed.

Numbers $\ldots, -3, -2, -1$ are called **negative integers**. Numbers 1, 2, 3, … are **positive integers. 0 is neither positive nor negative**. Notice that 0 together with the positive integers are natural numbers. Any natural number is also an integer. To get integers, we just add all the negative numbers to all natural numbers. The set of natural numbers is contained in the set of integers.

It looks something like this



Integers are an “extension” of natural numbers. All rules that are established for natural numbers (order of operation, commutative property of addition and multiplication, the fact that multiplication and division as well as addition and subtraction are opposite operations) are also true for integers.

Comparing integers

Any two integers can be compared. One integer is either greater than, less than, or equal to another integer.

We use symbols “<”, “>”, and, of course “=” .

$3 < 5$ means that 3 is less than 5.
$5 > 3$ means that 5 is greater than 3.

The symbol points to a smaller number.

Notice that if 3 is less than 5 then 5 is more than 3, thus
$3 < 5$ is equivalent to (has the same meaning as) $5 > 3$.

We probably all agree with the following statements.

A person who has 5 dollars has more than the one who has 3 dollars: $5 > 3$.
A person who has 5 dollars has more than the one who owes 3 dollars: $5 > -3$
A person who owes 3 dollars (is better off) has more than the one who owes 5 dollars. The more you owe, the less you have: $-3 > -5$.

Notice that $3 < 5$
but $-3 > -5$

We are going to adopt the following rules for comparison of integers.

HOW TO COMPARE INTEGERS
We compare two positive integers like we compare natural numbers. For example, $5 > 3$
Any positive number is greater than zero or any negative number. For example, $5 > -3$, $5 > 0$
To compare two negative numbers ‘drop’ the minus sign in front of each, compare the resulting positive integers and then reverse the inequality sign. For example, to compare -3 and -5, compare 3 and 5 first: $3 < 5$, and then switch “<” to “>” to get: $-3 > -5$.
Any negative number is less then zero. For example, $-5 < 0$

<u>Example 2.1</u> Fill in blanks using the following symbols ‘<”, ‘>’, ‘=’ as appropriate.

a) 0 _____ -7 b) -8 ____ -9

Solution :

a) Any negative number is always less than zero, thus $0 > -7$.

b) Since both numbers are negative, we first compare 8 and 9 to get $8 < 9$. Then, we reverse the inequality sign to obtain $-8 > -9$.

Example 2.2 Put the integers $-20, 0, 9\ -16, -11, 2$ in order from least to greatest.

Solution:
First let us put positive integers in order, $2 < 9$. To compare negative integers, we "drop" their minus signs and compare the resulting positive integers 11<16<20. To get the comparison of negative numbers, we reverse inequality signs. $-11 > -16 > -20$. Finally, any positive number is greater than 0 and any negative number is less than 0. Thus, $-20 < -16 < -11 < 0 < 2 < 9$.

Example 2.3 Replace each star with a digit so that the inequality is true.

a) $78 < *1$ b) $-52 > -5*$

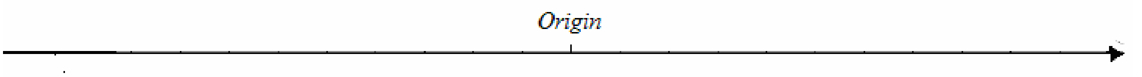
Solution:

a) All numbers less than 7 would not work. 7 does not work, because 78 is more than 71. We need to choose either 8 or 9 to get 81 or 91.

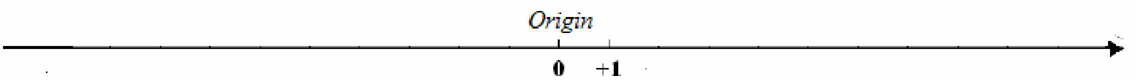
b) The inequality $-52 > -5*$ is true only if $52 < 5*$. For the inequality $52 < 5*$ we can choose any number greater than 2. Since numbers 3, 4, ... , 8, 9 satisfy the inequality $52 < 5*$, we can conclude that they also satisfy $-52 > -5*$.

Number line. Comparing integers with help of a number line

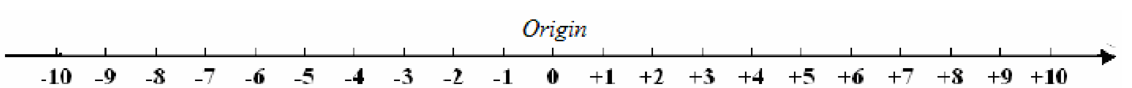
Integers can be ordered. We can visualize the order of integers with the help of a number line, a horizontal line that extends from a special point called the **origin** in both directions.



The origin corresponds to the number 0. The number 1 is placed to the right of 0.

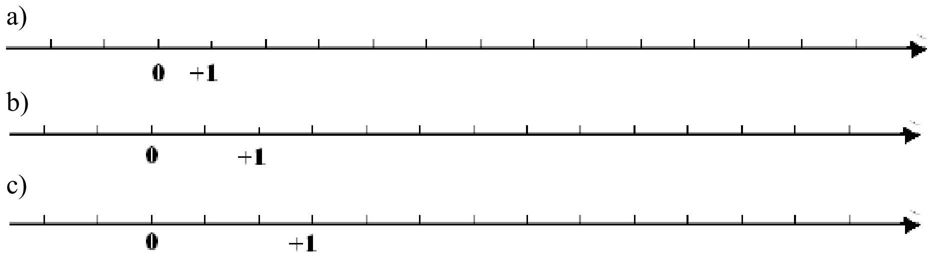


The distance between point 0 and 1 defines a unit of length. Once 0 and 1 have been placed on the line, this forces everything else. Each integer has a specific place on it. The number 2, for example, corresponds to a point 2 units away from the origin to the right, -3, corresponds to a point on the line 3 units away from the origin to the left.

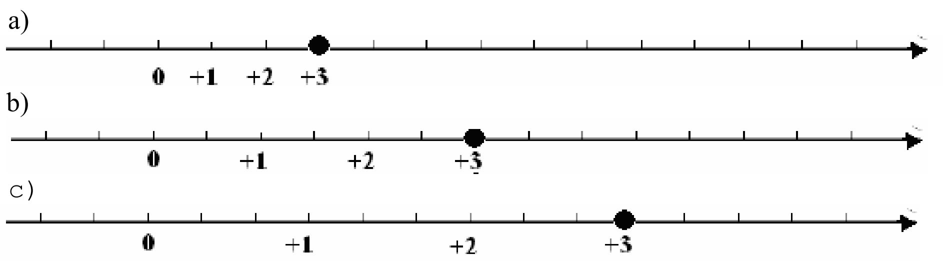


The distance between two successive integers is always the same and is equal to one unit. So, the distance between 1 and 2 is the same as between 2 and 3 or -5 and -6.

Example 2.4 Plot 3 on each of the following number lines.

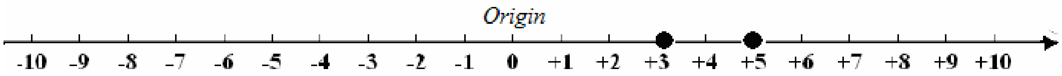


Solution:
The distance between 0 and 1 defines one unit. Number 3 should be placed 3 units to the right of 0. We get the following plots.



The arrowhead on the right-hand side of the line indicates the direction in which numbers increase. So, as we move from left to right on the number line, the numbers get larger. If one number is to the right of another number, it is larger than that number. So, when comparing two integers, we can do that by placing them on a number line.

For example, 3 and 5 are plotted on a number line below.

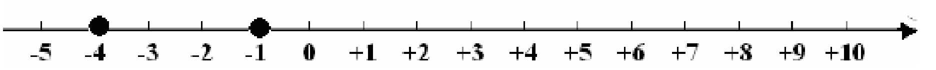


Since 3 is to the left of 5, it means 3 is less than 5, so we write $3 < 5$. We can also conclude that 5 is to the right of 3, thus 5 is greater than 3 and write $5 > 3$.

3 < 5 has the same meaning as 5 > 3.

Example 2.5 Plot -4 and -1, and determine which one is greater.

Solution:



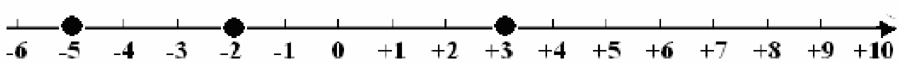
since -1 is to the right of -4, thus $-1 > -4$; -1 is greater.

Example 2.6 Plot the following three numbers $3, \; -2, \; -5$ on a number line and then find a number such that

a) it is greater than all the above three numbers.

b) it is greater than two of the the above three numbers and less than one of them.

Solution:

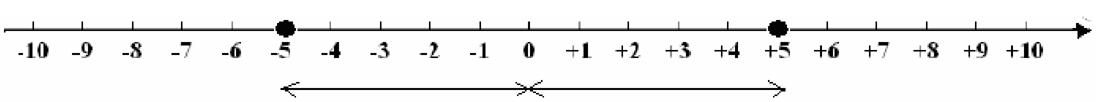


a) The numbers greater than $3, -2, -5$ must be to the right of the numbers $3, -2, -5$, thus any number $4, 5, 6, ...$ would be a correct answer.

b) The numbers greater than two of those numbers must be to the right of $-2, -5$. They must also be to the left of 3, as numbers less than 3. Thus any number among $-1, 0, 1, 2$ would be a correct answer.

Opposite numbers

For each non-zero integer, there is the integer, called opposite, that is the same distance from 0, but in opposite direction. For example, -5 is the opposite of 5.



The distance from 0 of 5 and −5 is the same.

The opposite of -3 is 3, the opposite of -6 is 6. The opposite of 0 is 0. Zero is the only number that is the opposite of itself.

Example 2.7 Find the opposite numbers of

a) 7 b) -8

Solution:

a) -7

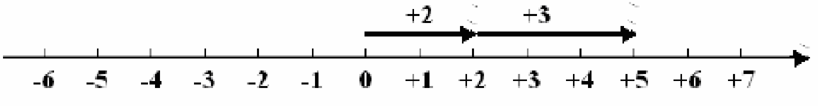
b) 8

We will first consider the following situations.

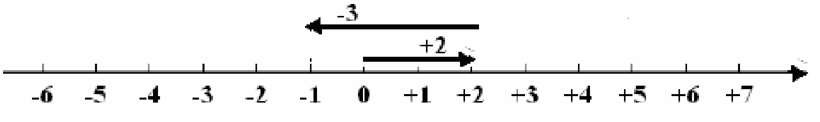
- Suppose that a gambler wins 2 dollars and then, later, he wins an additional 3 dollars. What is his total? The total amount won is 5 dollars. In other words, $2+3=5$
- Suppose that the gambler wins 2 dollars, and later he loses 3 dollars. What is his total? Notice that since 3 is more than 2, he lost more than he won, and thus the total will be with a minus sign (overall he lost). We subtract winnings from his losses ($3-2=1$), and use the minus sign in the answer, thus $2-3=-1$
- Now, the gambler lost 2 dollars and then, later, he won 3. To find his total, we subtract his losses from his winnings ($3-2=1$) and notice that, since 3 is more than 2, he won more than he lost, so the sign will be positive. We get, $-2+3=1$
- Finally, suppose that the gambler lost twice, first he lost 2 dollars and later, an additional 3 dollars. We need to add his losses and keep the minus sign (overall he lost!), $-2-3=-5$

Let us examine another interpretation of operations $2+3$, $2-3$, $-2+3$, and $-2-3$. This time we will use a number line and interpret any operations as 'a movement on the number line'. We will start the movement at zero. Any positive integer will indicate a movement to the right, any negative one, a movement to the left.

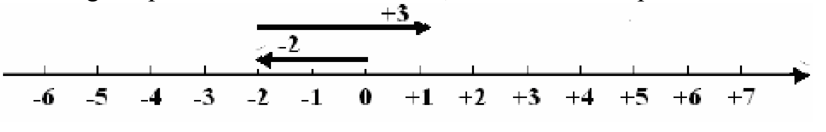
- $2+3=5$ Since the first number is 2 (a positive number), we will move two units to the right, starting from 0. The second number 3 moves us three units farther to the right. As a result we land on 5. Thus, $2+3=5$.



- $2-3=-1$ The signs of the numbers are different, so we move both left and right. The sign of the answer depends on which number is bigger. We first move 2 units to the right. Then, we reverse the direction and move 3 units to the left. It is like a tug-of-war. The bigger number wins, and "lends" its sign to the answer. -3 "won" over 2, so the answer is negative $2-3=-1$.



- $-2+3=1$ We move 2 units to the left, and then 3 units to the right. Once again the sign of the answer depends on which number is bigger. Because 3 is bigger than 2, we go to the right far enough to pass the zero and land on 1, so the answer is positive $-2+3=1$.



- $-2-3=-5$ Since this time we always go to the left (first 2 units, then 3 units to the left), the answer has to be negative. Indeed, we land on -5, so $-2-3=-5$.

Operations $2+3, -2+3$ are addition of integers but when we write $2-3$ or $-2-3$ addition is also assumed. Those statements are equivalent to

$$2-3=2+(-3)$$
$$-2-3=-2+(-3).$$

Before we proceed any further, let us observe how the parentheses are used in the above expressions. They are placed according to the following convention.

Any time two operation signs (+ and – in these cases) are next to each other, parentheses must be used to separate those two signs.

We write:	We *do not* write :
$2+(-1)$	$2+-1$

Now, we are ready to present the rules of additions of integers.

HOW TO ADD INTEGERS

When **adding two integers of the same sign** (two positive or two negative numbers), take the sum of the numbers. The sign of the result will be the same as the sign of numbers (if both positive, the result is positive, if both negative, the result is negative). For example,

$2+3=5$ We add 2 and 3 to get 5. Both numbers are positive, thus the result is positive.

$-2-3=-2+(-3)=-5$ We add 2 and 3 to get 5. Both numbers are negative, thus the result is negative.

When **adding a negative number to a positive number** (or a positive number to a negative number), take the difference of the numbers. The sign of the result will be the same as the sign of the 'larger' number. For example,

$-2+3=1$ We subtract 2 from 3 to get 1. Number 3 is larger than 2 and is preceded by a plus sign, thus the result is positive.

$2-3=2+(-3)=-1$ We subtract 2 from 3. Number 3 is larger than 2 and is preceded by a minus sign, thus the result is negative.

Example 2.7 Perform the indicated operations. Write your answer using '=' sign.

a) $-5-7$
b) $-4+5$
c) $3-9$
d) $0-8$

Solution:

a) $-5-7=-12$ Both numbers are negative, so we add 5 and 7 and the answer is negative.

b) $-4+5=1$ The sign of the answer is the same as the sign of the bigger number (5 is bigger than 4), so it is positive. We subtract the smaller number from the bigger $5-4=1$.

c) $3-9=-6$ The sign of the answer is the same as the sign of the bigger number, so it is negative. We subtract 3 from 9 to get 6.

d) $0-8=-8$

Addition and subtraction of integers

The statement $2-3=2+(-3)$ might be a little bit confusing, because in the case of $2-3$ you would probably say it is a subtraction, and if you see $2+(-3)$, addition. Well, you can view this problem as subtraction of 3, or as an addition of the opposite of 3, -3. **Any subtraction is equivalent to addition of the opposite.** Likewise, **any addition is equivalent to subtraction of the opposite.** So, are we adding or subtracting? Well, it depends how we look at it.

Consider $3-(-2)$

$3-(-2)=$ Recall, that subtraction is addition of the opposite and the opposite of -2 is 2.

$3+2=$

5

What we did, we replaced two adjacent minuses by the plus sign.

$$3-(-2) = 3+2$$

Two adjacent ("double") minus signs
are replaced by plus sign

By replacing two minuses with a plus, we were able to rewrite the subtraction problem as an addition. Since addition can be viewed as subtraction and subtraction as addition, from now on, when performing the operations, let us stop asking ourselves if we are adding or subtracting. Instead, we do the following.

HOW TO ADD/SUBTRACT INTEGERS

Step 1. Replace any adjacent ("double") signs according to the following rules.

$3-(-2) = 3+2$	$3+(-2) = 3-2$
"– and –" replace with "+"	"+ and –" replace with "–"

$3+(+2) = 3+2$	$3-(+2) = 3-2$
"+ and +" replace with "+"	"– and +" replace with "–"

Step 2. After replacing "double" signs, follow the rules for addition of integers, i.e.

If **adding two integers of the same sign** (two positive or two negative numbers), take the sum of the numbers. The sign of the result will be the same as the sign of numbers (if both are positive, the result is positive, if both are negative, the result is negative).
For example, $3+2=5$ $\quad -3-2=-5$

If **adding a negative number to a positive number** (or a positive number to a negative number), take the difference of the numbers. The sign of the result will be the same as the sign of the 'larger' number.
For example, $3-2=1$ $\quad -3+2=-1$

We hope that the table below will help with remembering the rules for replacement of "double' signs in addition/subtraction problems.

$(+)(+) \to (+)$ $\qquad (-)(+) \to (-)$
$(-)(-) \to (+)$ $\qquad (+)(-) \to (-)$

Example 2.8 Perform the indicated operations by first removing "double" signs. Write your answer using '=' sign.

a) $-3+(-7)$ b) $4-(-8)$
c) $-9-(+4)$ d) $-6+(+5)$

Solution:

a) $-3+(-7)=$ Replace "double" signs according to the rule $(+)(-)\to(-)$
$-3-7=$ Follow the rules for adding two negative integers.
-10

b) $4-(-8)=$ Replace “double” signs according to the rule **(−)(−) →(+)**

$4+8=$ Follow the rules for adding two positive integers.

12

c) $-9-(+4)=$ Replace “double” signs according to the rule **(−)(+) →(−)**

$-9-4=$ Follow the rules for adding two negative integers.

-13

d) $-6+(+5)=$ Replace “double” signs according to the rule **(+)(+)→(+)**

$-6+5=$ Follow the rules for adding integers of opposite signs.

-1

Addition of opposite numbers

We know that $3-3=3+(-3)=0$. Consider $-3+3$. If you move 3 to the left on the number line and then move 3 to the right, you are back to 0. Thus $-3+3=0$. Notice that in both cases we are adding opposite numbers.

Any number added to its opposite yields zero.

$$-5+5=5-5=0$$
$$-45+45=45-45=0$$
$$-3491+3491=3491-3491=0$$

This is actually a defining property of opposite numbers. **The opposite of a given number is a number such that the sum of those two is zero.**

Common mistakes and misconceptions

Mistake 2.1

0 is not less than any other number. It is greater than all negative numbers.

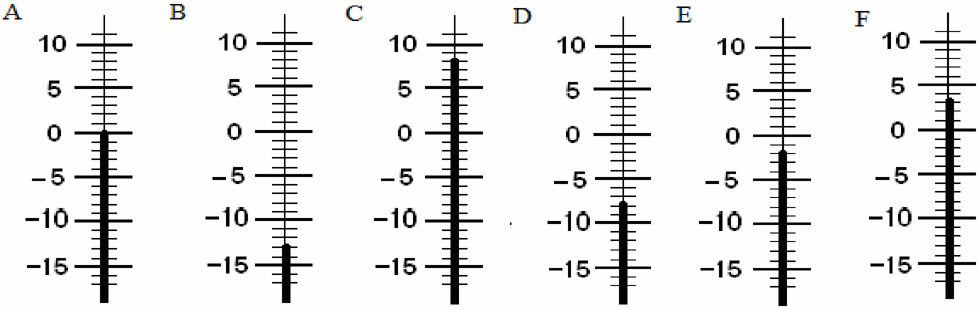
Exercises with Answers (For answers see Appendix A)

Ex.1 Fill in the blanks using the words “left”, “right” as appropriate.
All positive numbers are to the ___________of zero on a number line.
All negative numbers are to the __________ of zero on a number line.

Ex.2 Write the following statements using inequality symbols.

a) 3 is less than 5
b) 4 is greater than -2

Ex.3 Observe the temperature on each of the thermometers below (from A-F) and answer the following questions.



a) Which thermometer shows the highest temperature?
b) Which thermometer shows the lowest temperature?
c) What is the difference in temperature between the thermometers C and D?
d) What is the difference in temperature between the thermometers E and F?
e) Which thermometer(s) show(s) a temperature higher than -10?
f) Which thermometer(s) show(s) a temperature lower than -5?
g) List all temperatures from the lowest the highest.

Ex.4 The following temperatures were recorded at different locations at a given time.

Location A	Location B	Location C	Location D	Location E	Location F	Location G
-1	6	-12	-14	8	0	-5

a) Where was it warmer that day, in location A or F?
b) Where was it warmer that day, in location A or G?
c) Where was it warmer that day, in location C or D?
d) In what location was it the warmest?
e) In what location was it the coldest?
f) List all temperatures from the lowest to greatest.

Ex.5 Fill in blanks using the following symbols ‘<”, ‘>’, ‘=’ as appropriate.

a) 5 _____ 8 b) 3 ____ -4
c) 0 _____ 10 d) 0 _____ -1
e) -4 _____ -5 f) -3 _____ -2
g) -2 _____ -2 h) -99 ___ -100

Ex.6 Write the following integers in order from lowest to greatest.

a) $8, 5, -10, -3, 0, -7$
b) $-2, -5, 3, -7, -1, 2$
c) $-62, -51, -43, 48, -63, 49$

Ex.7 Replace each star with a digit such that the inequality is true (more than one answer is possible).

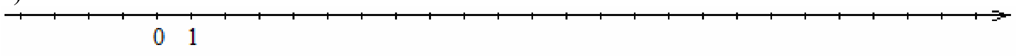
a) $3* < 36$ b) $-3* < -36$

c) $*5 > 45$ d) $-*5 > -45$

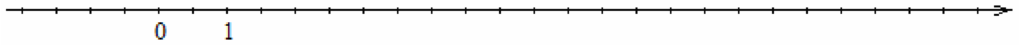
Ex.8 Plot the following numbers on a number line.
a) -2 b) -4
c) 3 c) 8

Ex.9 Plot 4 on each of the following number lines.

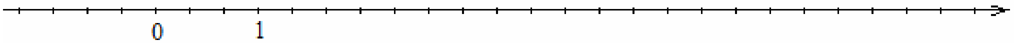
a)



b)



c)



Ex.10 Plot the following pairs of numbers on the same number line and determine which one is greater, which one is smaller. Use "<", ">" symbols when writing your answer.
a) 6 and 7 b) -1 and 2
c) -4 and -2 d) -5 and -7

Ex.11 List all integers that are less than 5 but greater than -3 (if it helps, use a number line).

Ex.12 Plot the following three numbers $-7, 1, 5$ on a number line and then find a number such that
a) It is greater than all the above three numbers.
b) It is less than all the above three numbers.
c) It is greater than two of the above three numbers and less than one.
d) It is less than two of the above three numbers and greater than one.

Ex.13 Find the opposite number of the following numbers.
a) 4 b) -16
c) -102 d) 0

Ex.14 Plot
a) 8 together with its opposite.
b) -3 together with its opposite.

Ex.15 Imagine a very high building with a very deep underground section and an elevator moving between all floors. The ground level will be called 0 level. The levels above the ground are levels 1, 2, 3, … and the underground levels are $-1, -2, -3$,....
Answer the following questions by writing a corresponding numerical statement and then evaluating it.

a) A person enters the building on the ground floor, takes the elevator 5 floors up and then 7 floors down. What is the level he will end up with?

b) A person enters the building on the ground floor, takes the elevator 2 floors down and then 6 floors up. What is the level he will end up with?

c) A person enters the building on the ground floor, takes the elevator 4 floors down and then 3 floors down again. What is the level he will end up with?

Ex.16 Answer the questions by first writing a corresponding numerical statement and then evaluating it.

a) Alice, when going to a store, had \$40. She spent (with the help of a credit card) \$60 in the store. What is her net worth now?

b) Alice does not have any money. She borrowed \$25 on Monday, and then \$25 on Tuesday. Assuming that she did not have any additional income, what is Alice's financial standing?

Ex.17 Answer the questions by first writing a corresponding numerical statement and then evaluating it.

a) The temperature in Alaska at 12am was $-6^0 F$. The temperature dropped by $8^0 F$ during the next 4 hours. What was the temperature at 4 am?

b) The temperature in Alaska at 12am was $-6^0 F$. The temperature rose by $10^0 F$ during the next 4 hours. What was the temperature at 4 am?

Ex.18 In April the water level of the Hudson River was -7 (seven inches below the average level). After May's precipitation, the water level rose by 12 inches. What was the water level then? Write a numerical statement corresponding to this situation and evaluate it.

Ex.19 Represent each of the following operations as a "movement on a number line" (see page 22) and then evaluate them.

a) $-9+11$	b) $-9-11$
c) $9-11$	d) $9+11$
e) $11-11$	f) $-9+9$

Ex.20 First, determine, if the final answer is positive or negative, and only then perform the indicated operations.

a) $-2+5$	b) $-7-12$	c) $7-10$

Ex.21 Perform the indicated operations.

a) $7-8$	b) $-2-3$	c) $-5+5$
d) $-1+4$	e) $-3+0$	f) $12-8$
g) $4-9$	h) $-5+6$	i) $4-8$
j) $-2-5$	k) $-9+10$	l) $-15-3$
m) $0-4$	n) $56-56$	o) $-6+8$
p) $3+2$	q) $-9+6$	r) $-1+1$

Ex.22 Rewrite the following expressions without 'double signs' and then evaluate them.

a) $3-(-2)$	b) $-10-(+3)$
c) $5+(-4)$	d) $-7-(+5)$
e) $12-(+16)$	f) $2-(-9)$
g) $4+(-1)$	h) $0-(+6)$
i) $41-(-1)$	j) $-2+(-6)$
k) $-9+(+1)$	l) $-15+(-6)$
m) $10-(-14)$	n) $12-(+12)$
o) $-9+(-3)$	p) $3-(-12)$

q) $2-(-2)$
r) $-15-(+15)$

Ex.23 Evaluate.

a) $3-7$
b) $0+(-4)$
c) $2-(-12)$
d) $-34-0$
e) $4-(-1)$
f) $8+(+2)$
g) $-10-10$
h) $-2-5$
i) $-32-(-32)$
j) $-28+9$
k) $14-16$
l) $-6-(-6)$

Ex.24 Choose any number. Perform the following operations on it.

a) take the opposite of the number
b) Increase the obtained number by 1
c) Take the opposite of the number obtained in the previous step above
d) Increase the number from the previous step by 1

What is the number you get after completing (a)-(d)? How about if you repeat steps (a)-(d) 3 times? Or 5 times?

Ex.25 Replace the triangle Δ with a number to make the statement true.

a) $\Delta+3=-2$
b) $1-\Delta=-9$
c) $\Delta-2=-4$
d) $31-\Delta=0$
e) $-2+\Delta=3$
f) $-5-\Delta=-7$
g) $-5-\Delta=-8$
h) $7-\Delta=2+3$
i) $\Delta-1=-3+1$

Lesson 3

Topics: Addition/subtraction of more than two integers; Multiplication and division of integers; Opposite of a number as multiplication by -1 .

<u>Addition/subtraction of more than two integers</u>

Sean has 40 dollars. He is planning to buy a CD for 15 dollars and a book for 30 dollars. He also knows that his grandma will give him 20 dollars for his birthday. What will be his final balance? To answer this question we need to evaluate the expression $40-15-30+20$. To this end, we perform the operations from the left to right (following the order of operations).

$$\begin{aligned} 40-15-30+20 &= && \text{perform the first operation } 40-15 \\ 25-30+20 &= && \text{perform the second operation } 25-30 \\ -5+20 &= && \text{perform the third operation } -5+20 \\ 15 & && \end{aligned}$$

We know that it does not matter whether Sean buys a CD or a book first, or even if his grandma gives him money before he buys all of those things, the final result will be the same. If, for instance, Sean buys a book before he buys a CD, the expression that needs to be evaluated would be $40-15-30+20$ and should be equal to the original expression.

$$40-15-30+20 = 15 = 40-30-15+20$$

The mathematical reason for this phenomena comes from the fact that any subtraction can be viewed as addition of the opposite number and thus the expression $40-15-30+20$ can be rewritten as repeated addition $40-15-30+20 = 40+(-15)+(-30)+20$. Since addition is commutative, **we can rearrange all terms of addition**. So for example, if Sean's grandma gives him the 20 dollars before, rather than after he spends the 15 and 30, it will make no difference to the final result.

$$40-15-30+20 = 40+(-15)+(-30)+20 = 40+20-15-30$$

Rearranging terms is like taking each integer *together* with the sign that precedes it and treating it as a 'block'.

$$\boxed{+40}\ \boxed{-15}\ \boxed{-30}\ \boxed{+20}$$

The blocks can be rearranged in any order.

$$40-15-30+20 = \boxed{+40}\ \boxed{-15}\ \boxed{-30}\ \boxed{+20} = \boxed{+40}\ \boxed{+20}\ \boxed{-15}\ \boxed{-30} = 40+20-15-30$$

Example 3.1 Write the numerical expression $4-5+6-7$ as an addition problem and then rewrite it in two different ways by changing the order of terms of addition.

Solution:

The "blocks" that we can rearrange are $+4, -5, +6, -7$. We get
$4-5+6-7 = 4+(-5)+6+(-7) = 4+6-5-7 = -5-7+6+4 = -5+6-7+4$
(other answers are possible).

Example 3.2 Among all expressions below find those that are equal to $5-6-9$.
a) $-6+5-9$ b) $-6-9+5$ c) $-9+5-6$ d) $-5-6-9$

Solution:
The only expression that is not equal to $5-6-9$ is (d). Notice that the signs preceding 5 in $5-6-9$ and $-5-6-9$ are different.

This possibility of rearranging the terms of additions means that we can perform addition in any order. For example, instead of performing operations from left to right, we can first add all negative and all positive integers separately, and then combine the two results.

$40-15-30+20=$ We rearrange the order grouping positive and negative numbers together
$40+20-15-30=$
$60-45=$
15

The answer, as it should, is the same as we got when evaluating from left to right.

Sometimes, we might get a 'tricky' evaluation that can be made much easier to evaluate if we rearrange terms in some special way. Consider,

$$-1234+7890+3579-7890+1234$$

Performing addition from the left to right would be quite cumbersome. Instead, we rearrange the terms of addition in the following way

$$-1234+7890+3579-7890+1234 = -1234+1234+7890-7890+3579.$$

Remembering that the sum of the opposite numbers is always zero ($-1234+1234=0$ and $7890-7890=0$), we get easy calculations

$$-1234+1234+7890-7890+3579 = 0+0+3579 = 3579$$

Let us summarize what we have learned.

In order to add/subtract integers do the following:

1. If there are any 'double signs' in the expression, replace them with one sign according to rules $(+)(+) \to (+)$ $(-)(+) \to (-)$
$(-)(-) \to (+)$ $(+)(-) \to (-)$

2 Perform the operations from left to right or, if you prefer, rearrange terms of addition according to your liking, and then perform the operations from left to right.

For example,

$-3-(-7)-4=$ replace "double" signs

$-3+7-4=$ add integers from the left to right

$4-4=0$

Or, we can do it differently,

$-3-(-7)-4=$ replace "double" signs

$-3+7-4=$ group all negative numbers together

$-3-4+7=$ add all negative numbers

$-7+7=0$ add the resulting numbers

Example 3.3 Perform the indicated operations.

a) $-(-3)+2+(-8)$ b) $-4+(+5)-(-6)-3$

Solution:

a) $-(-3)+2+(-8)=3+2-8=5-8=-3$

b) $-4+(+5)-(-6)-3=-4+5+6-3=-4-3+5+6=-7+11=4$

In (b), before adding, we grouped all positive and negative integers together.

Multiplication of integers

Multiplication of integers is similar to multiplication of natural numbers except that the sign of the product must be determined. Here are the rules for multiplication (for some justification for these rules, see (*)).

HOW TO MULTIPLY INTEGERS
When **multiplying two integers of the same sign** (two positive or two negative numbers), take the numbers without their signs and multiply them. The sign of the result will be positive. For example, $2\times 3=6$ or $-2\times(-3)=6$ We multiply 2 by 3 to get 6. Both numbers are of the same sign, thus the result is positive.
When **multiplying a negative number and a positive number** (or a positive number and a negative number), the numbers without their signs and multiply them. The sign of the result will be the negative. For example, $-2\times 3=-6$ or $2\times(-3)=-6$ We multiply 2 by 3 to get 6. The numbers are of the opposite sign, so the result is negative.

It might help to notice that rules for signs in multiplication are exactly the same rules we used in case of "double signs'. Namely, **$(+)(+)\ \to (+)$** **$(-)(+)\ \to (-)$**

$(-)(-)\ \to (+)$ $(+)(-)\ \to (-)$

(*) Here is one possible way we can justify the rules for multiplication (do not worry if you find it difficult to understand – you can still perform the operations - the understanding will come with time). If you win 3 dollars 2 times, you total is 6 (the product of two positive numbers is positive); If you lose 3 dollars 2 times, your total is -6 (the product of one positive and one negative number is negative); You have a 3 dollar penalty (-3) that you do not pay two times (-2) and the result is you are ahead by plus 6 dollars **(**the product of two negative numbers is positive).

Let us also recall important convention: Any time two operation signs (+, −, ×, or ÷) are next to each other, parentheses must be used to separate those two signs. It should be noticed that even if the multiplication sign is not explicitly displayed, we assume it is there, and place parentheses.

We write:	We *do not* write:
$2\times(-3)$	2×-3
$2\cdot(-3)$	$2\cdot -3$ Do not write $2-3$ since it would no
$2(-3)$	longer mean multiplication but subtraction.

<u>Example 3.4</u> Perform the indicated operations.

a) $(-9)(-5)$ b) $6(-7)$ c) $(-10)(53)$

Solution:

a) $(-9)(-5)=45$ use the rule "$(-)(-)\to(+)$"

b) $6(-7)=-42$ use the rule "$(+)(-)\to(-)$"

c) $(-10)(53)=-530$ use the rule "$(-)(+)\to(-)$"

<u>Example 3.5</u> Write the following operations using mathematical symbols and evaluate them. Remember about placing parentheses when needed.

a) -8 multiplied by -3 b) The product of 14 and -2

Solution:

a) $-8(-3)=24$

b) $14(-2)=-28$

<u>Division of integers</u>

The rules for division of integers are very similar to the rules for multiplication.

HOW TO DIVIDE INTEGERS
When **dividing two integers with the same sign** (two positive or two negative numbers), find the quotient of the numbers. The sign of the result will be positive. For example, $-6\div(-3)=\frac{-6}{-3}=2$ $6\div 3=\frac{6}{3}=2$ We divide 6 by 3 to get 2. Both numbers are of the same sign, thus the result is positive.
When dividing **a negative number by a positive number** (or a positive number by a negative number), find the quotient of the numbers. The sign of the result will be negative. For example, $6\div(-3)=\frac{6}{-3}=-2$ $-6\div 3=\frac{6}{-3}=-2$ We divide 6 by 3 to get 2. The numbers are of the opposite sign, so the result is negative.

Once again we follow the important conventions for parentheses.

We write: $12 \div (-3)$

We *do not* write: $12 \div -3$

Again, we have the same rules for determining the sign of the result of division.

(+)(+) → (+) **(−)(+) → (−)**

(−)(−) → (+) (+)(−) → (−)

Example 3.6 Perform the indicated operations.

a) $22 \div (-11)$ b) $\dfrac{-54}{-9}$ c) $-32 \div 8$

Solution:

a) $22 \div (-11) = -2$ use the rule "**(+)(−) → (−)**"

b) $\dfrac{-54}{-9} = 6$ use the rule "**(−)(−) → (+)**"

c) $-32 \div 8 = -4$ use the rule "**(−)(+) → (−)**"

Example 3.7 Write the following operations using mathematical symbols. Remember about parentheses.

a) 4 divided by -2

b) The quotient of -25 and -5

Solution:

a) $\dfrac{4}{-2} = -2$ or $4 \div (-2) = -2$

b) $\dfrac{-25}{-5} = 5$ or $25 \div (-5) = 5$

Multiplication of more than two integers

Let us perform the following operations.

$(-2)(-2) = 4$

$(-2)(-2)(-2) = 4(-2) = -8$

$(-2)(-2)(-2)(-2) = 4(-2)(-2) = -8(-2) = 16$

$(-2)(-2)(-2)(-2)(-2) = 4(-2)(-2)(-2) = -8(-2)(-2) = 16(-2) = -32$

(Notice the use of parentheses)

We can observe the pattern of the sign in the problems above. The sign alternates. If we multiply an even number of negative numbers, the result is positive (even numbers are 2, 4, 6, ...). If we multiply an odd number of negative numbers, the result is negative (odd numbers are 1, 3, 5, ...).

It happens because if we have an even number of minuses we can put them in pairs, and each pair can be replaced by plus, so at the end we have a positive answer.

$$\underbrace{-\quad-}_{+}\quad\underbrace{-\quad-}_{+}\quad\underbrace{-\quad-}_{+}$$

If we have an odd number of minuses

$$\underbrace{-\quad-}_{+}\quad\underbrace{-\quad-}_{+}\quad\underbrace{-\quad-}_{+}\quad\underset{-}{-}$$

one minus is "left" and the result is negative.

Now consider the product of not only negative, but also positive numbers. Notice that multiplication by a positive number does not change the sign of the result. What was positive stays positive. What was negative, stays negative. Thus, the sign of the product is determined only by the number of negative integers that are factors of the product.

To determine the sign of the product of two or more integers, count the number of 'minus signs'. If there are even number of 'minus signs', the result is positive; if odd, the result is negative.

For example,

$(-3)4(-1)=12$	Since there are 2 (even number) of negative factors, the result is positive.
$(-1)(-1)(-3)(3)(10) = -90$	Since there are 3 (odd number) of negative factors, the result is negative.

Please, notice that you do not have to remember this rule. You can always perform multiplication step by step, and you will get the same result. Eventually, if you perform a lot of multiplications, you will remember the rule anyway.

<u>Example 3.8</u> Perform the indicated operations.

a) $4(-1)(-2)(-6)$ b) $-10(3)(2)(-1)(2)$

Solution:

a) $4(-1)(-2)(-6) = -48$ The answer is negative since there are three (3 is odd) negative numbers in this product.

b) $-10(3)(2)(-1)(2) = 120$ The answer is positive since there are two (2 is even) negative numbers in the product.

Example 3.9 Write the expression “the product of $-1, -100$ and -3“ using mathematical symbols and evaluate it. Remember about placing parentheses when needed.

Solution:
$-1(-100)(-3) = -300$

Just like in case of addition, when we were able to change the order of terms of addition, **we can change the order of factors in multiplication.** For example,

$$3\times(-100) = -100\times 3$$
$$2\times(-1)\times 3\times(-4) = 2\times 3\times(-1)\times(-4)$$

Both, addition and multiplication have this nice property, NOT division, NOT subtraction.

Example 3.10 Write the expression $12(-5)(-6)$ in three different ways using the fact that factors of any product can be rearranged.

Solution:
$12(-5)(-6) = -6(-5)(12) = -6(12)(-5) = -5(-6)(12)$
Note that the product will be 360 regardless of how the factors were arranged.

Opposite numbers in the light of multiplication

Recall that for each integer we can find its opposite. For example, -7 is the opposite of 7, -45 is the opposite of 45. The opposite of 0 is 0. In general, to take the opposite of a number other than 0 means to reverse its sign. The operation that changes the sign is the multiplication by -1.

$$-1\times 2 = -2$$
$$-1\times(-2) = 2$$

To find the opposite of a number we multiply the number by −1.

Opposite of 8 is $-1\times 8 = -8$
Opposite of -8 is $-1\times(-8) = 8$

We will use the minus sign to mean “the opposite of”. So, we can say that the opposite of the opposite of 8 (let’s say) is: $-(-8)$
and what we mean by that is : $-(-8) = -1(-1)8 = 8$

So, now, the expressions as $-(-(-7))$ have not only some meaning (the opposite of opposite of -7), but also can be evaluated.

$$-(-(-7)) = -1(-1)(-1)7 = -7$$

Since there are 3 negative numbers in the product (or three minuses in front of 7), the answer is negative.

Example 3.11. Write the following expressions as a product of $-1's$ and the number. Evaluate.

a) $-(-4)$ b) $-(-(-13))$

Solution:

a) $-(-4) = -1(-1)4 = 4$

b) $-(-(-13)) = -1(-1)(-1)(13) = -13$

Example 3.12. Find the opposite of opposite of opposite of 12.

Solution:

$-(-(-12)) = -1(-1)(-1)(12) = -12$

Example 3.13 Among the following numbers find all pairs of the opposite numbers.

$-5,\quad -(-2),\quad -2,\quad 5,\quad -(-(-5)),\quad 3$

Solution:

$-(-2) = 2$ and $-(-(-5)) = -5$ Thus we have $-5,\ 2,\ -2,\ 5,\ 3$. The pairs of the opposite numbers are -2 and 2, -5 and 5.

Common mistakes and misconceptions

Mistake 3.1

The expressions $-3-3$ and $-3(-3)$ have different meanings. $-3-3$ means subtraction of 3 from -3 (or, equivalently, addition of -3 and -3) which is -6. The second one $-3(-3)$ means multiplication of -3 and -3 which is 9.

Mistake 3.2

The rule saying that if there are even number of 'minus signs' in *a product*, then the result is positive; if odd, the result is negative applies only to *multiplication.* It does not apply to addition/subtraction like for example, $-2-3-4-9$ is still negative, even though there are an even number of minus signs.

Exercises with Answers (For answers see Appendix A)

Ex.1 Write the following numerical expressions as addition problems and then rewrite each of them in two different ways by changing the order of the terms. Indicate that all of the resulting expressions are equal to each other by placing "=" sign between them.

a) $7-3+4$ b) $-2-5+12$

Ex.2 Circle all expressions that are equal to $-3+1-8+2$

$1-3-8+2$ $-8-3+2+1$ $2+3-8+1$ $2+1-3-8$ $-3+1-2+8$

Ex.3 Perform the indicated operations. Please, use "=" sign correctly when writing your answer.

a) $8-9+1$ b) $-7-13-7$

c) $10-8-9$ d) $12-5-7$

e) $-4-3+5$ f) $-13-3+1$

g) $-14+7-3$ h) $50-60-10$
i) $-3+4-7+2$ j) $-6-5-3+9$
k) $7-3+8-5$ l) $-123+41-41+123$

Ex.4 Perform the indicated operations. Show all your steps and use "=" sign correctly when writing your answer.

a) $1-(-9)-10$ b) $-7+(-6)-2$
c) $-(-3)+(-2)+9$ d) $8+(-7)+(-11)$
e) $-3+3-(-12)$ f) $-(-2)-1+4$
g) $9-10+(-11)$ h) $-6-(+4)+(-5)$
i) $-3-(-6)-(+7)-1$ j) $7-(-23)-1+(-23)$
k) $-80+10-(-20)+(-20)$ l) $-(-9)-9+(+1)-(-12)$

Ex.5 Perform the indicated operations. Show all your steps and use "=" sign correctly when writing your answer.

a) $9+4-10+2$ b) $6+(-2)-(-9)+12-3$
c) $11-(+2)-(-8)+(+3)$ d) $-(-2)+6-(+3)-7+(-3)$
e) $-6-7-1+4-3$ f) $9-12+4-6-1$
g) $-345-3-(-7)+345$ h) $6+2-9-2+1$
i) $-5+9-1+8$ j) $-(-1)+(-1)-(-1)-(+1)+6$
k) $289-12-9-289+13$ l) $-(+4)-(-3)+12-1+(-12)$

Ex.6 Write the following statements using mathematical symbols and then evaluate them. Remember about placing parentheses when appropriate.

a) 100 multiplied by -2
b) -3 multiplied by -9
c) The product of -20 and 6
d) The product of -100 and -10

Ex.7 Perform the indicated operations. Please, use "=" sign correctly when writing your answer.

a) $2(-3)$ b) $-100(-42)$
c) $4(-2)$ d) $-9(7)$
e) $-6(-3)$ f) $12(4)$
g) $77(-1000)$ h) $0(-124)$
i) $(-5)(-8)$ j) $-12(3)$
k) $7(8)$ l) $-7(9)$
m) $-9(-4)$ n) $-1(345)$

Ex.8 Write the following statements using mathematical symbols and then evaluate them. Remember about placing parentheses when appropriate.

a) 14 divided by -2
b) The quotient of -25 and -5
c) -36 divided by 6
d) The quotient of -40 and 8

Ex.9 Perform the indicated operations if possible, if not possible write "not defined". Please, use "=" sign correctly when writing your answer.

a) $35 \div (-5)$ b) $\frac{-4}{-2}$

c) $\frac{-63}{7}$ d) $-64 \div (-8)$

e) $-42 \div 7$ f) $-100 \div (-10)$

g) $0 \div 34$ h) $34 \div 0$

i) $\frac{15}{-5}$ j) $\frac{-56}{-8}$

k) $\frac{-28}{7}$ l) $\frac{36}{-4}$

Ex.10 Write the following statements using mathematical symbols and then evaluate them. Remember about placing parentheses when appropriate.

a) The product of 8, 5 and -10

b) The product of -1, -100 and -3

c) The product of -7, 3 and -2

Ex.11 Perform the indicated operations. Please, use "=" sign correctly when writing your answer.

a) $-2(-3)9$ b) $3(-3)(-3)$

c) $-3(-1)9$ d) $-50(2)(34)$

e) $8(2)(-3)$ f) $1000(-9)(6)$

g) $3(-5)4$ h) $-20(-20)0$

i) $(-1)(-1)(-1)(-1)$ j) $3(-2)(8)(1000)$

k) $2(2)(-2)(-2)(-2)$ l) $-1(2)(-1)(2)(-1)$

Ex.12 Determine the sign of the answer. Do not evaluate!

a) $12(-4)(-6)(-5)7$

b) $4(-5)(-14)6(7)(-23)$

c) $(-1)(2)(-3)(-4)(-5)$

d) $-5(-3)(-2)8(6)(-5)(-7)$

Ex.13 Perform the indicated operations, if possible. If not possible, write "not defined". Please, use "=" sign correctly when writing your answer.

a) $-6(-8)$ b) $-5(-1)(8)$

c) $-3 \div (-3)$ d) $4(-100)(-100)$

e) $-63 \div 9$ f) $2(-3)(7)$

g) $\frac{-48}{-8}$ h) $\frac{-24}{0}$

i) $30 \div (-5)$ j) $-1(-2)(-1)(-7)$

Ex.14 Using the fact that factors of any product can be rearranged, write the following expression in two different ways. Indicate that all of the resulting expressions are equal to each other by placing "=" sign between them.

a) $-7(8)(-9)$

b) $-2(-3)(14)(-8)$

Ex.15 Write the following expressions as a product of $-1's$ and a natural number. Evaluate them.

a) $-(-4)$

b) $-(-(-13))$

c) $-(-(-7))$

d) $-(-(-(-5)))$

Ex.16 Find the opposite of opposite of opposite of

a) 26

b) -45

Ex.17 Among the following numbers match all pairs of the opposite numbers.

$-7,\quad -(-9),\quad -(-(-(9))),\quad -(-12),\quad 0,\quad -(-(-7)),\quad -12,\quad 0$

Ex.18 Compare the following two expressions and write "=" between them, if they are equal, or "≠" if the expressions are not equal.

a) $-5-7$ ________ $5+7$

b) $12-7$ ________ $-7+12$

c) $5\times2\times4$ ______ $2\times4\times5$

d) $-(-5)$ _______ $-(-(-5))$

e) $(-3)(-7)(-8)$ _____ $-3(7)(8)$

f) $-7+3-2$ _____ $3-2-7$

g) $2-(-4)+(-5)$ ______ $2+4-5$

Ex.19 Replace the triangle Δ with a number to make the statement true.

a) $-479-832=\Delta-832$

b) $439-512=-512+\Delta$

c) $\Delta-6-1=-6-1+5$

d) $2-\Delta+3=3+2-8$

e) $-10\times\Delta=-560$

f) $543\times(-784)=-784\times\Delta$

g) $\Delta\div5=-4$

h) $\Delta\times7=-28$

i) $-7(-2)=10+\Delta$

j) $-1-1-1=-1\times\Delta$

Ex.20 If $-346\times45=-15570$, apply the principle "Equals can be substituted for equals" to evaluate

a) $-346\times(-45)$

b) -45×346

c) $346\times45\times10$

d) $-346\times(-1)\times(-45)$

Lesson 4

Topics: Exponentiation of integers; Order of operations; All operations combined.

Exponentiation of integers

Just as in the case of natural numbers, exponentiation of integers is a shorthand for repeated multiplication. For example,

$$(-2)^1 = -2$$
$$(-2)^2 = (-2)(-2)$$
$$\ldots$$
$$(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2)$$
$$\ldots$$

and so on. The number that is multiplied by itself is called the base. The exponent (a number slightly up to the right of a base) indicates how many times the base should be multiplied by itself. So, in the expression $(-2)^6$, -2 is the base, 6 is called the exponent (or power). **An exponent pertains only to the symbol that is closest to it.** This is why, if we wish to raise -2 to a given power, we need to place parentheses around it. If we do not, and instead we write, let us say, -2^4, the resulting expression would mean that only 2 is raised to the fourth power.

$(-2)^4 = -2(-2)(-2)(-2) = 16$ $\quad$ -2 is raised to the 4th power
$-2^4 = -2 \times 2 \times 2 \times 2 = -16$ $\quad$ only 2 is raised to the 4th power

Any time you wish to raise a negative number to a given power, you must place parentheses around it.

Example 4.1 $\quad$ Write using exponential notation**.**

a) Number -9 raised to the tenth power.
b) Number 13 cubed.
c) Exponential expression with the exponent 7 and base -3

Solution:

a) $(-9)^{10}$
b) 13^3
c) $(-3)^7$

Example 4.2 $\quad$ Expand.

a) -6^5 $\qquad$ b) $(-6)^3$

Solution:

a) $-6^5 = -6 \times 6 \times 6 \times 6 \times 6$
b) $(-6)^3 = (-6)(-6)(-6)$

<u>Example 4.3</u> Whenever possible, use exponential notation. Do not evaluate.

a) $-2(-2)(-2)$
b) $-2-2-2$
c) $3+3+3+(-3)(-3)(-3)(-3)$
d) $-7 \times (-7) \times (-7) \times 5 \times 5 \times 5 \times 5$

Solution:

a) Since -2 is multiplied by itself 3 times: $-2(-2)(-2) = (-2)^3$. We need to place the parentheses around -2, to make sure that -2 is cubed, not just 2.

b) $-2-2-2$ We cannot apply the exponential notation. This is a repeated subtraction not repeated multiplication.

c) $3+3+3+(-3)(-3)(-3)(-3) = 3+3+3+(-3)^4$

d) $-7 \times (-7) \times (-7) \times 5 \times 5 \times 5 \times 5 = (-7)^3 \times 5^4$ We need to have the same base to apply the exponential notation, thus we 'treat' -7's and 5's separately.

Exponentiation is a repeated multiplication and thus what we learned about determining the sign of the product of two or more integers can be applied to exponentiation. If the base (the number we multiply by itself) is a negative integer and the exponent is even, that means we multiply a negative integer an even number of times, and thus the result is positive. If the exponent is odd, we multiply a negative integer an odd number of times and the result is negative.

When a negative number is raised to an even power, the result is positive.
When a negative number is raised to an odd power, the result is negative.

<u>Example 4.4</u> Determine the sign of the result. Please, do not evaluate.

a) $(-234)^4$ b) $(-21)^5$ c) $-(-35)^7$

Solution:

a) $(-234)^4$ is positive, since the exponent is even.

b) $(-21)^5$ is negative, since the exponent, 5, is odd and the base, -21, is negative.

c) $-(-35)^7 = -1(-35)^7$, $(-35)^7$ is negative, since the power is odd, but the product of two negative numbers is positive, so $-(-35)^7$ is positive.

<u>Example 4.5</u> Evaluate the following expressions.

a) $(-10)^4$ b) -10^4
c) $(-1)^{24}$ d) $(-1)^{33}$

Solution:

a) $(-10)^4 = -10(-10)(-10)(-10) = 10000$

b) $-10^4 = -10 \times 10 \times 10 \times 10 = -10000$
c) $(-1)^{24} = 1$ since the power 24 is even
d) $(-1)^{33} = -1$ since the power 33 is odd

Performing more than one operation on integers

If more than one operation is to be performed on integers we follow the order of operations.

Now, we know all we need to be able to add, subtract, multiply (thus exponentiate) or divide integers. But before we start practicing performing those operations, let us go over the following two examples. They illustrate two traps many students fall in.

- Consider $3-(-5+4)$.

Do NOT be tempted to begin by replacing adjacent minuses by a plus. According to the order of operations, we should first perform all operations in parentheses.

$3-(-5+4) = 3-(-1) = 3+1 = 4$.

Notice, that we can replace two adjacent minuses by plus in $3-(-1)$, because inside parentheses there is no operation to perform (there is only -1 inside) .

- Consider $-(-5)^2$.

Do NOT be tempted to begin by replacing adjacent minuses by a plus. If you write $-(-5)^2 = -1 \times (-5)^2$, you can see that exponentiation should be performed before multiplication, thus

$-(-5)^2 = -1 \times (-5)^2 = -1 \times 25 = -25$

In other words, we raise -5 to the second power first and then take the opposite of the result.

Example 4.6 Perform the indicated operations one step at a time. Please, use equal signs in a proper way.

a) $-2-3\cdot(-2)$ b) $(-6-1)(2-4)$ c) -3×2^2

Solution:

a) $-2-3\cdot(-2) = -2-(-6) = -2+6 = 4$ (multiplication should be performed first).

b) $(-6-1)(2-4) = -7(-2) = 14$ (operations in parentheses should be performed first).

c) $-3\times 2^2 = -3\times 4 = -12$ (exponentiation should be performed first).

Example 4.7 Write each of the following as a single numerical expression, and then evaluate them.

a) The product of -1 and 3 and then add 5.
b) Subtract 3 from 1 and then multiply by the result by -2.

Solution:

a) $(-1)(3)+5=-3+5=2$
b) $(1-3)(-2)=-2(-2)=4$

Exercises with Answers (For answers see Appendix A)

Ex.1 Write using exponential notation.
a) Number -5 raised to the sixth power
b) Number 8 raised to the seventh power
c) Number -4 squared.
d) Exponential expression with power 8 and base -7

Ex.2 Identify the base of the following expressions.
a) -32^{45} b) $(-71)^{5}$

Ex.3 Whenever possible, use exponential notation. Do not evaluate.
a) $-8\times 8\times 8\times 8$ b) $(-8)(-8)(-8)$
c) $(-8)(-8)(-8)-8-8$ d) $-8\times 8\times 8-8\times 8\times 8\times 8$
e) $5\times 5\times 7\times 7\times 7\times 7$ f) $-(-8)(-8)(-8)(8)(8)(8)(8)$
g) $2(2)(2)-8(8)(8)$ h) $3\times 2\times 3\times 2\times 2$

Ex.4 Expand.
a) -11^{3} b) $(-11)^{4}$

Ex.5 Evaluate the following expressions.
a) $(-1000)^{2}$ b) -2^{4}
c) $(-123)^{1}$ d) $(-1)^{14}$
e) $(-4)^{3}$ f) $(-1)^{25}$
g) $-(-7)^{2}$ h) $-(-3)^{3}$

Ex.6 Determine the sign of the result. Please, do not evaluate.
a) $(-25)^{4}$ b) -32^{4}
c) $(-18)^{7}$ d) -88^{3}
e) 23^{47} f) $(-9)^{49}$

Ex.7 Name the first mathematical operation that should be performed according to the order of operation and then evaluate the expression starting with this operation.
a) $-5\times 3-4$ b) $-5(3-4)$
c) $(-2\times 3)^{2}$ d) -2×3^{2}
e) $4-(2+5)$ f) $4-2+5$
g) $-12\div 3-2$ h) $-12\div(3-2)$

Ex.8 Evaluate, when possible. Please, perform operations one step at a time and use the equal signs properly. If an expression cannot be evaluated, write “cannot be evaluated”.

a) $8-(-9+2)$
b) $(-3+2)^{18}$
c) $-2^4-(-2)^4$
d) $\dfrac{-3-7}{-4+4}$
e) $(-2\times 4)^2$
f) $-2-3\cdot(-2)$
g) $(-6-1)(2-9)$
h) $4+\dfrac{-36}{6}$
i) -2^2+6
j) $12\div(-4)\times(-1)$
k) $-(-2-4)$
l) $-4-4(-4)$
m) $4-(-3+5)$
n) $2-3\times 4$
o) $\dfrac{4-5}{6-5}$
p) $(-12+12)\div 4$
q) $\dfrac{-6}{-3}-2$
r) $-2^3\times(-3)^2$
s) $-8(-2)-8$
t) $4\div(-1+3-2)$
u) $-(-7)-7^2$
v) $(-1)^{25}(-1)^{26}$
w) $[-3+(-7)]^3$
x) $2-(4-5)$
y) $2-(-1)(-3)$
z) $-4-(-7)+(-2)$

Ex.9 Write each of the following as a single numerical expression, and then evaluate it.

a) The product of -3, -2, and 4
b) Subtract 7 from 2 and then add 5
c) Divide -4 by 2 and then subtract 9 from the result
d) The sum of 5 and the product of 7 and -3
e) Subtract 6 from 4 and then raise the result to the third power
f) Add 8 to -5 and divide the result by -3
g) Multiply -3 by 2 and then raise the result to the second power
h) Raise -3 to the third power and then subtract it from 30
i) Add 5 and 7 and subtract the result from -11
j) Raise -1 to the twenty second power and subtract the result from 1
k) Subtract 2 from -5 and then multiply the result by 1000

Ex.10 Replace the triangle Δ with a number to make the statement true. Use parentheses if needed.

a) $\Delta^3=-8$
b) $\Delta^{27}=1$
c) $-4\times 2+567=\Delta+567$
d) $67(-8+3)=67\times\Delta$

Ex.11 Replace x with a number to make the statement true. Use parentheses if needed.

a) $-7+(-2)^4=-7+x$
b) $(-5+10-2)^7=x^7$
c) $x^{14}=-1$
d) $-341+(-25)\div 5=-341+x$

Ex.12 Knowing that $-7(-25+345)=-2240$ evaluate the following expressions

a) $(-25+345)(-7)$
b) $-7(-25+345)(-10)$
c) $2240-7(-25+345)$

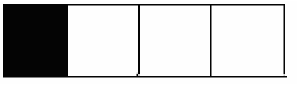
Lesson 5

Topics: Fractions; Rational numbers; Equivalent fractions; Reducing fractions; Converting fractions to an equivalent form with a given denominator.

If you cut a cake into several equal parts and eat one of them. How much of the cake did you eat? To answer this question, we need a new type of numbers. These numbers are called fractions.

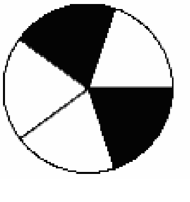
Fractions

A cake is cut into several **equal** parts. Each such part can be represented as a fraction. One out of four equal parts is $\frac{1}{4}$ (we read it "one-fourth").



One out of five equal parts is $\frac{1}{5}$ (read "one-fifth).

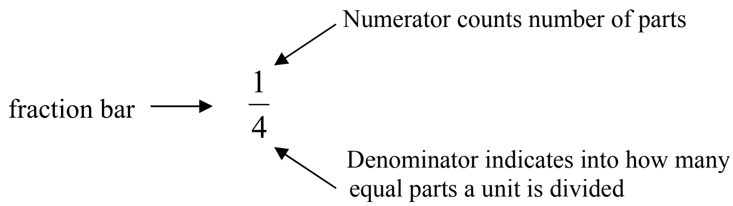
Two out of five equal parts is $\frac{2}{5}$ (two-fifths).



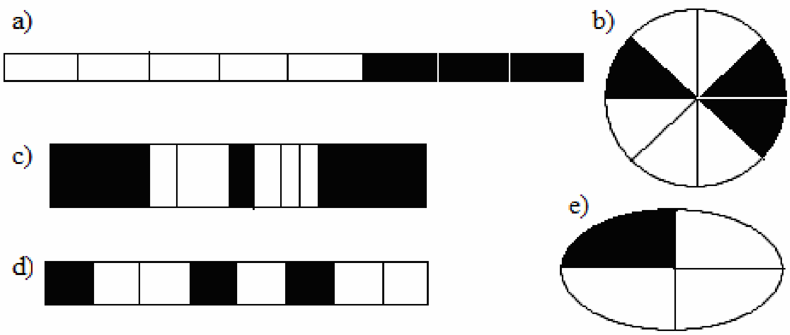
Each fraction consists of a numerator (the top number) and denominator (the bottom number).

$$\frac{\text{numerator}}{\text{denominator}}$$

The denominator names the number of *equal parts* into which a unit has been divided and the numerator names how many of those parts we take. The numerator and denominator are separated by a fraction bar.

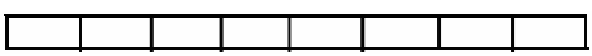


Example 5.1 Select all figures that have $\frac{3}{8}$ of them shaded.



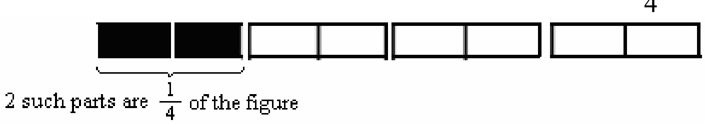
Solution:
Figures a, b, and d since each of the figures is divided into 8 equal parts and exactly 3 of them are shaded. Notice, that in figure (c), 3 parts out of 8 are shaded, however, the parts are not equal, and the shaded area is definitely more than $\frac{3}{8}$.

Example 5.2 Shade $\frac{1}{4}$ of the figure below in three distinct ways (assume the figure represents one whole and parts are equal).

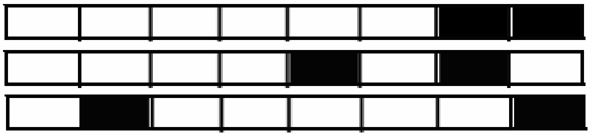


Solution:

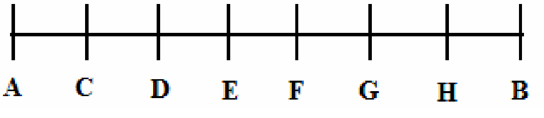
The figure is divided into 8 equal parts, so 2 such parts are $\frac{1}{4}$ of it.



We have to shade any 2 parts. The following are three distinct ways (other ways are possible, for example the one above).



Example 5.3 The marks on the line segment AB (AB begins at point A and ends at point B) are evenly spaced.



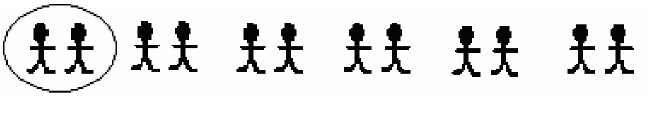
a) What fraction of the segment AB is the segment AC?
b) What fraction of the segment AB is the segment DG?
c) What fraction of the segment CH is the segment EG?

Solution:

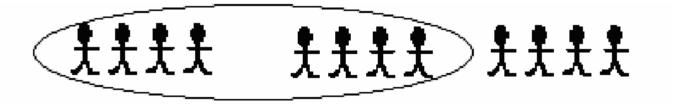
a) Segment AB is divided into 7 equal parts and AC is one of them, $\frac{1}{7}$.
b) Segment AB is divided into 7 equal parts and DG consists of 3 of them. Thus, the fraction is $\frac{3}{7}$.
c) Segment CH is divided into 5 equal parts and EG consists of 2 of them. Thus, the fraction is $\frac{2}{5}$.

Fractions as part of collection

We often hear statements like "Five sixths of cars …", " One third of students …". We have a collection of objects. We treat the collection as one unit, and take "a fraction" of it. The numerator represents the number of objects we have and the denominator the total number of parts in the collection. For example, if we have 40 students in a class and on a given day 3 of these students are absent, we could say "$\frac{3}{40}$ of students are absent". Or, if there are 10 cars in a parking lot and exactly 7 of them are red, we could say $\frac{7}{10}$ of cars in the parking lot are red. On the other hand, if somebody says one-sixth of students in a class is sick, and we know that the class consists of 12 students, we can find the number of students that are sick. We split (divide) the collection (in this case, collection of students) *equally* into 6 parts. Since $12 \div 6 = 2$, each such part consists of 2 students. We take one such part, so $\frac{1}{6}$ of 12 students is 2 students.



What would be $\frac{2}{3}$ of the class with 12 students? This time, we split the collection into 3 equal parts (as indicated by the denominator). $12 \div 3 = 4$, thus each part consists of 4 students. We take two such parts, as indicated by the numerator. If we take 2 such parts, we have $2 \times 4 = 8$, 8 students. $\frac{2}{3}$ of 12 students is 8 students.



Example 5.4 There are 20 pairs of shoes in a closet. 6 pairs are men's shoes and the remaining are women's shoes. What fraction of shoes in the closet are women's shoes?

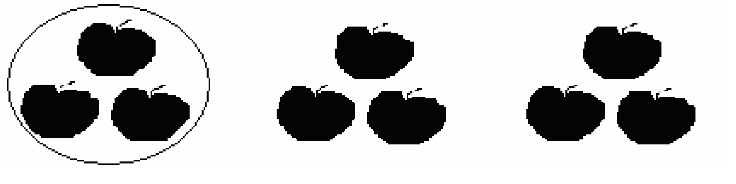
Solution:
There are 14 remaining pairs of shoes $(20 - 6 = 14)$, so the fraction of women's pairs of shoes in the closet is $\frac{14}{20}$.

Example 5.5 Lisa grabbed a handful of M&M's. She got 3 brown, 2 blue, 4 red, and 2 yellow. What fraction of M&M's are blue?

Solution:
Lisa grabbed $3 + 2 + 4 + 2 = 11$ M&M's. 2 out of them are blue. The fraction of the M&M's that are blue is $\frac{2}{11}$.

Example 5.6 There are 9 apples in a basket. $\frac{1}{3}$ of them are red. How many red apples are in the basket?

Solution:
The denominator 3 indicates that we need to divide all apples into 3 equal groups.



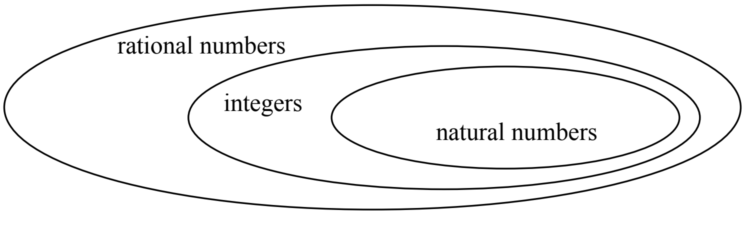
Each such group consists of 3 apples. ($9 \div 3 = 3$). If we take one such group (1, since the numerator is equal to 1), we will have 3 apples. Thus, there are 3 red apples.

Fractions as division; Rational Numbers

In the previous sections, a fraction was identified with a *part* of a whole or a *part* of a collection. But a fraction is not simply a part of any whole or any collection. A fraction is a *number* that is a part of the unit of measure, which is 1. A fraction bar represents division. Recall, that $1 \div 4 = \frac{1}{4}$. We can interpret the fraction $\frac{1}{4}$ as a number that is the result of division of 1 by 4. When we use only integers, the operation of division of 1 by 4 cannot be performed (there is no integer that is equal to 1 divided by 4). To fix this problem we extend integers to, so called, rational numbers.

All numbers that can be written in the form of a fraction, where the numerator and denominator are integers (as long as the denominator is different from zero), are called rational numbers.

Numbers like $\frac{1}{4}$, $\frac{2}{3}$, $\frac{-5}{7}$ or $\frac{8}{-9}$ are examples of rational numbers. With rational numbers we can perform all the divisions $1 \div 4 = \frac{1}{4}$, $2 \div 3 = \frac{2}{3}$, $-5 \div 7 = \frac{-5}{7}$ and $8 \div (-9) = \frac{8}{-9}$. Notice also, that **any integer can be written in the form of a fraction**. For example, $2 = \frac{2}{1}$ (indeed, 2 divided by 1 is 2), $4 = \frac{4}{1}$ and $-5 = \frac{-5}{1}$. This means that any integer is also a rational number (but not all rational numbers are integers). Just like integers are an extension of natural numbers, rational numbers are an extension of integers. We can represent this fact as follows.



Rational numbers are an extension of integers (and as such they are an extension of natural numbers). All rules that we established for integers are preserved for rational numbers. Multiplication and addition are commutative, division by zero is not defined, addition is the opposite operation of subtraction, and multiplication is the opposite of division.

<u>Example 5.7</u> Write the following integers as a fraction.

a) 3 b) -23 c) 0

Solution:

a) $3 = \frac{3}{1}$.

b) $-23 = \frac{-23}{1}$

c) $0 = \frac{0}{1}$

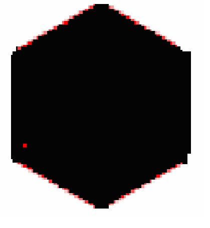
Example 5.8 Among the following rational numbers, select all that are integers.

$$\frac{5}{1} \quad \frac{16}{4} \quad \frac{9}{-3} \quad \frac{5}{4} \quad \frac{12}{11} \quad \frac{-20}{4} \quad \frac{4}{-7} \quad \frac{1}{3}$$

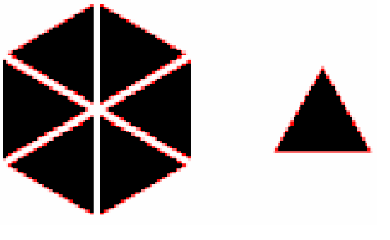
Solution:
If the numerator can be divided evenly by the denominator, the fraction represents an integer. $\frac{5}{1} = 5 \div 1 = 5$, $\frac{16}{4} = 16 \div 4 = 4$, $\frac{9}{-3} = 9 \div (-3) = -3$, $\frac{-20}{4} = -20 \div 4 = -5$. Thus $\frac{5}{1}$, $\frac{16}{4}$, $\frac{9}{-3}$, $\frac{-20}{4}$ represent integers.

Fractions less or greater than 1; Proper and improper fractions

Consider a fraction $\frac{7}{6}$. Let us represent it as a part of the unit below.



We divide the unit into 6 equal parts and take 7 parts of that size.



Since one unit consists of 6 parts, and we take 7 of them, we have more than one unit, $\frac{7}{6} > 1$. In general, **any time a positive fraction has the numerator greater than the denominator, the value of the fraction is more than 1**. Also, **any time a fraction's numerator is less than its denominator the fraction is less than 1** (there are not enough pieces to "have" one unit).

For example, $\frac{3}{5} < 1$ since $3 < 5$

$\frac{9}{5} > 1$ since $9 > 5$.

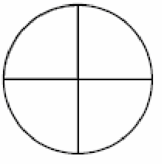
What happens if the numerator is equal to the denominator, for example $\frac{2}{2}$? Since $\frac{2}{2} = 2 \div 2 = 1$, we have $\frac{2}{2} = 1$. **When the numerator is equal to the denominator, the fraction is equal to 1.**

$$\frac{1}{1} = 1 \div 1 = 1,\ \frac{2}{2} = 2 \div 2 = 1,\ \frac{3}{3} = 3 \div 3 = 1, \cdots$$

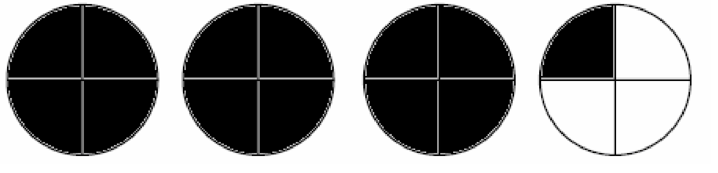
Number 1 can be written as a fraction whose denominator is equal to its numerator.

$$1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \ \dots \ = \frac{251}{251} = \frac{252}{252} = \dots$$

Example 5.9 This following represents one unit.



What fraction shows how much is shaded?



Solution:

The unit is divided into 4 equal parts, and we take 13 of such parts. The fraction $\frac{13}{4}$ shows how much is shaded.

Example 5.10 Determine which of those fractions is greater than, less than or equal to 1.

a) $\frac{19}{17}$ b) $\frac{6}{11}$ c) $\frac{23}{23}$

Solution:

a) The numerator 19 is more than the denominator 17, thus $\frac{19}{17} > 1$.

b) The numerator 6 is less than the denominator 11, thus $\frac{6}{11} < 1$.

c) The numerator 23 is equal to the denominator 23, thus $\frac{23}{23} = 1$.

We will use the following definitions: **A proper fraction is a fraction in which the numerator is less than the denominator. If the numerator is greater than or equal to the denominator, then the fraction is called an improper fraction**.

Fractions $\frac{5}{13}, -\frac{2}{3}, \frac{60}{61}$ are examples of proper fractions. Their numerators are less than their denominators. Fractions $\frac{7}{3}, -\frac{16}{15}, -\frac{12}{12}$ are examples of improper fractions. Their numerators are greater than or equal to their denominators.

<u>Example 5.11</u> Determine which fractions are proper and which are improper fractions.

$$\frac{2}{5} \qquad \frac{39}{39} \qquad \frac{12}{11} \qquad \frac{1}{25} \qquad \frac{25}{1} \qquad \frac{6}{7}$$

Solution:
The following fractions are proper (their numerator is less than the denominator).

$$\frac{2}{5}, \frac{1}{25}, \frac{6}{7}.$$

The remaining are improper: $\frac{39}{39}, \frac{12}{11}, \frac{25}{1}$.

<u>The place of the minus sign in fractions</u>

Suppose that we divide -3 by 4. We know that it is equal to 3 divided by -4, and that the result is a negative number. Thus $\frac{-3}{4} = \frac{3}{-4} = -\frac{3}{4}$.

A minus sign can be placed in the numerator, in the denominator or in front of a fraction and all resulting expressions are equal. For example, $\frac{-3}{4} = \frac{3}{-4} = -\frac{3}{4}$

Let us also take this opportunity to notice that since any time we divide two negative integers, the result is positive, $\frac{-3}{-4} = \frac{3}{4}$. The expression $\frac{-3}{-4}$ represents a division of -3 by -4, and the result of this division is a rational number $\frac{3}{4}$.

<u>Example 5.12</u> Determine if the two given fractions are equal or not equal to each other. Write your answer using "=" or "$\neq$" sign.

a) $-\frac{5}{7}$ and $\frac{5}{-7}$ b) $-\frac{7}{9}$ and $\frac{-7}{-9}$ c) $\frac{-2}{3}$ and $\frac{2}{-3}$

Solution:

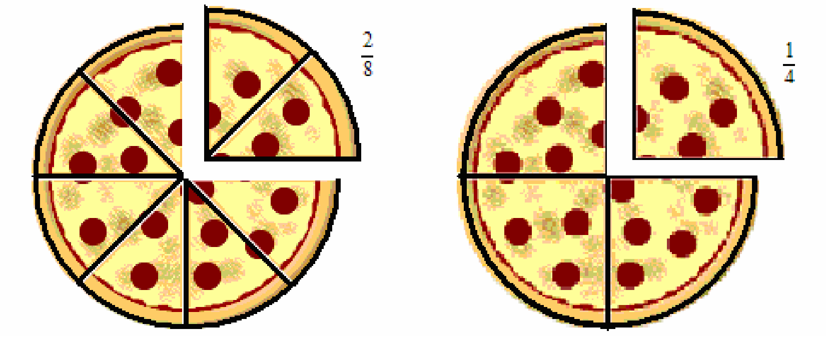
a) $-\frac{5}{7} = \frac{5}{-7}$

b) $-\frac{7}{9} \neq \frac{-7}{-9}$ Because, when we divide two negative numbers (like -7 and -9) the result is positive, not negative.

c) $\frac{-2}{3} = \frac{2}{-3}$

Equivalent fractions

What would you rather have, two slices of pizza cut into 8 pieces or, one piece of an identical pizza, which was cut into 4 equal parts?



Unless the size of a pizza is not the only consideration, it does not matter. So, although $\frac{1}{4}$ and $\frac{2}{8}$ look different, they both represent the same value $\frac{1}{4} = \frac{2}{8}$. We would say that $\frac{1}{4}$ and $\frac{2}{8}$ are equivalent (or equal).

Two fractions are equivalent if they represent the same number.

Slices of a pizza that is cut into 8 equal parts are twice as small as those of an identical pizza cut into 4 equal parts. Therefore we need two times as many of the smaller slices for the same amount.

$$\frac{1}{4} = \frac{1 \times 2}{4 \times 2} = \frac{2}{8}$$

Or, we can look at it this way: Slices of a pizza cut into 4 are twice as large as slices of a pizza cut into 8 parts. Thus, we only need to eat half as many, which causes us to divide by 2.

$$\frac{2}{8} = \frac{2 \div 2}{8 \div 2} = \frac{1}{4}$$

We can always multiply or divide a numerator and denominator by the same number different from zero, and the value of the fraction will stay the same (the resulting fraction is equivalent to the original one).

Example 5.13 The following pairs of fractions are equal (equivalent). Show that, indeed, they are equivalent by finding a number by which the numerator and denominator of the first fraction was multiplied or divided in order to get the second fraction.

a) $\frac{4}{7} = \frac{400}{700}$ b) $\frac{14}{35} = \frac{2}{5}$

Solution:

a) Both numerator and denominator were multiplied by 100.

$$\frac{4}{7} = \frac{4 \times 100}{7 \times 100} = \frac{400}{700}$$

b) Both numerator and denominator were divided by 7.

$$\frac{14}{35} = \frac{14 \div 7}{35 \div 7} = \frac{2}{5}$$

Expanding a fraction by multiplying its numerator and denominator by the same number

Suppose that we have a fraction $\frac{2}{3}$ and we would like to find the equivalent fraction with the denominator 15, $\frac{2}{3} = \frac{?}{15}$. We will use the fact that if we multiply both numerator and denominator by the same nonzero number, we get an equivalent fraction. We will expand the fraction. We ask ourselves the question "By what number do we multiply 3 to get 15?". Since the answer is 5, we multiply the numerator and denominator by 5.

Multiply the numerator by 5

$$\frac{2}{3} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15}$$

Multiply the denominator by 5

Example 5.14 Find the fraction with the denominator 27, equivalent to $\frac{7}{9}$.

Solution:

We need to find the numerator such that $\frac{7}{9} = \frac{?}{27}$. Since 9 in the denominator was multiplied by 3 to get 27, the same operation must be performed in the numerator.

$$\frac{7}{9} = \frac{7 \times 3}{9 \times 3} = \frac{21}{27}$$

The fraction we are looking for is $\frac{21}{27}$.

Reducing a fraction by dividing its numerator and denominator by the same number

We often want to write fractions using the smallest numbers possible. The process is called **simplifying or reducing a fraction.** We do that because it gives a better sense of the fraction value, and it also makes for simpler operations. While reducing a fraction, we use the fact that if we divide both numerator and denominator by the same nonzero number we get an equivalent fraction.

Reduce $\frac{9}{12}$.

We look for a number that both 9 *and* 12 could be divided by without any remainder. We will refer to such a number as **a common factor** (*). Since a common factor of 9 and 12 is 3, we divide the numerator and denominator by 3.

Divide the numerator by 3

$$\frac{9}{12}=\frac{9\div 3}{12\div 3}=\frac{3}{4}$$

Divide the denominator by 3

Since, there is no other number, different from 1, by which we can divide both 3 and 4 (or, equivalently, the only common factor of 3 and 4 is 1), the fraction is in its reduced form or, we can say, it is **reduced to its lowest terms.**

We will now present the previous simplification in a slightly different way.

$\frac{9}{12}=$	Find a common factor of the numerator and denominator. Common factor of 9 and 12 is 3. Replace 9 with the product of 3, and 12 with the product of 4 and 3.
$\frac{3\times 3}{4\times 3}=$	Divide the numerator by the common factor 3. Cross out each 3 in the numerator and denominator and place 1's above them.
$\frac{3\times \overset{1}{\cancel{3}}}{4\times \underset{1}{\cancel{3}}}=\frac{3\times 1}{4\times 1}=$	Multiply 3 and 1 in the numerator and 4 and 1 in the denominator.

$\frac{3}{4}$ or, simply, $\frac{9}{12}=\frac{3\times 3}{4\times 3}=\frac{3}{4}$

(*) Recall that numbers that are multiply together are called factors. Since $9=3\times 3$ and $12=3\times 4$, we call 3 a common factor of 9 and 12. For example, 7 and 14 have a common factor 7. We can write $7=7\times 1$ and $14=7\times 2$. Or, 24 and 32 have a common factor 8, $24=8\times 3$, $32=8\ \times 4$

Reduce $\frac{30}{10}$.

Like in the previous example, we look for a common factor. 10 is a common factor of both 30 and 10. We write,

$$\frac{30}{10}=\frac{3\times10}{1\times10}=\frac{3\times\cancel{10}^{1}}{1\times\cancel{10}_{1}}=\frac{3}{1}=3$$

Equivalently, we could have noticed right from the beginning that $\frac{30}{10}=30\div10=3$.

To reduce a negative fraction, keep the minus and follow the algorithm for reducing positive fractions.

$$-\frac{18}{21}=-\frac{6\times3}{7\times3}=-\frac{6\times\cancel{3}^{1}}{7\times\cancel{3}_{1}}=-\frac{6}{7}$$

Notice, that we **only cancel factors**, so for example in the expression $\frac{2+7}{7}$ we *cannot* cancel 7, because, in the numerator, 7 is not a factor but a term (operand of addition). $\frac{2+7}{7}=\frac{9}{7}$ not 2 (the result we would have gotten if we had cancelled).

Example 5.15 Reduce the following fractions.

a) $\frac{20}{35}$ b) $\frac{32}{144}$ c) $-\frac{20}{24}$ d) $\frac{10}{5}$ e) $\frac{6}{18}$

Solution:

a) $$\frac{20}{35}=\frac{4\times5}{7\times5}=\frac{4\times\cancel{5}^{1}}{7\times\cancel{5}_{1}}=\frac{4}{7}$$

b) Let us simplify $\frac{32}{144}$. Since both are even numbers, we known that 2 is a common factor of 32 and 144. Thus,

$$\frac{32}{144}=\frac{16\times2}{72\times2}=\frac{16\times\cancel{2}^{1}}{72\times\cancel{2}_{1}}=\frac{16}{72}$$

Since the numerator and denominator still have a common factor (for example 2, but also 8), we keep dividing both terms by a common factor. We could have divided by 2 again, but it takes fewer steps if we use the larger factor.

$$\frac{16}{72}=\frac{2\times 8}{9\times 8}=\frac{2\times \cancel{8}^{1}}{9\times \cancel{8}_{1}}=\frac{2}{9}$$

We cannot divide 2 and 9 any further. Thus $\frac{32}{144}=\frac{2}{9}$ We did this simplification in two steps. If we had noticed from the very beginning that 32 and 144 have a common factor of 16, we could have done it in just one step.

c) We keep the minus sign and simplify the fraction as usual.

$$-\frac{20}{24}=-\frac{5\times 4}{6\times 4}=-\frac{5\times \cancel{4}^{1}}{6\times \cancel{4}_{1}}=-\frac{5}{6}$$

d) $\frac{10}{5}=\frac{2\times 5}{1\times 5}=\frac{2\times \cancel{5}^{1}}{1\times \cancel{5}_{1}}=\frac{2}{1}=2$

Remember that $\frac{2}{1}=2$ and thus 2 (not $\frac{2}{1}$) should be the final answer.

e) $\frac{6}{18}=\frac{6}{3\times 6}=\frac{1}{3}$ We did not go through the process of crossing out 6's. If you choose to present your work this way, please remember that when you cancel 6 (divide the numerator and denominator by 6) the number in the numerator that "stays" is 1 (since $6\div 6=1$).

Example 5.16 Which of the following expressions can be simplified by canceling the number 3?

a) $\frac{3+4}{3}$ b) $\frac{3}{3\times 4}$ c) $\frac{3\times 5}{3-4}$

Solution:

Only (b), $\frac{3}{3\times 4}=\frac{1}{4}$ Remember, only factors (operands of multiplication) can be canceled. In (c) 3 is a factor in the numerator, but not in the denominator.

Exercises with Answers (For answers see Appendix A)

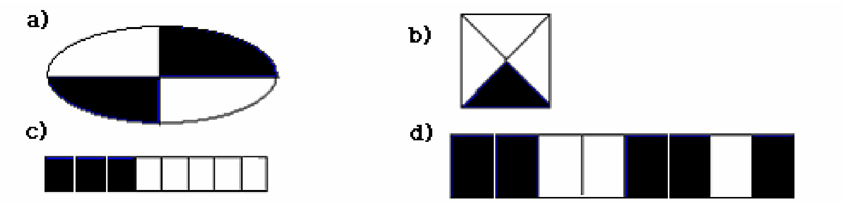
Ex.1 Identify the denominator and the numerator in $\frac{2}{3}$.

Ex.2 Write a fraction whose denominator is 354 and numerator is 29.

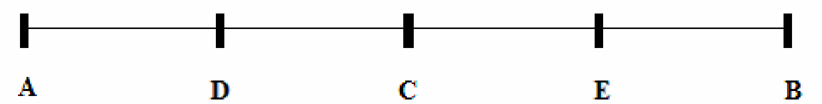
Ex.3 Fill in blanks with words "denominator" or "numerator" as appropriate.
The top number of a fraction is called ____________ , the bottom number is called ____________ .

The _____________ names the number of equal parts the unit has been divided and the ____________ names how many of those parts we take.

Ex.4 Write a fraction represented by the shaded regions. Assume each figure represents one whole.

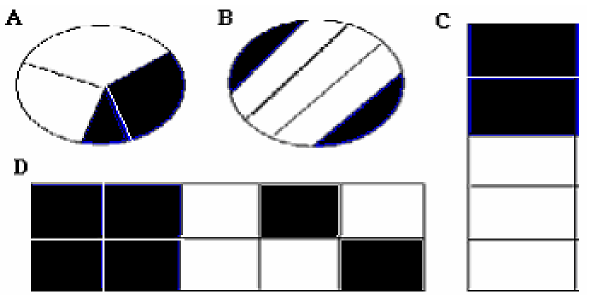


Ex.5 All marks are equally spaced. Please, answer the following questions.

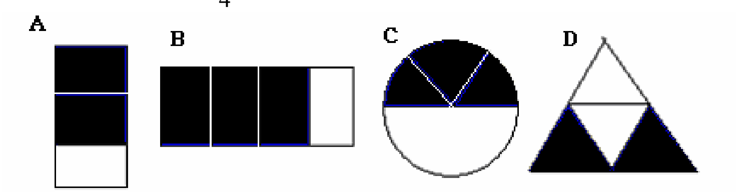


a) What fraction of the segment AB is the segment AC?
b) What fraction of the segment AB is the segment DE?
c) What fraction of the segment CB is the segment EB?
d) What fraction of the segment DB is the segment CB?
e) What fraction of the segment DE is the segment DC?

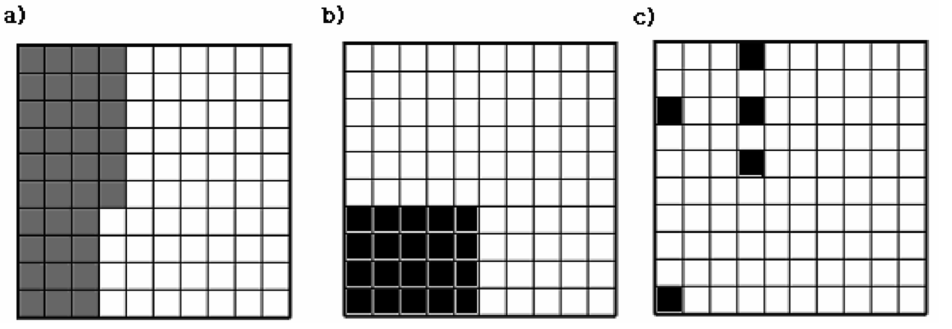
Ex.6 Which figure shows $\frac{2}{5}$ shaded? Assume each figure represents one whole.



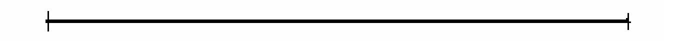
Ex.7 Which figure is $\frac{3}{4}$ shaded? Assume each figure represents one whole.



Ex.8 Write fraction which represents the NOT shaded regions. Assume each figure represents one whole and parts are equal.

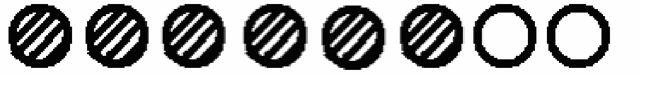


Ex.9 To find $\frac{1}{8}$ of the segment below

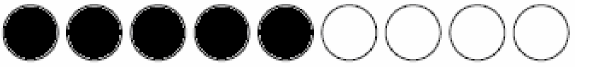


one has to divide it into _________ equal parts and take ______ of them. To find $\frac{2}{8}$ one has to divide it into _________ equal parts and take ______ of them.

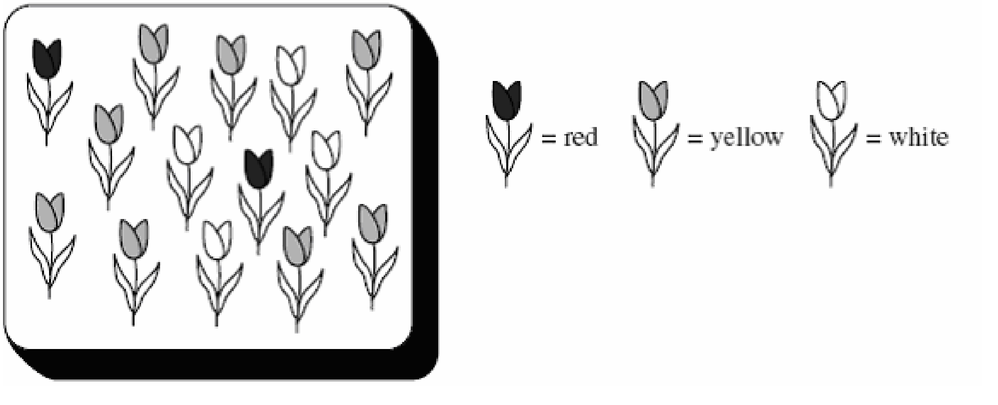
Ex.10 Which fraction shows the part of the set of balls that has stripes? In your answer identify the denominator and numerator.



Ex.11 What fraction of balls is NOT white?

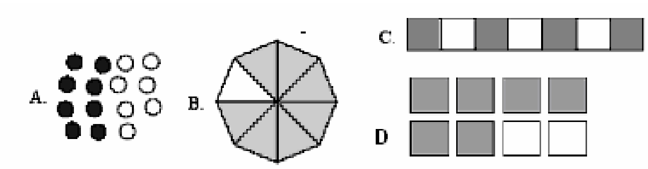


Ex.12 This is Ann's flower garden.

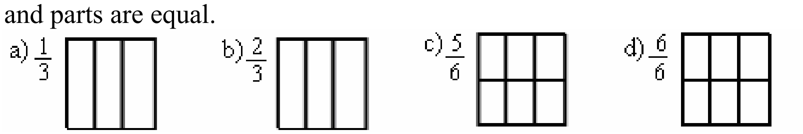


a) What fraction of the tulips is white?
b) What fraction of the tulips is red?
c) What fraction of the tulips is NOT white?

Ex.13 Which picture correctly shows $\frac{7}{8}$?



Ex.14 Shade the area corresponding to the following fraction. Assume each figure represents one whole and parts are equal.

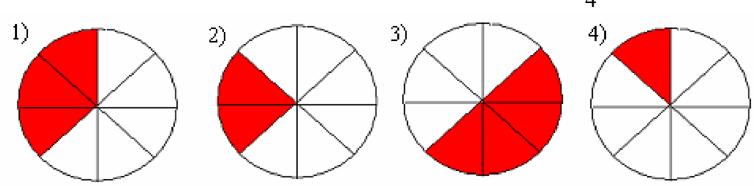


Ex.15 a) Which fractional part of a circle below is equal to $\frac{1}{8}$?

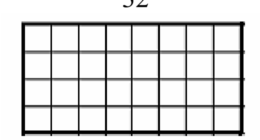
b) Which fractional part of a circle below is equal to $\frac{3}{8}$?

c) Which fractional part of a circle below is equal to $\frac{1}{2}$?

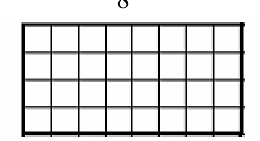
d) Which fractional part of a circle below is equal to $\frac{1}{4}$?



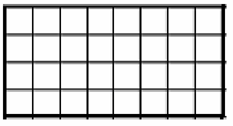
Ex.16 a) Shade $\frac{5}{32}$ of the figure below in three distinct ways.



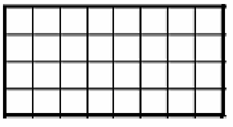
b) Shade $\frac{1}{8}$ of the figure below in three distinct ways.



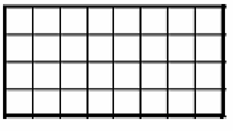
c) Shade $\frac{3}{4}$ of the figure below in three distinct ways.



d) Shade $\frac{1}{16}$ of the figure below in three distinct ways.

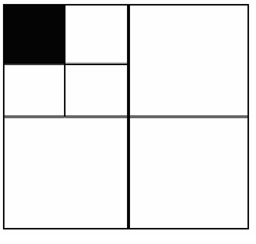


e) Shade $\frac{3}{16}$ of the figure below in three distinct ways.



Ex.17 Select the fraction which shows the part of the largest square that is shaded.

a) $\frac{1}{4}$ b) $\frac{1}{7}$ c) $\frac{1}{16}$ d) $\frac{1}{8}$



Ex.18 I ate $\frac{1}{3}$ of my 24 M&M's and I gave the rest to my sister. She ate $\frac{5}{8}$ of her M&M's. How many M&M's did my sister eat?

Ex.19 This is a chest of drawers.

$\frac{1}{2}$ of all drawers is to be used by John and $\frac{1}{6}$ by Tom. Assign appropriate number of drawers to Tom and John by labeling the drawers. What fraction of all drawers is still not used?

Ex.20 Austin plans to spend $\frac{3}{10}$ of all days in October studying math, $\frac{7}{30}$ studying biology, $\frac{1}{5}$ studying French, and $\frac{2}{15}$ studying chemistry. Plan the month of October for Austin by assigning each subject of study on appropriate number of days. Does Austin have any days when he does not have to study? What fraction of all days in October he does not have to study?

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
		1				
			30			

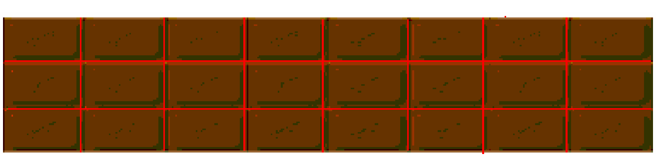
Ex.21 This is Ann. 

She is 4 feet tall. Her son Oliver is $\frac{3}{4}$ of Ann's height. Draw Oliver to the right of Ann (of appropriate height, of course). Ann's daughter, Lauren, is $\frac{1}{2}$ of Ann's height. Draw Lauren to the right of Oliver. The family dog is $\frac{1}{4}$ of Ann's height. Draw the dog to the right of Lauren.

Answer the following questions:

a) What fraction of Oliver's height is the dog?
b) What fraction of Oliver's height is Lauren?
c) What fraction of Laurens's height is the dog?
d) Their neighbor's cat is $\frac{1}{2}$ of height of Oliver's height. Is it taller than the family dog?

Ex.22 This is a chocolate bar divided into 24 equal parts.



a) You are supposed to give $\frac{1}{2}$ of it to your teacher and $\frac{1}{4}$ of it to your best friend. Draw pieces of chocolate (of appropriate size) for both of them. The rest of the chocolate bar is for you. What fraction of the chocolate bar do you get?

b) You are supposed to give $\frac{1}{2}$ of it to your teacher and $\frac{1}{4}$ *of the remaining* bar to your best friend. Draw pieces of chocolate (of appropriate size) for both of them. The rest of the chocolate bar is for you. What fraction of the chocolate bar do you get?

Ex.23 I took $\frac{1}{4}$ of a pizza, cut it into two equal parts, and ate one such part. What fraction of the whole pizza did I eat?

Ex.24 Mr. X divided a cake into 5 equal pieces and then he divided each of those pieces again into 6 even smaller parts. What fraction of the whole cake is such one small part?

Ex.25 There are 100 students who eat lunch. Of these, 80 students drink milk. What fraction of students drink milk at lunch?

Ex.26 Sean collects posters. He has 3 animal posters, 4 posters of sports teams, and 2 posters of musical bands. What fraction of his posters are of sports teams?

Ex.27 There are 5 red, 7 blue, and 3 green marbles in a jar. What fraction of all marbles in the jar is green?

Ex.28 Three-fiftieths of all states in the United States of America has names starting with the letter A. How many states have names starting with the letter A?

Ex.29 A class consists of 16 children.

a) How many children are $\frac{1}{2}$ of the class?

b) How many children are $\frac{1}{4}$ of the class?

Ex.30 Alice worked on her homework 60 minutes. Toby worked $\frac{1}{2}$ of what Alice worked. How many minutes did Toby worked?

Ex.31 To get to place A from place B, Mr. Smith has to walk 25 yards. If Mr. Smith walked $\frac{1}{5}$ of the way, how many yards did Mr. Smith walk?

Ex.32 Jane read $\frac{3}{4}$ of a 100 page book. How many pages did Jane read?

Ex 33 There are 30 miles between town A and town B. The distance between the town C and D is $\frac{2}{3}$ of what is between A and B. What is the distance in miles between C and D?

Ex.34 How many halves are equal to one whole? How many thirds are equal to one whole?

Ex.35 Write the following fraction $\frac{67}{89}$ as a division problem using '$\div$' sign.

Ex.36 If number 1 is divided into 5 equal parts, and we count 4 of them, what fraction is that?

Ex.37 a) If we divide 1 by 2, the result is equal to ___________ .
b) If we divide 1 by 3, the result is equal to ___________ .
c) If we divide 19 by 782, the result is equal to ____________ .

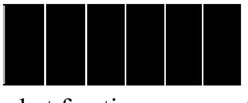
Ex.38 Rewrite in fraction form and express as either a positive or negative fraction.
a) $3 \div 14$ b) $-4 \div (-17)$ c) $-7 \div 26$ d) $5 \div (-7)$

Ex.39 Write the following integers as fractions $3, 1, -23, 7, 0$.

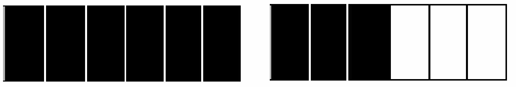
Ex.40 Among the following fractions circle all that are integers.

$\frac{103}{103}, \quad \frac{4}{8}, \quad \frac{1235}{1235}, \quad -\frac{30}{15}, \quad -\frac{60}{12}, \quad \frac{3}{5}, \quad \frac{15}{12}$

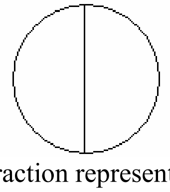
Ex.41 This is one unit



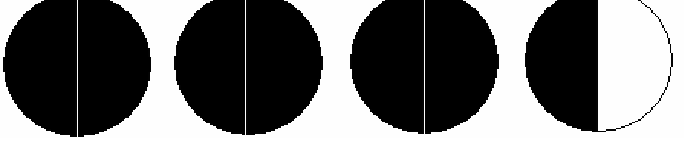
what fraction represents the shaded area?



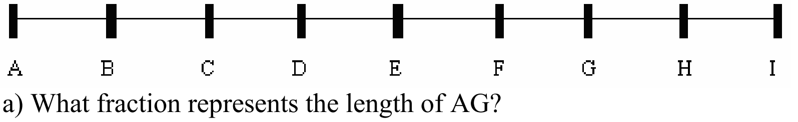
Ex.42 If this is one unit,



what fraction represents shaded area?



Ex.43 All marks are equally spaced. Assume that the length of the segment *AF* represents one unit.



a) What fraction represents the length of AG?

b) What fraction represents the length of AH?
c) What fraction represents the length of AI?

Ex.44 I have 4 shrubs in my garden: rhododendron 3 feet high, boxwood that is $\frac{5}{3}$ of rhododendron's height, azalea that is $\frac{4}{3}$ of rhododendron's height, and juniper that is $\frac{1}{2}$ of the height of azalea. Draw boxwood, azalea, and juniper next to rhododendron (of appropriate height, of course).

Answer the following questions:

a) What fraction of rhododendron's height is juniper?
b) What fraction of boxwood's height is rhododendron?
c) What fraction of azalea's height is rhododendron?
d) If I am $\frac{7}{6}$ of azalea's height, am I shorter or taller than azalea?

Ex.45 List three fractions that are greater than 1.

Ex.46 For each fraction determine if it is more than, less than or equal to 1. Write your answer using "<", ">", or "=".

a) $\frac{7}{5}$ b) $\frac{86}{85}$ c) $\frac{4}{5}$

d) $\frac{13}{5}$ e) $\frac{14}{14}$ f) $\frac{4}{89}$

Ex.47 Replace X with an integer to make a true statement.

a) $\frac{X}{8}$ is greater than 1. b) $\frac{7}{X}$ is greater than 1.

c) $\frac{X}{123}$ is less than 1. d) $\frac{X}{123}$ is equal to 1

Ex.48 Determine which fractions are proper and which are improper fractions.

$$\frac{7}{17} \qquad \frac{8}{5} \qquad \frac{3}{3} \qquad \frac{19}{51} \qquad \frac{8}{8} \qquad \frac{34}{33}$$

Ex.49 Replace ______ with either "=" or "$\neq$" sign to make a true statement.

a) $\frac{-7}{-4}$ ______ $\frac{7}{4}$ b) $\frac{-1}{2}$ ______ $-\frac{1}{2}$

c) $\dfrac{5}{-6}$ ______ $\dfrac{-5}{-6}$

d) $\dfrac{-3}{8}$ ______ $\dfrac{3}{-8}$

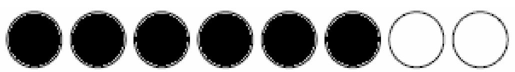
e) $-\dfrac{4}{9}$ ______ $\dfrac{-4}{-9}$

f) $\dfrac{3}{5}$ ______ $-\left(\dfrac{-3}{5}\right)$

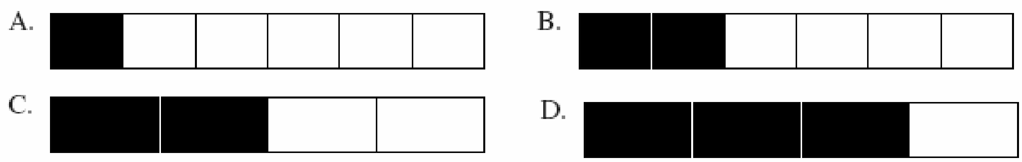
Ex.50 Write the following division problem $-35 \div 478$ using fraction notation. Write it in three equivalent forms by placing the minus sign differently.

Ex.51 Find the opposite of $\dfrac{9}{11}$ and write it in three different ways by placing minus sign differently.

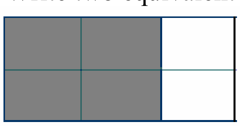
Ex.52 A fraction of the group of marbles below are black.



Which figure below is shaded to represent a fraction with the same value as the fraction that represents the black marbles?



Ex.53 Write two equivalent fractions corresponding to the shaded area.



Ex.54 The following pairs of fractions are equal (equivalent). Show that, indeed, they are equivalent by finding a number by which the numerator and denominator of the first fraction was multiplied or divided in order to get the second fraction.

a) $\dfrac{3}{7} = \dfrac{15}{35}$

b) $\dfrac{50}{60} = \dfrac{5}{6}$

c) $\dfrac{28}{36} = \dfrac{7}{9}$

d) $\dfrac{3}{2} = \dfrac{18}{12}$

Ex.55 Name three equivalent fractions to $\dfrac{4}{7}$.

Ex.56 Is it true that $\dfrac{3}{4}$ and $\dfrac{6}{8}$ are equal? Why? Write $\dfrac{3}{4}$ in two different equivalent forms.

Ex.57 Among the following find all fractions that are equivalent (equal) to $\frac{3}{5}$.

$$\frac{-9}{15}, \qquad \frac{30}{50}, \qquad \frac{5}{3}, \qquad -\frac{3}{5}, \qquad \frac{21}{25}, \qquad \frac{24}{40}, \qquad \frac{-6}{-10}$$

Ex.58 Find a fraction with the denominator 32 that is equivalent to $\frac{5}{8}$.

Ex.59 Find a fraction with the denominator 16 that is equivalent to $-\frac{3}{4}$.

Ex.60 Find a fraction with the denominator 64 that is equivalent to $\frac{3}{8}$.

Ex.61 Find the missing numerator.

a) $\frac{3}{16} = \frac{?}{32}$ b) $\frac{5}{9} = \frac{?}{900}$ c) $5 = \frac{?}{6}$

d) $\frac{5}{9} = \frac{?}{27}$ e) $\frac{3}{9} = \frac{?}{90}$ f) $-7 = -\frac{?}{8}$

Ex. 62 Reduce the following fractions.

a) $\frac{25}{100}$ b) $\frac{100}{2000}$

c) $\frac{27}{36}$ d) $\frac{9}{30}$

e) $-\frac{14}{49}$ f) $\frac{12}{4}$

g) $\frac{24}{36}$ h) $-\frac{50}{15}$

Ex. 63 Which of the following expressions can be simplified by cancelling number 7? Any time 7 can be cancelled, cancel it.

a) $\frac{7}{2\times 7}$ b) $\frac{2\div 7}{7}$ c) $\frac{7}{9+7}$

d) $\frac{3\times 7}{3-7}$ e) $\frac{4+7}{7\times 4}$ f) $\frac{-7(8)}{7}$

Lesson 6

Topics: Addition and subtraction of rational numbers.

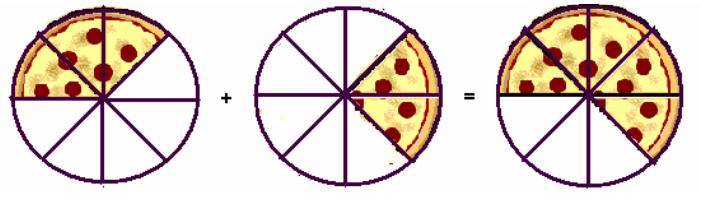
Addition of fractions with the same denominator

We will now learn how to add fractions whose denominators are the same. Suppose that we wish to add $\frac{3}{8}$ and $\frac{2}{8}$. The quantity $\frac{3}{8}$ represents 3 parts of a unit divided into 8 equal parts. So, for example, if we cut a pizza into 8 equal pieces, then 3 such pieces are $\frac{3}{8}$ of the pizza:

Similarly, 2 such pieces represent $\frac{2}{8}$ of the pizza

Since pieces are of the same size, to find what the sum $\frac{3}{8}+\frac{2}{8}$ is, we must add the number of pieces: 3 pieces and 2 pieces are 5 pieces; 5 pieces of pizza cut into 8 equal parts. Thus,

$$\frac{3}{8} \quad + \quad \frac{2}{8} \quad = \quad \frac{5}{8}$$



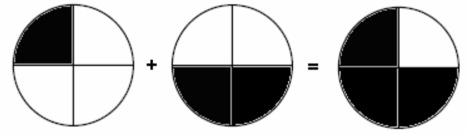
This above gives us the following rule.

To add two fractions with the same denominator, we add their numerators and keep the denominator unchanged.

$$\frac{3}{8}+\frac{2}{8}=\frac{\overbrace{3+2}^{\text{ADD NUMERATORS}}}{8}=\frac{5}{8}$$

Example 6.1 Illustrate the operation $\frac{1}{4}+\frac{2}{4}$ using a graphical representation of fractions of your choice and then perform the addition.

Solution:



$$\frac{1}{4}+\frac{2}{4}=\frac{1+2}{4}=\frac{3}{4}$$

Example 6.2 Add the following fractions.

a) $\frac{4}{9}+\frac{1}{9}$ b) $\frac{2}{7}+\frac{4}{7}+\frac{8}{7}$

Solution:

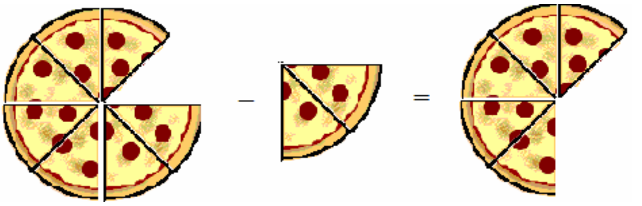
a) $\frac{4}{9}+\frac{1}{9}=\frac{4+1}{9}=\frac{5}{9}$

b) $\frac{2}{7}+\frac{4}{7}+\frac{8}{7}=\frac{2+4+8}{7}=\frac{14}{7}=14\div 7=2$. **If the answer can be expressed as an integer, you must write the integer for the final answer** (do not leave it as a fraction).

Subtracting fractions with the same denominator

The principle behind subtraction of fractions with the same denominator is the same as in the case of addition. Consider the following.

Suppose that we had 7 pieces of pizza cut into 8 equal pieces, that is $\frac{7}{8}$ of the pizza and we ate 2 such pieces, $\frac{2}{8}$. To find how much of the pizza is left, we must perform the subtraction $\frac{7}{8}-\frac{2}{8}$.



Seven eighths − two eighths = five eighths

$$\frac{7}{8} - \frac{2}{8} = \frac{5}{8}$$

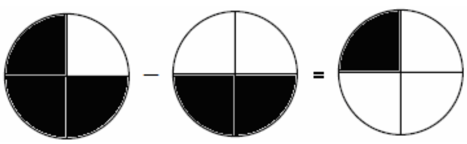
We have the following rule.

To subtract fractions with the same denominator, we subtract their numerators and keep the denominator unchanged.

$$\frac{7}{8}-\frac{2}{8}=\frac{\overbrace{7-2}^{\text{SUBTRACT NUMERATORS}}}{8}=\frac{5}{8}$$

<u>Example 6.3</u> Illustrate the operation $\frac{3}{4}-\frac{2}{4}$ using a graphical representation of fractions of your choice and then perform the subtraction.

Solution:



$$\frac{3}{4}-\frac{2}{4}=\frac{3-2}{4}=\frac{1}{4}$$

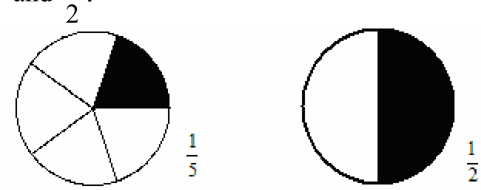
<u>Example 6.4</u> Perform the following operation $\frac{9}{13}-\frac{2}{13}$.

Solution:

Fractions have the same denominators. Therefore, we keep the denominator and perform the subtraction on numerators: $\frac{9}{13}-\frac{2}{13}=\frac{9-2}{13}=\frac{7}{13}$

<u>Adding fractions with different denominators</u>

We can only add objects of the same type. Fractions with the same denominators are objects of the same type: if denominators are the same, we divide units into the same number of pieces and thus "pieces" are of the same size. But if the denominators are different, the size of "pieces" is not the same Consider, for example, $\frac{1}{5}$ and $\frac{1}{2}$.



The size of "pieces" is different and thus we cannot simply add number of pieces. It is like in the old saying, one cannot add apples and bananas.

The main rule of addition of fractions is that we cannot do anything until the denominators are the same. Thus, to add fractions, we need the same (common) denominator. To find a **common denominator means to find a number that can be evenly divided by the denominators of both fractions.** The choice of **a common denominator that will *always* work is the product of denominators of the fractions.** In this case $5 \times 2 = 10$.

$\frac{1}{5}+\frac{1}{2}=$	Find a common denominator $5 \times 2 = 10$ and expand both fractions to their equivalent form with the new denominator 10. To this end, multiply the numerator and denominator of the fraction $\frac{1}{5}$ by 2, and the numerator and denominator of $\frac{1}{2}$ by 5.
$\frac{1\times 2}{5\times 2}+\frac{1\times 5}{2\times 5}=$	Perform the indicated multiplications.
$\frac{2}{10}+\frac{5}{10}=$	Once we have a common denominator, we add numerators and keep the denominator unchanged.
$\frac{2+5}{10}=$	Perform the indicated addition.
$\frac{7}{10}$	

Let us consider another example.

$\frac{1}{3}+\frac{5}{6}=$	As we have discussed, we can always use the product of denominators as a common denominator (in this case it would be $3\times 6=18$), but it will not necessarily be the least common denominator. For instance, in this example, the better choice for a common denominator is 6. **If we can quickly find a common denominator that is less than the product of denominators, we should use it** because smaller numbers make calculations easier (if we do not use it, we will still get the right answer). Let us choose a common denominator of 6 (number 6 can be divided by both 6 and 3 without a remainder). The next step is to rewrite the fractions in their equivalent form with the denominator 6. Notice, that the fraction $\frac{5}{6}$ is already written in the desired form.
$\frac{1\times 2}{3\times 2}+\frac{5}{6}=$	Perform the indicated multiplications.
$\frac{2}{6}+\frac{5}{6}=$	Add numerators and keep the denominator the same.
$\frac{2+5}{6}=$	Perform the indicated addition.
$\frac{7}{6}$	

<u>Example 6.5</u> Add the following fractions.

a) $\frac{2}{5}+\frac{3}{4}$. b) $\frac{3}{8}+\frac{1}{6}$ c) $\frac{3}{7}+\frac{2}{5}+\frac{1}{10}$

Solution:

a) We find a common denominator. It is $5 \times 4 = 20$.

$$\frac{2}{5}+\frac{3}{4}=\frac{2\times 4}{5\times 4}+\frac{3\times 5}{4\times 5}=\frac{8}{20}+\frac{15}{20}=\frac{8+15}{20}=\frac{23}{20}$$

b) We choose the common denominator to be 24. You could have chosen 48 (8×6), but 24 is simpler to use.

$$\frac{3}{8}+\frac{1}{6}=\frac{3\times 3}{8\times 3}+\frac{1\times 4}{6\times 4}=\frac{9}{24}+\frac{4}{24}=\frac{9+4}{24}=\frac{13}{24}$$

c) When adding three or more fractions, we find a common denominator for all of them. Multiplying all three denominators will always give us a common denominator (but not necessarily the least common denominator). In this case, we can use 70 instead of $7\times 5\times 10=350$.

$$\frac{3}{7}+\frac{2}{5}+\frac{1}{10}=\frac{3\times 10}{7\times 10}+\frac{2\times 14}{5\times 14}+\frac{1\times 7}{10\times 7}=\frac{30}{70}+\frac{28}{70}+\frac{7}{70}=\frac{30+28+7}{70}=\frac{65}{70}=\frac{13\times 5}{14\times 5}=\frac{13}{14}$$

Notice, that the last step was reducing the fraction. Any final answer should be given in its reduced form; thus, **we should get into the habit of always reducing the final answer**.

Adding/subtracting rational numbers

To add/subtract rational numbers, all of them must have a common denominator.

Consider $\frac{3}{4}-\frac{5}{3}$.

$\frac{3}{4}-\frac{5}{3}=$	Find a common denominator. In this case a common denominator is $4\times 3=12$. Rewrite both fractions in their equivalent form with the new denominator 12.
$\frac{3\times 3}{4\times 3}-\frac{5\times 4}{3\times 4}=$	Perform the indicated multiplications.
$\frac{9}{12}-\frac{20}{12}=$	Subtract numerators and keep the denominator the same.
$\frac{9-20}{12}=$	Perform the indicated operations.
$\frac{-11}{12}=$	We usually place the minus sign in front of a fraction.
$-\frac{11}{12}$	

Consider $-\frac{1}{5}-\left(-\frac{1}{2}\right)$.

$-\frac{1}{5}-\left(-\frac{1}{2}\right)=$	First, remove all “double signs”. Minus followed by a minus changes to plus.
$-\frac{1}{5}+\frac{1}{2}=$	Find a common denominator, $5\times 2=10$, rewrite fractions in their equivalent form with the denominator 10.
$-\frac{1\times 2}{5\times 2}+\frac{1\times 5}{2\times 5}=$	Perform the indicated multiplications.
$-\frac{2}{10}+\frac{5}{10}=$	Use the fact that $-\frac{2}{10}=\frac{-2}{10}$.
$\frac{-2}{10}+\frac{5}{10}=$	Write the expression as one fraction.
$\frac{-2+5}{10}=$	Perform the indicated operations.
$\frac{3}{10}$	

Add the following $\frac{3}{7}+\frac{-2}{5}$.

$\frac{3}{7}+\frac{-2}{5}=$	Use the fact $\frac{-2}{5}=-\frac{2}{5}$ to rewrite the expression.
$\frac{3}{7}+\left(-\frac{2}{5}\right)=$	Replace the “double signs”. Plus followed by a minus changes to minus.
$\frac{3}{7}-\frac{2}{5}=$	Find a common denominator: $7\times 5=35$.
$\frac{3\times 5}{7\times 5}-\frac{2\times 7}{5\times 7}=$	Perform the indicated multiplications.
$\frac{15}{35}-\frac{14}{35}=$	Write the expression as one fraction.
$\frac{15-14}{35}=$	Perform the indicated operations.
$\frac{1}{35}$	

<u>Example 6.6</u> Perform the indicated operations.

a) $-\frac{1}{2}+\frac{3}{8}$ b) $\frac{1}{6}+\left(\frac{-2}{5}\right)$ c) $-\frac{2}{3}-\left(-\frac{3}{4}\right)-\left(\frac{1}{6}\right)$

Solution:

a) $-\frac{1}{2}+\frac{3}{8}=-\frac{1\times 4}{2\times 4}+\frac{3}{8}=-\frac{4}{8}+\frac{3}{8}=\frac{-4}{8}+\frac{3}{8}=\frac{-4+3}{8}=\frac{-1}{8}=-\frac{1}{8}$

b) $\frac{1}{6}+\left(\frac{-2}{5}\right)=\frac{1}{6}+\left(-\frac{2}{5}\right)=\frac{1}{6}-\frac{2}{5}=\frac{1\times 5}{6\times 5}-\frac{2\times 6}{5\times 6}=\frac{5}{30}-\frac{12}{30}=$

$$\frac{5-12}{30}=\frac{-7}{30}=-\frac{7}{30}$$

c) $-\frac{2}{3}-\left(-\frac{3}{4}\right)-\left(\frac{1}{6}\right)=-\frac{2}{3}+\frac{3}{4}-\frac{1}{6}=-\frac{2\times 8}{3\times 8}+\frac{3\times 6}{4\times 6}-\frac{1\times 4}{6\times 4}=$

$-\frac{16}{24}+\frac{18}{24}-\frac{4}{24}=\frac{-16+18-4}{24}=\frac{-2}{24}=\frac{-1\times 2}{12\times 2}=\frac{-1}{12}=-\frac{1}{12}$ **Always reduce!**

A summary is presented in the following table.

HOW TO ADD/SUBTRACT RATIONAL NUMBERS

Step 1. If there are any 'double signs' in the expression, replace them with one sign according to the rules: **(+)(+) → (+)** **(−)(+) → (−)**
(−)(−) → (+) (+)(−) → (−)

Step 2. If the fractions do not have the same denominator, find a common denominator (a number that can be evenly divided by the denominators of all fractions). The product of denominators can always be used as a common denominator. If you can quickly find a smaller one, use the smaller one.

Step 3. Rewrite the fractions in their equivalent form using a common denominator (multiply each denominator by a number such that the product is equal to the desired common denominator; you must also multiply the corresponding numerator by the same number as in the denominator).

Step 4. Put all numerators (together with signs in front of them) over one fraction bar, keep the denominator unchanged.

Step 5. Perform operations indicated in the numerator.

Step 6. Reduce the fraction, if possible.

For example, $-\frac{1}{6}-\left(-\frac{3}{5}\right)=-\frac{1}{6}+\frac{3}{5}=-\frac{1\times 5}{6\times 5}+\frac{3\times 6}{5\times 6}=-\frac{5}{30}+\frac{18}{30}=\frac{-5}{30}+\frac{18}{30}=\frac{-5+18}{30}=\frac{13}{30}$

Common mistakes and misconceptions

Mistake 6.1

Do not confuse the rules for multiplication with those for addition and subtraction. When adding, it is NOT correct to simply add both the numerators and denominators**:** $\frac{1}{2}+\frac{3}{7}\neq\frac{1+3}{2+7}$**.**

Remember that if the denominators are the same, you keep the denominator and if the denominators are not the same, you must find a common denominator. NEVER ADD the DENOMINATORS.

Exercises with Answers (For answers see Appendix A)

Ex.1 Rational numbers preserve the commutative property of multiplication. Illustrate the commutative property of addition using the fractions $\frac{3}{7}$ and $\frac{4}{5}$. Is subtraction of rational numbers commutative?

Ex.2 Make an appropriate drawing, write the corresponding numerical statement and perform the operation to answer the following questions.

a) I ate $\frac{2}{9}$ of a cake in the morning and $\frac{5}{9}$ of the cake in the afternoon. How much cake did I eat altogether?

b) Michael and Charles were paining a fence. On Monday Michael painted $\frac{1}{3}$ of the fence and Charles also painted $\frac{1}{3}$ of it. What fraction of the fence was painted on Monday?

c) Clarissa drank $\frac{2}{5}$ glass of milk from a glass with $\frac{3}{5}$ glass of milk in it. How much milk is still in the glass?

d) $\frac{4}{10}$ of a pizza was left after the party. I ate additional $\frac{1}{10}$ of a pizza. How much pizza is still left?

Ex.3 Add or subtract the following fractions. Show all your steps and use "=" correctly. Always reduce if possible.

a) $\frac{1}{5}+\frac{7}{5}$ b) $\frac{2}{27}+\frac{7}{27}$ c) $\frac{3}{10}+\frac{5}{10}$

d) $\frac{11}{6}+\frac{5}{6}+\frac{7}{6}$ e) $\frac{2}{9}+\frac{5}{9}+\frac{2}{9}$ f) $\frac{1}{4}+\frac{5}{4}+\frac{9}{4}+\frac{7}{4}$

g) $\frac{4}{5}-\frac{1}{5}$ h) $\frac{11}{8}-\frac{3}{8}$ i) $\frac{7}{12}-\frac{1}{12}$

Ex.4 Add the following fractions. Show all your steps and use "=" correctly. Always reduce if possible.

a) $\frac{4}{5}+\frac{1}{6}$ b) $\frac{2}{9}+\frac{3}{8}$

c) $\frac{4}{15}+\frac{1}{6}$ d) $\frac{13}{20}+\frac{5}{10}$

e) $\frac{5}{6}+\frac{7}{12}$

f) $\frac{3}{4}+\frac{7}{9}$

g) $\frac{5}{8}+\frac{1}{3}+\frac{3}{4}$

h) $\frac{3}{7}+\frac{5}{14}+\frac{2}{7}$

Ex.5 Perform the following operations. As a first step, write each expression as a single fraction, for example: $-\frac{1}{5}+\frac{-2}{5}=\frac{-1+(-2)}{5}$. Pay attention to signs and parentheses. Use "=" correctly. Always reduce if possible.

a) $-\frac{1}{6}+\frac{5}{6}$

b) $\frac{2}{8}-\frac{3}{8}$

c) $-\frac{4}{25}-\frac{1}{25}$

d) $-\frac{13}{7}+\frac{13}{7}$

e) $\frac{-5}{7}+\frac{1}{7}$

f) $-\frac{3}{4}+\frac{-1}{4}$

g) $-\frac{5}{3}+\frac{1}{3}-\frac{2}{3}$

h) $\frac{-3}{14}-\frac{-5}{14}-\frac{3}{14}$

i) $-\frac{-2}{9}+\frac{4}{9}-\frac{-1}{9}$

j) $\frac{3}{2}-\frac{-1}{2}+\frac{-3}{2}$

Ex.6 Perform the following operations. Show all your steps and use "=" correctly. Always reduce if possible.

a) $-\frac{3}{14}+\frac{5}{7}$

b) $-\frac{2}{3}-\frac{3}{11}$

c) $\frac{2}{3}-\frac{4}{5}$

d) $-\frac{2}{9}-\frac{3}{7}$

e) $\frac{-4}{5}-\frac{13}{15}$

f) $-\frac{7}{10}+\frac{2}{5}-\frac{3}{20}$

g) $\frac{4}{9}+\frac{1}{2}-\frac{11}{18}$

h) $\frac{-3}{8}-\frac{1}{4}-\frac{5}{2}$

Ex.7 Perform the following operations. Show all your steps and use "=" correctly. Always reduce if possible.

a) $-\frac{2}{3}+\left(-\frac{1}{4}\right)$

b) $\frac{3}{7}-\left(-\frac{3}{8}\right)$

c) $-\frac{7}{12}-\frac{-5}{18}$

d) $\frac{-3}{8}+\left(+\frac{5}{12}\right)$

e) $-\frac{2}{7}-\left(+\frac{5}{9}\right)$

f) $-\left(-\frac{4}{7}\right)-\left(-\frac{3}{49}\right)$

g) $-\frac{2}{5}+\frac{7}{2}-\left(-\frac{3}{25}\right)$

h) $-\left(\frac{-3}{4}\right)+\left(\frac{-6}{5}\right)$

i) $-\frac{1}{6}-\left(+\frac{7}{9}\right)-\left(-\frac{2}{3}\right)$

j) $\frac{-2}{15}-\left(\frac{3}{-10}\right)$

Ex.8 Perform the following operations. Show all your steps and use "=" correctly. Always reduce if possible.

a) $-\frac{5}{4}-\frac{3}{5}$

b) $\frac{2}{3}+\left(-\frac{3}{7}\right)$

c) $-\frac{4}{9}+\frac{-5}{12}$

d) $-\left(-\frac{5}{6}\right)-\frac{3}{8}$

e) $\frac{-3}{14}-\frac{9}{2}$

f) $-\frac{7}{23}+\left(-\frac{15}{23}\right)$

g) $-\left(-\frac{4}{7}\right)+\left(-\frac{2}{9}\right)$

h) $-\frac{3}{4}-\frac{10}{9}$

i) $\frac{7}{18}-\left(\frac{-4}{9}\right)$

j) $-\left(-\frac{3}{4}\right)+\left(\frac{-5}{7}\right)$

k) $-\frac{4}{7}-\frac{15}{21}-\frac{1}{7}$

l) $-\left(\frac{-2}{11}\right)+\frac{-3}{22}$

m) $\frac{4}{-9}+\frac{-1}{5}$

n) $-\left(\frac{-3}{6}\right)-\frac{5}{9}+\frac{1}{3}$

o) $-\frac{3}{7}-\left(\frac{-3}{7}\right)$

p) $-\left(\frac{3}{-8}\right)-\left(-\frac{5}{6}\right)$

q) $-\left(\frac{3}{10}\right)-\left(+\frac{9}{30}\right)-\left(-\frac{6}{20}\right)$

r) $-\frac{5}{6}+\frac{7}{12}-\left(-\frac{1}{8}\right)$

s) $-\frac{1}{5}-\frac{7}{10}+\left(-\frac{3}{2}\right)$

t) $-\frac{5}{66}+\frac{5}{66}+\frac{77}{121}-\frac{77}{121}$

Ex.9 Replace x with a number to make the statement true.

a) $\frac{2+x}{8}=\frac{3}{8}$

b) $\frac{3}{13}+\frac{x}{13}=\frac{9}{13}$

c) $\frac{3}{13}+x=\frac{9}{13}$

d) $\frac{x}{5}-\frac{1}{5}=\frac{3}{5}$

e) $-\frac{2}{9}-\frac{x}{9}=-\frac{4}{9}$

f) $-\frac{7}{11}+x=0$

g) $-\frac{2}{7}-\frac{3}{7}=-\frac{5}{x}$

h) $\frac{1}{5}+\frac{3}{10}=\frac{x}{10}+\frac{3}{10}$

i) $\frac{2}{3}-\frac{1}{5}=\frac{x}{15}-\frac{3}{15}$

j) $\frac{1}{3}+x=1$

k) $x+\frac{2}{5}=1$

l) $\frac{11}{8}-x=1$

Lesson 7

Topics: Multiplication of rational numbers; Exponentiation; Reciprocals; Division of rational numbers.

Multiplication of fractions and integers

Recall that multiplication of integers is a shortcut for addition.

$$4 \times 2 = \underbrace{2+2+2+2}_{4\ times}.$$

Similarly, the multiplication of a fraction by 4 is equivalent to adding the fraction 4 times.

$$4 \times \frac{2}{5} = \underbrace{\frac{2}{5}+\frac{2}{5}+\frac{2}{5}+\frac{2}{5}}_{4\ times}$$

And so,

$$4 \times \frac{2}{5} = \frac{2}{5}+\frac{2}{5}+\frac{2}{5}+\frac{2}{5} = \frac{2+2+2+2}{5} = \frac{4\times 2}{5}$$

The above gives us the following rule.

To multiply a fraction by an integer multiply the numerator of the fraction by the integer and keep the denominator unchanged. $9 \times \frac{2}{7} = \frac{9\times 2}{7} = \frac{18}{7}$

Notice that since multiplication is commutative (i.e. order of factors can be changed) we also have

$$\frac{2}{7}\times 9 = 9 \times \frac{2}{7} = \frac{9\times 2}{7} = \frac{18}{7}$$

Example 7.1 Write the multiplication $3 \times \frac{5}{11}$ as an addition and then perform the operation.

Solution:

$$3 \times \frac{5}{11} = \frac{5}{11}+\frac{5}{11}+\frac{5}{11} = \frac{5+5+5}{11} = \frac{3\times 5}{11} = \frac{15}{11}$$

Example 7.2 Perform the following multiplications.

a) $4 \times \frac{2}{3}$ b) $\frac{2}{7}\times 100$

Solution:

a) $4\times\frac{2}{3}=\frac{4\times 2}{3}=\frac{8}{3}$

b) $\frac{2}{7}\times 100=100\times\frac{2}{7}=\frac{100\times 2}{7}=\frac{200}{7}$.

Multiplication of fractions without canceling

To multiply fractions we multiply the numerators to get the new numerator and multiply the denominators to get the new denominator. We determine the sign of the result according to the same rules we use for the product of integers.

$(+)(+)\to(+)$ $\quad$ $(-)(+)\to(-)$

$(-)(-)\to(+)$ $\quad$ $(+)(-)\to(-)$

For example,

$$\frac{1}{3}\times\frac{2}{5}=\frac{\overbrace{1\times 2}^{\substack{\text{MULTIPLY}\\ \textit{NUMERATORS}}}}{\underbrace{3\times 5}_{\substack{\text{MULTIPLY}\\ \textit{DENOMINATORS}}}}=\frac{2}{15}$$

Multiply $\frac{1}{6}\times\left(-\frac{5}{7}\right)$.

$\frac{1}{6}\times\left(-\frac{5}{7}\right)=$	Since we are multiplying fractions with opposite signs, the product is negative. Put the minus sign in front, and drop all other minus signs. Rewrite the minus sign in each consecutive step.
$-\frac{1}{6}\times\frac{5}{7}=$	Multiply the numerator by the numerator and the denominator by the denominator.
$-\frac{1\times 5}{6\times 7}=$	Perform the indicated multiplications.
$-\frac{5}{42}$	

Example 7.3 Perform the indicated operation $-\frac{2}{3}\times\left(-\frac{4}{5}\right)$.

Solution:

$-\frac{2}{3}\times\left(-\frac{4}{5}\right)=\frac{2}{3}\times\frac{4}{5}=\frac{2\times 4}{3\times 5}=\frac{8}{15}$

The result is positive because we are multiplying two negative numbers.

Multiplication of fractions with canceling

Often, after performing multiplication, we must reduce the resulting fraction. We could shorten the procedure if, before multiplying numerators and denominators, we "**cancel**" fractions. **Canceling means to divide one factor of the numerator and one factor of the denominator by the same number (**like in case of reducing**).**

Multiply $\frac{4}{7}\times\frac{7}{5}$.

$\frac{4}{7}\times\frac{7}{5}=$ We first determine the sign. Both fractions are positive, the product is also positive. Multiply the numerators. Multiply the denominators.

$\frac{4\times 7}{7\times 5}=$ A common factor 7 in the denominator and numerator can be cancelled.

$\frac{4\times\overset{1}{\cancel{7}}}{\underset{1}{\cancel{7}}\times 5}=$ We cancel 7 by dividing the numerator and denominator by 7. We replace 7's with 1's. Multiply 4 and 1 in the numerator and 1 and 5 in the denominator.

$\frac{4\times 1}{1\times 5}=\frac{4}{5}$

Multiply $\frac{6}{7}\times\left(-\frac{14}{9}\right)$.

$\frac{6}{7}\times\left(-\frac{14}{9}\right)=$ Multiplication of one positive and one negative number results in a negative answer. Write a minus sign in front of the expression (and omit the minus in front of the second fraction). Other than making sure to rewrite the minus sign in every consecutive step of the calculations, we no longer have to worry about the sign of the final answer.

$-\frac{6}{7}\times\frac{14}{9}=$ Multiply numerators and multiply denominators.

$-\frac{6\times 14}{7\times 9}=$ Look for common factors of the numerator and denominator. 3 is a common factor of 6 and 9, 7 is a common factor of 7 and 14.

$-\frac{2\times 3\times 2\times 7}{1\times 7\times 3\times 3}=$ Cancel fraction by dividing the numerator and denominator by 3 and 7.

$-\frac{2\times\overset{1}{\cancel{3}}\times 2\times\overset{1}{\cancel{7}}}{1\times\underset{1}{\cancel{7}}\times\underset{1}{\cancel{3}}\times 3}=$ Multiply all numbers in the numerator and denominator

$-\frac{2\times 1\times 2\times 1}{1\times 1\times 3\times 1}=-\frac{4}{3}$

Example 7.4 Perform the indicated operations.

a) $\frac{3}{2}\times\frac{1}{3}$ b) $-\frac{4}{9}\times\left(-\frac{3}{5}\right)$ c) $-\frac{25}{4}\times\frac{8}{5}$

Solution:

a) $\frac{3}{2}\times\frac{1}{3}=\frac{3\times 1}{2\times 3}=\frac{\overset{1}{\cancel{3}}\times 1}{2\times \underset{1}{\cancel{3}}}=\frac{3\times 1}{2\times 3}=\frac{1}{2}$

b) $-\frac{4}{9}\times\left(-\frac{3}{5}\right)=\frac{4\times 3}{9\times 5}=\frac{4\times \overset{1}{\cancel{3}}}{\underset{1}{\cancel{3}}\times 3\times 5}=\frac{4\times 1}{1\times 3\times 5}=\frac{4}{15}$

c) $-\frac{25}{4}\times\frac{8}{5}=-\frac{25\times 8}{4\times 5}=-\frac{5\times 5\times 2\times 4}{4\times 5}=-\frac{\overset{1}{\cancel{5}}\times 5\times 2\times \overset{1}{\cancel{4}}}{\underset{1}{\cancel{4}}\times \underset{1}{\cancel{5}}}=-\frac{1\times 5\times 2\times 1}{1\times 1}=$

$-\frac{10}{1}=-10$

Multiplication of fractions and integers revisited

We learned that to multiply an integer by a fraction we multiply the integer by the numerator. For example, $5\times\frac{2}{7}=\frac{5\times 2}{7}=\frac{10}{7}$. Notice, that if instead we decide to write the integer as a fraction and apply the rules for multiplication of fractions, then, as expected, we get the same result.

$$5\times\frac{2}{7}=\frac{5}{1}\times\frac{2}{7}=\frac{5\times 2}{1\times 7}=\frac{10}{7}$$

Example 7.5 Perform the multiplication $3\times\frac{2}{5}$ by representing the integer as a faction.

Solution:

Write 3 as a fraction $\frac{3}{1}$ and perform multiplication $3\times\frac{2}{5}=\frac{3}{1}\times\frac{2}{5}=\frac{3\times 2}{1\times 5}=\frac{6}{5}$.

When multiplying integers and fractions, canceling should be done whenever possible. For example,

$$15\times\frac{7}{10}=\frac{15}{1}\times\frac{7}{10}=\frac{\overset{3}{\cancel{15}}\times 7}{1\times \underset{2}{\cancel{10}}}=\frac{3\times 7}{1\times 2}=\frac{21}{2}$$

Notice, that equivalently, we can cancel 15 with 10 before bringing 15 to the numerator.

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$$15\times\frac{7}{10}=\overset{3}{\cancel{15}}\times\frac{7}{\underset{2}{\cancel{10}}}=\frac{3\times 7}{2}=\frac{21}{2}$$

Example 7.6 Perform the multiplication $4 \times \frac{3}{8}$ by canceling the integer with the denominator before performing the multiplication.

Solution:

$$4 \times \frac{3}{8} = \overset{1}{\cancel{4}} \times \frac{3}{\underset{2}{\cancel{8}}} = \frac{1 \times 3}{2} = \frac{3}{2}$$

Multiplication of more than two fractions or integers

Multiply $\frac{10}{3}\left(-\frac{2}{7}\right)\left(-\frac{9}{25}\right)$.

$\frac{10}{3}\left(-\frac{2}{7}\right)\left(-\frac{9}{25}\right) =$	Recall that an even number of negative numbers in the product makes the result positive. The product above contains two negative numbers, so the result is positive.
$\frac{10}{3} \times \frac{2}{7} \times \frac{9}{25} =$	Multiply numerators and denominators.
$\frac{10 \times 2 \times 9}{3 \times 7 \times 25} =$	Notice that 3 and 9 have a common factor. Also, 25 and 10 have a common factor.
$\frac{5 \times 2 \times 2 \times 3 \times 3}{3 \times 7 \times 5 \times 5} =$	Cancel common factors. Notice that **numbers that are cancelled do not have to be adjacent to each other**. Also, **there is no specific order that must be used in cancellation as long as both a numerator and a denominator are divided by the same number.**
$\frac{\overset{1}{\cancel{5}} \times 2 \times 2 \times \overset{1}{\cancel{3}} \times 3}{\underset{1}{\cancel{3}} \times 7 \times 5 \times \underset{1}{\cancel{5}}} =$	Multiply all numbers in the numerator and denominator.
$\frac{1 \times 2 \times 2 \times 1 \times 3}{1 \times 7 \times 5 \times 1} = \frac{12}{35}$	

It is worth mentioning that if we perform all possible cancellations, the answer will already be in a reduced form. If we "miss" some cancellations, we will have to reduce the answer at the end. **Always give your final answer in a reduced form.**

Example 7.7 Perform the indicated operations.

a) $\left(-\frac{15}{8}\right) \times \left(-\frac{1}{7}\right) \times \left(-\frac{12}{5}\right)$ b) $\frac{24}{15} \times 5 \times \left(-\frac{7}{8}\right)$

Solution:

$$a)\ \left(-\frac{15}{8}\right)\times\left(-\frac{1}{7}\right)\times\left(-\frac{12}{5}\right)=-\frac{15}{8}\times\frac{1}{7}\times\frac{12}{5}=-\frac{15\times1\times12}{8\times7\times5}=-\frac{3\times5\times1\times4\times3}{2\times4\times7\times5}=$$

$$-\frac{3\times5\times1\times4\times3}{2\times4\times7\times5}=-\frac{3\times\overset{1}{\cancel{5}}\times1\times\overset{1}{\cancel{4}}\times3}{2\times\underset{1}{\cancel{4}}\times7\times\underset{1}{\cancel{5}}}=-\frac{3\times1\times1\times1\times3}{2\times1\times7\times1}=-\frac{9}{14}$$

The result of the multiplication is negative because the product consists of an odd number of negative factors.

$$b)\ -\frac{24}{15}\times5\times\left(-\frac{7}{8}\right)=-\frac{24}{15}\times\frac{5}{1}\times\left(-\frac{7}{8}\right)=\frac{24}{15}\times\frac{5}{1}\times\frac{7}{8}=\frac{24\times5\times7}{15\times1\times8}=\frac{3\times8\times5\times7}{3\times5\times1\times8}$$

$$=\frac{\overset{1}{\cancel{3}}\times\overset{1}{\cancel{8}}\times\overset{1}{\cancel{5}}\times7}{\underset{1}{\cancel{3}}\times\underset{1}{\cancel{5}}\times1\times\underset{1}{\cancel{8}}}=\frac{1\times1\times1\times7}{1\times1\times1\times1}=\frac{7}{1}=7$$

This time the result of the multiplication is positive. It is because there are an even number of negative factors in the product.

Multiplication of rational numbers

The table below summarizes the necessary steps.

HOW TO MULTIPLY RATIONAL NUMBERS
Step 1. Determine the sign of the result using: $(+)(+)\to(+)$ $(-)(+)\to(-)$ $(-)(-)\to(+)$ $(+)(-)\to(-)$ Step 2 Find all common factors of the numerator and denominator and cancel them. Step 3. Multiply all numerators and put them over the product of all denominators. For example, $\frac{3}{5}\times\left(-\frac{1}{9}\right)=-\frac{3}{5}\times\frac{1}{9}=-\frac{3\times1}{5\times9}=-\frac{3\times1}{5\times3\times3}=-\frac{\overset{1}{\cancel{3}}\times1}{5\times3\times\underset{1}{\cancel{3}}}=-\frac{1\times1}{5\times3\times1}=-\frac{1}{15}$
If one of the numbers is an integer, multiply the integer by a numerator or equivalently write it as a fraction with the denominator 1, and proceed with steps 1–3 from above. For example, $-2\times\frac{4}{3}=-\frac{2\times4}{3}=-\frac{8}{3}$ or equivalently $-2\times\frac{4}{3}=-\frac{2}{1}\times\frac{4}{3}=-\frac{2\times4}{1\times3}=-\frac{8}{3}$
If a product consists of more than two numbers, count the number of negative numbers. If the number of negative numbers is even, the result is positive; if odd, the result is negative. Before multiplying all numerators together and all denominators together, cancel common factors if possible. For example, $-\frac{2}{11}\left(-\frac{3}{2}\right)\left(\frac{22}{5}\right)=\frac{2}{11}\left(\frac{3}{2}\right)\left(\frac{22}{5}\right)=\frac{1\times3\times2}{1\times1\times5}=\frac{6}{5}$ or $-2\left(-\frac{3}{5}\right)\left(-\frac{1}{7}\right)=-\frac{2\times3\times1}{5\times7}=-\frac{6}{35}$

Equivalent fractions revisited

Multiplication rules give us a different look at the procedure of reducing and expanding fractions. Below we are expanding the fraction $\frac{2}{7}$ to an equivalent fraction whose denominator is 21, using the properties of multiplication.

$\frac{2}{7} =$	Multiply the fraction by 1. We can do that, since this multiplication does not change the value of fraction.
$\frac{2}{7} \times 1 =$	Since $1 = \frac{3}{3}$ and "equals can be substituted for equals", we can replace 1 with $\frac{3}{3}$. .
$\frac{2}{7} \times \frac{3}{3} =$	Apply the rules of multiplication.
$\frac{2 \times 3}{7 \times 3} =$	Multiply the numerator by numerator and the denominator by denominator.
$\frac{6}{21}$	

Similarly, we can apply the rules of multiplication to reducing of fractions. Let us reduce $\frac{6}{21}$.

$\frac{6}{21} =$	Notice that 6 and 21 have a common factor 3. Apply the rules of multiplication ("backwards").
$\frac{2}{7} \times \frac{3}{3} =$	But $\frac{3}{3} = 1$ and "equals can be substituted for equals" so, we replace $\frac{3}{3}$ with 1.
$\frac{2}{7} \times 1 =$	And, since the multiplication by 1 does not change the result, we get
$\frac{2}{7}$	

Notice, that simplifying $\frac{6}{21}$ to $\frac{2}{7}$ is a reverse process to expanding of $\frac{2}{7}$ to $\frac{6}{21}$.

Finding a fraction of a given quantity

What is $\frac{1}{3}$ of 12? To answer, we treat 12 as one unit, divide it into 3 equal parts and take one such part. Thus $\frac{1}{3}$ of 12 is 4. We have learned a lot since we first encountered this type of problem. At this point we can explicitly name the operation that we need to perform to get the answer. To find what is $\frac{1}{3}$ of 12, we multiply the numbers: $\frac{1}{3} \times 12 = \frac{1}{\cancel{3}_1} \times \cancel{12}^4 = 1 \times 4 = 4$

Example 7.8 A recipe calls for $\frac{1}{4}$ pound of sugar. If I want to prepare only $\frac{1}{2}$ of the portion that is in the recipe, how much sugar do I have to use?

Solution:
We must perform the multiplication.

$$\frac{1}{2}\times\frac{1}{4}=\frac{1\times1}{2\times4}=\frac{1}{8}$$

I need to use $\frac{1}{8}$ pounds of sugar.

Example 7.9 What is $\frac{5}{6}$ of 7?

Solution:
We must perform the following multiplication.

$$\frac{5}{6}\times7=\frac{5\times7}{6}=\frac{35}{6}$$

$\frac{5}{6}$ of 7 is equal to $\frac{35}{6}$.

Exponentiation of fractions

Exponential expressions, such as $\left(\frac{3}{2}\right)^4$ have the same meaning as in the case of integers. The exponent 4 indicates how many times the base $\frac{3}{2}$, is multiplied by itself. Hence,

$$\left(\frac{3}{2}\right)^4=\underbrace{\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}\times\frac{3}{2}}_{4\ times}$$

One important fact to remember is that if we wish to raise a fraction to any power, the **fraction must be enclosed by parentheses to have the exponent apply to the entire fraction.**

If we wrote $\frac{3^4}{2}$, instead of $\left(\frac{3}{2}\right)^4$, it would mean that only 3 is raised to the fourth power.

Example 7.10 Expand, that is write without exponential notation. Do not evaluate.

a) $\left(\frac{2}{5}\right)^3$ b) $\frac{2^3}{5}$ c) $\left(-\frac{2}{5}\right)^3$

Solution:

a) $\left(\frac{2}{5}\right)^3 = \left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)$

b) $\frac{2^3}{5} = \frac{2\times 2\times 2}{5}$

c) $\left(-\frac{2}{5}\right)^3 = \left(-\frac{2}{5}\right)\left(-\frac{2}{5}\right)\left(-\frac{2}{5}\right)$.

Example 7.11 Whenever possible, rewrite the expression using exponential notation.

a) $\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)$ b) $\left(-\frac{7}{8}\right)\left(-\frac{7}{8}\right)+\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)$ c) $-\left(\frac{1}{9}\right)\left(\frac{1}{9}\right)\left(\frac{1}{9}\right)\left(-\frac{4}{5}\right)\left(-\frac{4}{5}\right)$

Solution:

a) $\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)\left(\frac{7}{8}\right)\left(\frac{7}{8}\right) = \left(\frac{7}{8}\right)^4$

b) $\left(-\frac{7}{8}\right)\left(-\frac{7}{8}\right)+\left(\frac{7}{8}\right)\left(\frac{7}{8}\right) = \left(-\frac{7}{8}\right)^2 + \left(\frac{7}{8}\right)^2$

c) $-\left(\frac{1}{9}\right)\left(\frac{1}{9}\right)\left(\frac{1}{9}\right)\left(-\frac{4}{5}\right)\left(-\frac{4}{5}\right) = -\left(\frac{1}{9}\right)^3\left(-\frac{4}{5}\right)^2$

To evaluate an exponential expression, we may first expand and then apply the rules for multiplication in order to arrive at a value. For example,

$$\left(\frac{3}{7}\right)^2 = \left(\frac{3}{7}\right)\left(\frac{3}{7}\right) = \frac{3\times 3}{7\times 7} = \frac{3^2}{7^2} = \frac{9}{49}$$

or

$$\left(-\frac{1}{2}\right)^5 = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = -\frac{1\times 1\times 1\times 1\times 1}{2\times 2\times 2\times 2\times 2} = -\frac{1^5}{2^5} = -\frac{1}{32}$$

(in the last evaluation, we used the rule that if an odd number of negative numbers is multiplied, the result of the multiplication is negative)

Equivalently, instead of expanding (which might sometimes be tedious), **to raise a fraction to a given power determine the sign of the result and raise the numerator and denominator to the indicated power.** For example,

$$\left(-\frac{4}{9}\right)^2 = \frac{4^2}{9^2} = \frac{16}{81}$$

(in this evaluation, we used the rule that if an even number of negative numbers is multiplied, the result of the multiplication is positive).

Example 7.12 Perform the indicated operations.

a) $\left(-\frac{5}{4}\right)^2$ b) $-\left(\frac{5}{4}\right)^2$

Solution:

a) Recall that since the minus sign is in parentheses, $-\frac{5}{4}$ is raised to the second power and this means that a negative number is multiplied twice. The result is positive.

$$\left(-\frac{5}{4}\right)^2 = \frac{5^2}{4^2} = \frac{25}{16}$$

b) Since the minus sign is outside the parentheses it is NOT "involved" in the exponentiation as occurred in the previous example. We rewrite the minus sign and square $\frac{5}{4}$ by squaring the numerator and denominator.

$$-\left(\frac{5}{4}\right)^2 = -\frac{5^2}{4^2} = -\frac{25}{16}$$

Reciprocals

The number 0 "plays" a special role in addition. If you add 0 to any number, the result will be the same number. The number that "plays" an analogous role in multiplication is the number 1. If you multiply any number by 1, the result will be the same number. We will now introduce the concept of reciprocals. The reciprocals in multiplication are like opposites in addition.

If the product of two numbers is equal to 1, we say that the numbers are reciprocals.
(Compare to: If the sum of any two numbers is equal to 0, we say that the numbers are opposites)
For example,

$\frac{3}{4}$ is the reciprocal of $\frac{4}{3}$. This is because $\frac{3}{4}\times\frac{4}{3} = \frac{3\times 4}{4\times 3} = \frac{1\times 1}{1\times 1} = 1$

$-\frac{2}{7}$ is the reciprocal of $-\frac{7}{2}$. This is because $-\frac{2}{7}\times\left(-\frac{7}{2}\right) = \frac{2\times 7}{7\times 2} = \frac{1\times 1}{1\times 1} = 1$

Notice that **the reciprocal of any fraction is obtained by "flipping" (inverting) the fraction** (changing its numerator with its denominator). If the number is not written as a fraction, rewrite it as a fraction and then flip if. For example, $4 = \frac{4}{1}$, so after flipping we get $\frac{1}{4}$. Thus $\frac{1}{4}$ is the reciprocal of 4. Indeed, $\frac{1}{4}\times 4 = 1$.

Zero is the only number that does not have a reciprocal.

Example 7.13 Find the reciprocal of the following numbers.

a) $\frac{7}{8}$ b) 3

Solution:

a) Flip the fraction to get the reciprocal $\frac{8}{7}$.

b) Write 3 as a fraction and then flip it. $3 = \frac{3}{1}$, so the reciprocal is $\frac{1}{3}$.

Division of fractions

To divide fractions determine the sign of the result following the same rules that we have for division of integers:

$(+)(+) \rightarrow (+)$ $(-)(+) \rightarrow (-)$

$(-)(-) \rightarrow (+)$ $(+)(-) \rightarrow (-)$

The result is obtained by replacing the divisor by its reciprocal and then performing multiplication instead of division.

For example, let us divide $-\frac{3}{5} \div \frac{4}{11}$.

First, we determine the sign of the division. Since one fraction is positive and the other negative, the answer is negative. We find the reciprocal of $\frac{4}{11}$ by "flipping" it, $\frac{11}{4}$. We replace operation of division with operation of multiplication.

Divisor is replaced by its reciprocal

$$-\frac{3}{5} \div \frac{4}{11} = -\frac{3}{5} \times \frac{11}{4} = -\frac{3 \times 11}{5 \times 4} = -\frac{33}{20}$$

Division is changed to multiplication

Divide $-\frac{10}{3} \div \left(-\frac{15}{7}\right)$.

$-\frac{10}{3} \div \left(-\frac{15}{7}\right) =$ Determine the sign. It is positive, since we are dividing negative numbers.

$\frac{10}{3} \div \frac{15}{7} =$ "Flip" $\frac{15}{7}$ to get $\frac{7}{15}$. Change division to multiplication.

$\frac{10}{3} \times \frac{7}{15} =$ Multiply numerators and denominators

$\frac{10\times 7}{3\times 15}=$ Notice that 10 and 15 have a common factor 5.

$\frac{2\times 5\times 7}{3\times 3\times 5}=$ Cancel 5 in the numerator and denominator.

$\frac{2\times \cancel{5}^{1}\times 7}{3\times 3\times \cancel{5}_{1}}=$ Multiply all numbers in the numerator and denominator.

$\frac{2\times 1\times 7}{3\times 3\times 1}=\frac{14}{9}$

<u>Example 7.14</u> Perform the indicated division. $\frac{2}{5}\div\left(-\frac{8}{15}\right)$.

Solution:

$$\frac{2}{5}\div\left(-\frac{8}{15}\right)=-\frac{2}{5}\times\frac{15}{8}=-\frac{2\times 15}{5\times 8}=-\frac{2\times 3\times 5}{5\times 2\times 4}=-\frac{1\times 3\times 1}{1\times 1\times 4}=-\frac{3}{4}.$$

<u>Division of fractions and integers</u>

Divide $3\div\frac{1}{2}$.

Replace the second fraction with its reciprocal and division with multiplication.

$$3\div\frac{1}{2}=3\times\frac{2}{1}=3\times 2=6$$

Divide $\frac{1}{2}\div 2$

Replace the second fraction with its reciprocal and division with multiplication.

$$\frac{1}{2}\div 2=\frac{1}{2}\div\frac{2}{1}=\frac{1}{2}\times\frac{1}{2}=\frac{1\times 1}{2\times 2}=\frac{1}{4}$$

<u>Division denoted by a fraction bar</u>

Perform the following operation $\frac{\frac{3}{8}}{-\frac{21}{4}}$.

Fraction bar means division. Rewrite the expression using "$\div$" sign and proceed as usual.

$$\frac{\frac{3}{8}}{-\frac{21}{4}} = \frac{3}{8} \div \left(-\frac{21}{4}\right) =$$

Division of two opposite sign numbers gives a negative result.

$$-\frac{3}{8} \div \frac{21}{4} =$$

Flip the second fraction and change division to multiplication.

$$-\frac{3}{8} \times \frac{4}{21} =$$

Multiply numerator by numerator and denominator by denominator.

$$-\frac{3 \times 4}{8 \times 21} =$$

Notice that 3 and 21 have a common factor 3,.while 4 and 8 have a common factor 4.

$$-\frac{3 \times 4}{4 \times 2 \times 7 \times 3} =$$

Cancel 3's and 4's

$$-\frac{\overset{1}{\cancel{3}} \times \overset{1}{\cancel{4}}}{\underset{1}{\cancel{4}} \times 2 \times 7 \times \underset{1}{\cancel{3}}} =$$

Multiply all numbers in the numerator and denominator.

$$-\frac{1 \times 1}{1 \times 2 \times 7 \times 1} = -\frac{1}{14}$$

Divide $\dfrac{\frac{1}{2}}{6}$

The long fraction bar indicates that we divide $\frac{1}{2} \div 6$ (not $1 \div \frac{2}{6}$). We get,

$$\frac{\frac{1}{2}}{6} = \frac{1}{2} \div 6 = \frac{1}{2} \div \frac{6}{1} = \frac{1}{2} \times \frac{1}{6} = \frac{1 \times 1}{2 \times 6} = \frac{1}{12}$$

Example 7.15 Perform the following division $\dfrac{-4}{\frac{2}{5}}$.

Solution:

$$\frac{-4}{\frac{2}{5}} = -4 \div \frac{2}{5} = -4 \times \frac{5}{2} = -\frac{4 \times 5}{2} = -\frac{2 \times 2 \times 5}{2} = -\frac{10}{1} = -10\,.$$

<u>*Division of rational numbers*</u>

We present a summary in the following table.

HOW TO DIVIDE RATIONAL NUMBERS
Step 1. Determine the sign of the result using (+)(+) → (+) (−)(+) → (−) (−)(−) → (+) (+)(−) → (−) Step 2. Replace the divisor (the second fraction) with its reciprocal ("flip it") and change the division to multiplication. Step 3. Follow the rules for multiplication, i.e. cancel as much as possible, multiply all numerators and put them over the product of all denominators. For example, $\frac{7}{5} \div \left(-\frac{8}{3}\right) = -\frac{7}{5} \times \frac{3}{8} = -\frac{7 \times 3}{5 \times 8} = -\frac{21}{40}$
If one of the numbers is an integer not written as a fraction, write it as a fraction with the denominator 1, and proceed with steps 1–3 above. For example, $-3 \div \frac{3}{5} = -\frac{3}{1} \div \frac{3}{5} = -\frac{3}{1} \times \frac{5}{3} = -\frac{3}{1} \times \frac{5}{3} = -\frac{5}{1} = -5$
If instead of "$\div$" sign, the operation of division is indicated by a fraction bar, rewrite it using "$\div$" sign and follow the rules for division of fractions. Longest fraction bar indicates the place where the sign "$\div$"should be used. For example, $\frac{\frac{3}{5}}{\frac{4}{7}} = \frac{3}{5} \div \frac{4}{7} = \frac{3}{5} \times \frac{7}{4} = \frac{3 \times 7}{5 \times 4} = \frac{21}{20}$ or $\frac{\frac{5}{7}}{2} = \frac{5}{7} \div 2 = \frac{5}{7} \times \frac{1}{2} = \frac{5 \times 1}{7 \times 2} = \frac{5}{14}$

Exercises with Answers (For answers see Appendix A)

Ex.1 Rational numbers preserve the commutative property of multiplication. Illustrate the commutative property of multiplication using the fractions $\frac{3}{7}$ and $\frac{4}{5}$. Is division of rational numbers commutative?

Ex.2 Write the multiplication as an addition and then perform the following operations.

a) $4\times\dfrac{1}{5}$ b) $\dfrac{3}{13}\times5$ c) $2\times\dfrac{23}{107}$

Ex.3 Perform the following operations, if possible. Otherwise, write "undefined". Show all your steps and use "=" sign correctly.

a) $\dfrac{1}{5}\times\dfrac{2}{3}$ b) $-\dfrac{9}{7}\times\dfrac{8}{5}$ c) $\dfrac{7}{3}\times\left(-\dfrac{2}{3}\right)$

d) $\left(-\dfrac{5}{4}\right)\left(\dfrac{9}{2}\right)$ e) $\left(-\dfrac{19}{3}\right)\left(\dfrac{-1}{8}\right)$ f) $\dfrac{8}{7}\times\dfrac{10}{9}$

Ex.4 Perform the following operations, if possible. Otherwise, write "undefined". Show all your steps and use "=" sign correctly.

a) $\dfrac{9}{7}x\dfrac{4}{3}$ b) $-\dfrac{5}{2}\times\dfrac{8}{15}$

c) $\left(-\dfrac{10}{11}\right)\left(-\dfrac{22}{5}\right)$ d) $\left(-\dfrac{5}{4}\right)\left(\dfrac{6}{10}\right)$

e) $\dfrac{7}{2}x\left(-\dfrac{4}{21}\right)$ f) $\left(-\dfrac{24}{3}\right)\left(-\dfrac{19}{8}\right)$

Ex.5 Perform the following operations, if possible. Otherwise, write "undefined". Show all your steps and use "=" sign correctly.

a) $2\times\dfrac{1}{3}$ b) $\dfrac{3}{5}\times7$

c) $100\times\dfrac{23}{27}$ d) $-7\times\dfrac{4}{7}$

e) $\dfrac{-8}{9}\times(-18)$ f) $12\left(-\dfrac{2}{15}\right)$

Ex.6 Perform the following operations, if possible. Otherwise, write "undefined". Show all your steps and use "=" sign correctly.

a) $\dfrac{11}{7}\times\dfrac{4}{3}\times\dfrac{1}{11}$ b) $\dfrac{-5}{2}\times\dfrac{4}{6}\times\left(-\dfrac{4}{35}\right)$

c) $-\dfrac{5}{12}\times\left(-\dfrac{3}{10}\right)\times\left(-\dfrac{4}{8}\right)$ d) $\left(-\dfrac{24}{7}\right)\left(-\dfrac{1}{6}\right)\left(\dfrac{4}{32}\right)$

e) $\dfrac{7}{321}\times\left(\dfrac{-9}{14}\right)\times\left(\dfrac{321}{6}\right)$ f) $\dfrac{31}{3}\times\dfrac{10}{6}\times\left(-\dfrac{4}{31}\right)\times\dfrac{18}{5}$

Ex.7 Perform the following operations, if possible. Otherwise, write "undefined". Show all your steps and use "=" sign correctly.

a) $-4\times\left(-\dfrac{2}{13}\right)$ b) $\left(-\dfrac{6}{11}\right)\left(-\dfrac{2}{5}\right)$

c) $\frac{15}{7}\times\frac{10}{9}$ d) $\left(\frac{4}{-35}\right)\times 70$

e) $-6\times\left(\frac{2}{15}\right)$ f) $\frac{2}{9}\times\left(-\frac{4}{5}\right)$

g) $-\frac{4}{7}\times\frac{14}{8}$ h) $\left(-\frac{1}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{20}\right)$

i) $\frac{5}{18}\left(-\frac{12}{5}\right)$ j) $\frac{1}{3}\times\left(-\frac{13}{8}\right)\times\left(-\frac{1}{31}\right)\times\left(-\frac{12}{13}\right)$

Ex.8 Austin has $\frac{5}{8}$ pizza left in the fridge. At lunch he ate $\frac{1}{10}$ of it. What fraction of the original pizza did he eat?

Ex.9 Charles gave $\frac{1}{8}$ of a chocolate bar to Noah. What fraction of the chocolate bar did Noah eat, if he ate $\frac{2}{5}$ of what he got from Charles?

Ex.10 There are 48 marbles in a jar. $\frac{5}{16}$ of them are blue. How many blue marbles are in the jar?

Ex.11 Jon studied for $\frac{1}{2}$ of an hour. If $\frac{1}{3}$ of the time he studied biology, how much time did he study biology?

Ex.12 What is $\frac{5}{6}$ of 7?

Ex.13 Expand, that is write without exponential notation. Do not evaluate.

a) $\left(\frac{1}{9}\right)^4$ b) $\left(-\frac{9}{8}\right)^3$ c) $-\left(\frac{9}{8}\right)^3$ d) $\frac{9^3}{8}$

Ex.14 Whenever possible, rewrite using exponential notation.

a) $\frac{2}{9}\times\frac{2}{9}\times\frac{2}{9}\times\frac{2}{9}\times\frac{2}{9}\times\frac{2}{9}\times\frac{2}{9}$ b) $-\left(\frac{5}{12}\right)\left(\frac{5}{12}\right)\left(\frac{5}{12}\right)-\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)$

c) $\left(\frac{-7}{8}\right)\left(-\frac{7}{8}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)$ d) $-\frac{5\times 5}{6\times 6\times 6}$

Ex.15 Perform the indicated operations.

a) $\left(-\frac{9}{8}\right)^2$ b) $-\left(\frac{1}{2}\right)^4$ c) $\frac{4^3}{9}$

d) $\dfrac{-5^2}{6}$ e) $\left(-\dfrac{1}{2}\right)^4$ f) $-\left(\dfrac{-2}{3}\right)^3$

Ex.16 Find the reciprocal of the following numbers.

a) $\dfrac{2}{7}$ b) $\dfrac{-9}{5}$ c) 4

d) $-\dfrac{14}{15}$ e) -6 f) $\dfrac{1}{-3}$

Ex.17 Perform the following operations, if possible. Otherwise, write "undefined". Show all your steps and use "=" sign correctly.

a) $\dfrac{2}{5}\div\dfrac{5}{3}$ b) $-\dfrac{9}{7}\div\dfrac{8}{14}$

c) $\dfrac{5}{3}\div\left(-\dfrac{2}{3}\right)$ d) $\left(-\dfrac{19}{4}\right)\div\left(-\dfrac{19}{8}\right)$

e) $\left(-\dfrac{15}{4}\right)\div\left(\dfrac{10}{6}\right)$ f) $\left(-\dfrac{9}{13}\right)\div\left(-\dfrac{2}{5}\right)$

g) $\dfrac{16}{9}\div\left(-\dfrac{1}{6}\right)$ h) $\dfrac{8}{7}\div\dfrac{10}{49}$

Ex.18 Perform the following operations, if possible. Otherwise, write "undefined". Show all your steps and use "=" sign correctly.

a) $2\div\dfrac{5}{3}$ b) $-14\div\dfrac{28}{3}$

c) $\dfrac{5}{3}\div(-5)$ d) $\left(-\dfrac{1}{4}\right)\div 4$

e) $(-1)\div\left(-\dfrac{19}{8}\right)$ f) $\dfrac{3}{7}\div(-3)$

g) $\dfrac{16}{9}\div(-8)$ h) $-20\div\left(-\dfrac{2}{5}\right)$

Ex.19 Perform the following operations, if possible. Otherwise, write "undefined". Show all your steps and use "=" sign correctly.

a) $\dfrac{2}{\dfrac{1}{5}}$ b) $\dfrac{-\dfrac{3}{10}}{\dfrac{1}{5}}$

c) $\dfrac{-\dfrac{2}{21}}{-\dfrac{1}{14}}$ d) $\dfrac{-\dfrac{14}{3}}{\dfrac{7}{9}}$

e) $\dfrac{\frac{2}{15}}{-6}$ f) $\dfrac{-\frac{4}{11}}{10}$

g) $\dfrac{-9}{\frac{-12}{13}}$ h) $\dfrac{-\frac{8}{9}}{\frac{5}{6}}$

Ex.20 Perform the following operations, if possible. Otherwise, write "undefined". Show all your steps and use "=" sign correctly.

a) $-2\div\left(-\frac{2}{3}\right)$ b) $-28\times\left(-\frac{4}{21}\right)$

c) $\dfrac{-\frac{2}{9}}{-6}$ d) $\dfrac{\frac{4}{7}}{\frac{-1}{28}}$

e) $(-50)\div\left(-\frac{30}{7}\right)$ f) $\left(\frac{7}{9}\right)^2$

g) $\frac{64}{11}\div(-8)$ h) $\dfrac{2}{\frac{3}{20}}$

i) $\frac{4^2}{7}$ j) $\frac{6}{7}\times\frac{3}{5}$

k) $-10\times\frac{2}{5}$ l) $\left(-\frac{27}{8}\right)\left(\frac{-2}{35}\right)$

m) $\frac{2}{3}\div(-7)$ n) $-\left(\frac{1}{100}\right)^3$

o) $12\times\frac{5}{18}$ p) $0\div\left(-\frac{2}{7}\right)$

q) $\frac{18}{4}\times\left(-\frac{4}{16}\right)$ r) $\frac{31}{3}\times 9\times\left(-\frac{4}{36}\right)\times\frac{18}{31}$

s) $\left(-\frac{50}{7}\right)\left(-\frac{1}{6}\right)\left(\frac{7}{100}\right)$ t) $-\left(-\frac{3}{4}\right)^3$

u) $\dfrac{-\frac{1}{5}}{0}$ v) $-5\left(\frac{-35}{15}\right)\times\left(-\frac{3}{14}\right)$

w) $\left(-\frac{27}{3}\right)\left(-\frac{1}{18}\right)$ x) $\frac{1}{5}\times(-5)$

y) $\dfrac{-\dfrac{4}{5}}{\dfrac{1}{25}}$ z) $\dfrac{1}{7}\times\left(\dfrac{-25}{8}\right)\times\left(\dfrac{-14}{3}\right)\times\left(\dfrac{-1}{15}\right)$

Ex.21 Write the following statement using the correct mathematical language and then evaluate them.

a) The product of $\dfrac{2}{7}$ and -14.

b) $\dfrac{9}{10}$ to the second power.

c) The product of $-\dfrac{3}{5}$ and $-\dfrac{10}{7}$.

d) The quotient of -3 and $\dfrac{7}{8}$.

e) The quotient of $-\dfrac{2}{9}$ and $-\dfrac{1}{27}$.

f) The quotient of $-\dfrac{3}{11}$ and 4.

g) The product of $\dfrac{1}{7}, -2,$ and $-\dfrac{21}{8}$.

h) The quotient of $-\dfrac{3}{16}$ and $\dfrac{9}{2}$.

i) $-\dfrac{2}{5}$ raised to the third power.

j) The product of $-\dfrac{33}{8}, -\dfrac{12}{21},$ and $-\dfrac{28}{33}$.

Ex.22 Replace x with a number to make the statement true.

a) $\dfrac{1}{2}\cdot x=\dfrac{3}{2}$ b) $\dfrac{3}{5}\div\dfrac{7}{8}=\dfrac{3}{5}\cdot x$ c) $\dfrac{8}{3}\times\dfrac{x}{7}=\dfrac{16}{21}$

d) $\left(\dfrac{x}{3}\right)^2=\dfrac{4}{9}$ e) $-\dfrac{2}{5}\cdot x=-\dfrac{8}{25}$ f) $\dfrac{7}{123}\cdot x=\left(\dfrac{7}{123}\right)^2$

g) $\dfrac{5}{6}\div x=\dfrac{5}{6}$ h) $\dfrac{5}{6}\div x=1$ i) $4\div x=8$

j) $x\div\dfrac{1}{2}=6$ k) $\dfrac{3}{\dfrac{5}{7}}=3\div x$ l) $\dfrac{3}{\dfrac{5}{7}}=3\cdot x$

Lesson 8

Topics: All operations combined on rational numbers; Comparing rational numbers.

Performing more than one operation on rational numbers

When the expression involves more than one operation, as always, we follow the order of operations**.** Recall that the rules for operations on fractions are

> **Fraction addition/subtraction.** Check to see if the fractions have the same denominator. If not, find a common denominator. Convert all fractions to have this common denominator. Add/subtract the numerators, and use the same denominator.
>
> **Fraction multiplication.** Multiply the numerators, and the denominators.
>
> **Fraction division.** Find the reciprocal of the divisor, and multiply by it.

Remember, **a common denominator is required only when performing addition or subtractions**.

Example 8.1 Perform the following operations.

$$\text{a) } \left(-\frac{3}{5}-\frac{1}{10}\right)\div\frac{6}{7} \qquad \text{b) } \frac{3}{4}\times\frac{2}{5}+\left(-\frac{1}{2}\right) \qquad \text{c) } \frac{2}{3}+\left(\frac{1}{3}\right)^2-\frac{5}{9}$$

Solution:

$$\text{a)}\left(-\frac{3}{5}-\frac{1}{10}\right)\div\frac{6}{7}=\left(-\frac{3\times 2}{5\times 2}-\frac{1}{10}\right)\div\frac{6}{7}=\left(-\frac{6}{10}-\frac{1}{10}\right)\div\frac{6}{7}=\left(\frac{-6-1}{10}\right)\div\frac{6}{7}=$$
$$\left(\frac{-7}{10}\right)\div\frac{6}{7}=\frac{-7}{10}\times\frac{7}{6}=-\frac{7\times 7}{10\times 6}=-\frac{49}{60}\ .$$

$$\text{b) }\frac{3}{4}\times\frac{2}{5}+\left(-\frac{1}{2}\right)=\frac{3\times 2}{4\times 5}+\left(-\frac{1}{2}\right)=\frac{3\times 2}{2\times 2\times 5}+\left(-\frac{1}{2}\right)=\frac{3\times 1}{2\times 5}+\left(-\frac{1}{2}\right)=$$
$$\frac{3}{10}+\left(-\frac{1}{2}\right)=\frac{3}{10}-\frac{1}{2}=\frac{3}{10}-\frac{1\times 5}{2\times 5}=\frac{3}{10}-\frac{5}{10}=\frac{3-5}{10}=\frac{-2}{10}=\frac{-1\times 2}{5\times 2}=\frac{-1}{5}=-\frac{1}{5}$$

$$\text{c)}\frac{2}{3}+\left(\frac{1}{3}\right)^2-\frac{5}{9}=\frac{2}{3}+\frac{1^2}{3^2}-\frac{5}{9}=\frac{2}{3}+\frac{1\times 1}{3\times 3}-\frac{5}{9}=\frac{2}{3}+\frac{1}{9}-\frac{5}{9}=$$
$$\frac{2\times 3}{3\times 3}+\frac{1}{9}-\frac{5}{9}=\frac{6}{9}+\frac{1}{9}-\frac{5}{9}=\frac{6+1-5}{9}=\frac{2}{9}$$

Comparing rational numbers

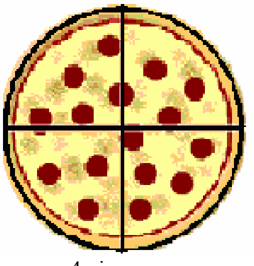
Suppose we have two fractions and we need to know which one is bigger. How do we do it? Well, sometimes it is easy and we can tell just by looking and sometimes we need to do some additional operations.

- Let us look at the "easy" case. Which one is greater $\frac{2}{9}$ or $\frac{8}{7}$?

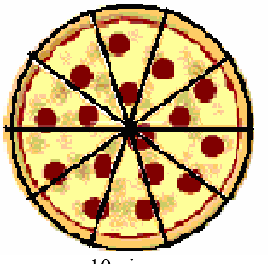
Since the numerator is less than the denominator, fraction $\frac{2}{9}$ is less than 1. But $\frac{8}{7}$ is more then 1. So, $\frac{2}{9} < \frac{8}{7}$.

- Fractions with the same numerators. Which one is greater $\frac{1}{4}$ or $\frac{1}{10}$?

Remember that the denominator tells us how many pieces the unit is cut into. Suppose that we can have 1 piece of one of these pizzas.


4 pieces

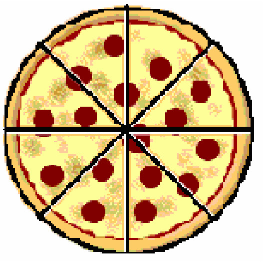
or


10 pieces

If we are very hungry which should we choose? $\frac{1}{4}$ of a pizza is much bigger than $\frac{1}{10}$ of a pizza. So, $\frac{1}{4} > \frac{1}{10}$. **If numerators are the same, the fraction with lower denominator is bigger.**

- Fractions with the same denominator. Which one is greater $\frac{3}{8}$ or $\frac{7}{8}$?

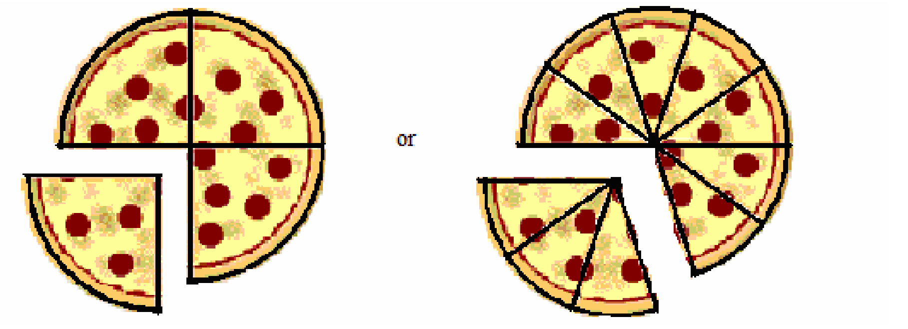
The denominators are both 8, so let us look at a pizza cut into 8 pieces.



If we are very hungry ... Do we want 3 pieces or 7 pieces? Of course, 7 is much more, so $\frac{3}{8} < \frac{7}{8}$.

If denominators are the same, the fraction with bigger numerator is bigger.

- Fractions with different denominators and numerators. Which is greater $\frac{3}{4}$ or $\frac{7}{10}$?



3 out of 4 pieces　　　　7 out of 10 pieces

In the example above, we can see that $\frac{3}{4}$ is greater $\frac{7}{10}$ than, but in general we cannot rely on the drawing. But, if we could rename the fractions in such a way that both have the same denominator, then we would be able to compare them easily. To do that, we find a common denominator (a number that can be evenly divided by the denominators of both fractions). A common denominator for 4 and 10 is 20. Once we have a common denominator, we rewrite the fractions in their equivalent form with the common denominator.

$$\frac{3}{4}=\frac{3\times 5}{4\times 5}=\frac{15}{20} \qquad \frac{7}{10}=\frac{7\times 2}{10\times 2}=\frac{14}{20}$$

Since $\frac{15}{20} > \frac{14}{20}$ we conclude that $\frac{3}{4} > \frac{7}{10}$.

- Comparing negative fractions. Which is greater $-\frac{5}{6}$ or $-\frac{2}{3}$?

First, compare fractions without minus signs, $\frac{5}{6}$ and $\frac{2}{3}$. Find a common denominator, 6. Name equivalent fractions with the new denominator. $\frac{5}{6}$ is already in a desired form and $\frac{2}{3}=\frac{2\times 2}{3\times 2}=\frac{4}{6}$. Since, $\frac{5}{6} > \frac{4}{6}$, we conclude that $\frac{5}{6} > \frac{2}{3}$. To obtain the comparison of the negative fractions, “reverse’ the inequality sign to get $-\frac{5}{6} < -\frac{2}{3}$.

Let us summarize all of the above.

HOW TO COMPARE RATIONAL NUMBERS

Any positive rational number is greater than zero or any negative rational number.

For example, $\frac{2}{3} > 0$, $\frac{3}{5} > -\frac{9}{2}$

If both rational numbers are positive and have the same numerator, the one with the smaller denominator is bigger.

For example, $\frac{3}{5} > \frac{3}{7}$ (since 5 is less than 7).

If both rational numbers are positive and have the same denominator, the one with the bigger numerator is bigger.

For example, $\frac{7}{9} > \frac{5}{9}$ (since 7 is more than 5).

If both rational numbers are positive but have different numerators and denominators, find equivalent fractions with a common denominator and compare them.

For example, compare $\frac{3}{7}$ and $\frac{1}{2}$.

Find a common denominator: 14 $(7 \times 2 = 14)$

Rewrite fractions: $\frac{3}{7} = \frac{3 \times 2}{7 \times 2} = \frac{6}{14}$ $\quad \frac{1}{2} = \frac{1 \times 7}{2 \times 7} = \frac{7}{14}$

Since $\frac{6}{14} < \frac{7}{14}$ we conclude that $\frac{3}{7} < \frac{1}{2}$

If both rational numbers are negative, “drop” their minus signs and compare the resulting positive rational numbers. To get the comparison of the original (negative) numbers, reverse the inequality sign (“$<$” becomes “$>$”; “$>$” becomes “$<$”).

For example, compare $-\frac{3}{7}$ and $-\frac{1}{2}$.

Compare $\frac{3}{7}$ and $\frac{1}{2}$ following the rules for comparison of positive numbers to get $\frac{3}{7} < \frac{1}{2}$.

Reverse the inequality sign. $-\frac{3}{7} > -\frac{1}{2}$

Any negative rational number is less than zero.

For example, $-\frac{2}{3} < 0$

Example 8.2 Compare the following fractions. Replace "and" with $<$ or $>$ symbol.

a) $\frac{4}{7}$ and $\frac{2}{7}$ b) $\frac{5}{3}$ and $\frac{5}{2}$

c) $\frac{1}{3}$ and $\frac{7}{5}$ d) $\frac{3}{7}$ and $\frac{2}{5}$

e) $\frac{7}{3}$ and $-\frac{2}{9}$ f) $-\frac{5}{6}$ and $-\frac{7}{9}$

Solution:

a) The denominators are equal (i.e. "size of pieces" are the same). We compare the number of pieces, i.e. numerators. Since 4 > 2, we conclude $\frac{4}{7} > \frac{2}{7}$.

b) The numerators are equal, so we have "the same number of pieces". If we divide a unit into more pieces, the pieces are going to be smaller. It means the smaller the denominator, the bigger the fraction. Thus $\frac{5}{3} < \frac{5}{2}$.

\` c) Notice, that $\frac{1}{3}$ is less than 1 (since the numerator is less than the denominator). At the same time, $\frac{7}{5}$ is greater than 1 (the numerator is greater than the numerator). Thus $\frac{1}{3} < \frac{7}{5}$.

d) Both numerators and denominators are different. We need to find a common denominator. We choose $7 \times 5 = 35$ as a common denominator. We name the equivalent fractions with the denominator 35.

$$\frac{3}{7} = \frac{3 \times 5}{7 \times 5} = \frac{15}{35}, \qquad \frac{2}{5} = \frac{2 \times 7}{5 \times 7} = \frac{14}{35}$$

Now, both fractions have the same denominator, we compare their numerators and conclude $\frac{3}{7} > \frac{2}{5}$.

e) $\frac{7}{3}$ is a positive number, while $-\frac{2}{9}$ is a negative one. A positive number is always greater than a negative one, so $\frac{7}{3} > -\frac{2}{9}$.

f) We first compare fractions without negative signs $\frac{5}{6}$ and $\frac{7}{9}$. We can use 18 as a common denominator. Since $\frac{5}{6} = \frac{5 \times 3}{6 \times 3} = \frac{15}{18}$ and $\frac{7}{9} = \frac{5 \times 2}{9 \times 2} = \frac{10}{18}$, we get $\frac{5}{6} > \frac{7}{9}$ (by comparing the numerators of $\frac{15}{18}$ and $\frac{10}{18}$). To compare the original fractions, we reverse the inequality sign $-\frac{5}{6} < -\frac{7}{9}$.

Exercises with Answers (For answers see Appendix A)

Ex.1 Perform the indicated operations, if possible. If not possible, write "undefined". Perform one operation at a time. Show all your steps. Use correctly "=" sign. Always reduce the answer.

a) $\left(-\frac{2}{3}\right)\left(\frac{1}{4}\right)+\frac{1}{2}$ b) $-3\times\left(\frac{1}{3}\right)^3$ c) $\left(\frac{2}{5}+\frac{1}{4}\right)\times\left(-\frac{10}{13}\right)$

d) $\frac{7}{8}\div\left(-\frac{3}{2}\right)\times\frac{2}{5}$ e) $\frac{3}{8}+\left(\frac{5}{8}-\frac{1}{2}\right)^2$ f) $9\div\frac{15}{8}-\frac{3}{50}$

Ex.2 Name the operation that has to be performed first according to the order of operations. Then evaluate the expressions performing *one operation at a time.* Please, make sure that you display your answer in a correct way, using the '=' sign.

a) $\frac{3}{5}\div\left(\frac{1}{2}-\frac{3}{5}\right)$ b) $\frac{-2}{5}\times 8-\frac{2}{3}$

c) $\frac{8}{7}-\frac{1}{4}\div\frac{3}{8}$ d) $-\frac{5}{6}\div\frac{3}{4}\div 2$

e) $\left(-\frac{2}{7}\times\frac{14}{5}\right)^2$ f) $\frac{3}{20}-\frac{4}{17}\times\frac{17}{5}$

g) $-2\div\left(-\frac{2}{7}-\left(-\frac{2}{7}\right)\right)$ h) $\left(-\frac{4}{5}\right)\left(-\frac{4}{5}\right)-\frac{4}{5}-\frac{4}{5}$

i) $\left(-\frac{3}{5}+\frac{1}{10}\right)\div\frac{5}{2}$ j) $\dfrac{\frac{3}{4}-\frac{1}{4}+\frac{5}{4}}{-\frac{3}{14}}$

k) $-\frac{8}{3}\left(\frac{1}{2}\right)^3\left(-\frac{3}{4}\right)$ l) $\left(\frac{-3}{5}-\frac{-3}{20}\right)\times\frac{4}{3}$

m) $-\frac{3}{4}+\frac{2}{3}\left(\frac{1}{2}+\frac{2}{5}\right)$ n) $-\left(\frac{2}{3}\right)^2-\frac{1}{3}\times\left(-\frac{5}{4}\right)$

o) $-\frac{2}{7}-\left(-\frac{1}{14}\right)+\frac{5}{2}$ p) $-\frac{1}{2}+\left(\frac{1}{2}\right)^2-\frac{3}{8}$

q) $-\left(-\frac{-3}{4}+\frac{1}{4}\right)^{25}$ r) $\left(\frac{-5}{4}\right)^2+\left(\frac{2}{3}-\frac{7}{4}\right)$

s) $-4\left(-\frac{1}{5}\right)-\left(\frac{1}{4}\right)\left(-\frac{1}{2}\right)$ t) $\dfrac{\frac{1}{5}\div 3}{-\frac{7}{20}}$

u) $-3\times\left(-\frac{1}{2}\right)^4\times\frac{4}{33}$ v) $-\left(-\frac{4}{3}+\frac{5}{6}\right)\div 2$

Ex.3 Write each of the following as a single numerical expression, and then evaluate them.

a) First find the sum of $\frac{4}{9}$ and $\frac{5}{9}$, and then multiply the result by $-\frac{4}{13}$.

b) First find the quotient of -8 and $\frac{3}{4}$, and then multiply the result by $-\frac{1}{16}$

c) Find the product of $\frac{5}{7}$ and $-\frac{2}{5}$, and then add it to $\frac{4}{21}$.

d) Subtract $\frac{3}{5}$ from $\frac{4}{9}$, and then divide 7 by the result.

e) Multiply $\frac{-12}{13}$ and $\frac{13}{12}$, then raise the result to the seventh power.

f) Cube $-\frac{1}{3}$, and then multiply the result by 10.

g) Divide $\frac{5}{6}$ by $-\frac{25}{12}$, and then subtract the result from $\frac{-3}{20}$.

Ex.4 Replace x with a number to make the statement true.

a) $-\frac{2}{7}\times\frac{14}{5}+\frac{2}{3}=x+\frac{2}{3}$ b) $\left(\frac{2}{7}-\frac{1}{14}\right)^5=x^5$ c) $-3\div\frac{1}{5}\cdot\frac{2}{5}=x\cdot\frac{2}{5}$

d) $\frac{3}{7}-\left(\frac{1}{9}\right)^2=\frac{3}{7}-x$ e) $\left(\frac{1}{4}-\frac{7}{8}\right)\div 2=x\div 2$ f) $-\frac{1}{2}+\frac{5}{16}-\frac{11}{2}=-\frac{1}{2}-\frac{11}{2}+x$

Ex.5 Compare the following fractions.

a) $\frac{3}{11}$ and $\frac{8}{11}$ b) $\frac{3}{2}$ and $\frac{1}{5}$ c) $\frac{8}{9}$ and $\frac{8}{3}$

d) $\frac{7}{5}$ and $\frac{2}{5}$ e) $\frac{3}{4}$ and $\frac{1}{4}$ f) $\frac{5}{2}$ and $\frac{5}{11}$

Ex.6 List all fractions that are greater than $\frac{4}{5}$?

a) $\frac{-4}{5}$ b) 0 c) $\frac{9}{5}$ d) $\frac{3}{5}$ e) $\frac{4}{7}$ f) $\frac{5}{6}$

Ex.7 Compare the following fractions.

a) $\frac{2}{5}$ and $\frac{1}{3}$ b) $\frac{11}{8}$ and $\frac{9}{7}$ c) $\frac{3}{7}$ and $\frac{10}{21}$

d) $\frac{2}{9}$ and $\frac{1}{6}$ e) $\frac{3}{10}$ and $\frac{1}{5}$ f) $\frac{5}{4}$ and $\frac{8}{7}$

Ex.8 Compare the following fractions.

a) $-\frac{2}{9}$ and $-\frac{1}{3}$ b) $-\frac{7}{9}$ and $-\frac{3}{4}$

c) $-\frac{3}{8}$ and $\frac{-2}{5}$ d) $\frac{-1}{4}$ and $-\frac{2}{5}$

e) $\frac{3}{-8}$ and $-\frac{1}{6}$ f) $-\frac{3}{10}$ and $-\frac{29}{100}$

Ex.9 Compare the following fractions.

a) $-\frac{5}{4}$ and $-\frac{9}{8}$ b) $\frac{2}{11}$ and $\frac{3}{22}$

c) $-\frac{2}{8}$ and $\frac{-2}{5}$ d) $\frac{-9}{7}$ and $-\frac{2}{5}$

e) $-\frac{5}{123}$ and $-\frac{123}{5}$ f) $\frac{8}{6}$ and $\frac{10}{9}$

g) $-\frac{13}{4}$ and $-\frac{13}{7}$ h) $-\frac{1}{2}$ and 0

Ex.10 Which girl hit the ball more than $\frac{1}{2}$ of her times at bat?

a) Suzanne hit the ball 5 times out of 8 times at bat.
b) Samantha hit the ball 2 times out of 5 times at bat.
c) Jan hit the ball 2 times out of 6 times at bat.
d) Karen hit the ball 7 times out of 15 times at bat.

Ex.11 The recipe requires

Spice	Amount
nutmeg	$\frac{1}{4}$ tsp
ginger	$\frac{3}{4}$ tsp
cinnamon	$\frac{1}{2}$ tsp
mace	$\frac{3}{8}$ tsp

Which spice does the recipe require the greatest amount?

Ex.12 Write the following rational numbers in order from lowest to greatest.

$-\frac{4}{5}$, $\frac{14}{15}$, $\frac{4}{5}$, $\frac{9}{7}$, 1, $\frac{8}{7}$, $-\frac{5}{6}$

Ex.13 Replace each x with a digit such that the inequality is true (more than one answer is possible).

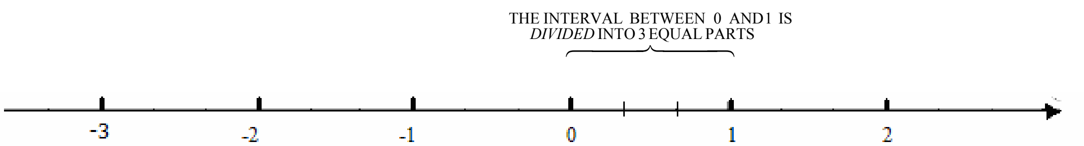
a) $\frac{x}{4} < \frac{5}{4}$ b) $\frac{7}{6} < \frac{7}{x}$ c) $\frac{7}{3} > \frac{x}{8}$ d) $-\frac{5}{12} > -\frac{x}{12}$

Lesson 9

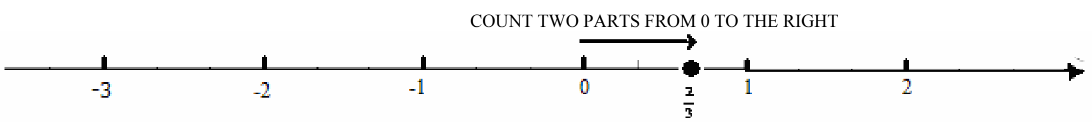
Topics: Mixed numbers; Converting mixed numbers to fractions and fractions to mixed numbers; Plotting fractions and mixed numbers on a number line; Arithmetic operations on mixed numbers.

Plotting proper fractions on a number line

Just as for each integer there is a unique point on a number line corresponding to it, there is also a unique point corresponding to each fraction. Consider a fraction $\frac{2}{3}$. It is a positive number, so the point representing it is to the right of 0. Since it is less than 1, the point must be to the left of 1, and so, must be between 0 and 1. The denominator is equal to 3, thus the unit between 0 and 1 should be divided into 3 equal parts.

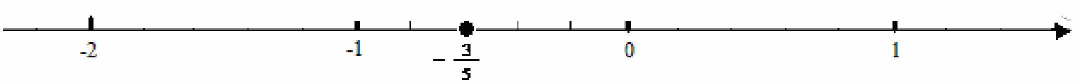


The numerator 2 indicates how many parts should be taken. We count them from the number 0 to the right.



Plot $-\frac{3}{5}$ on a number line.

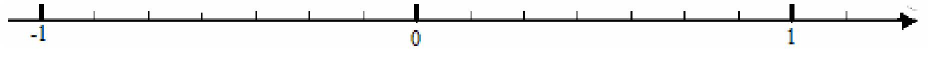
The fraction is negative, so the point is to the left of 0. It is to the right of -1, since $-\frac{3}{5}$ is greater than -1. Thus we divide the unit between -1 and 0 into 5 equal parts (5 because the denominator is 5). We count 3 parts from 0 to the left.



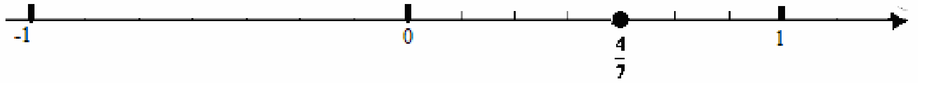
Example 9.1 Below is the line with equally spaced marks on it.

Choose the point corresponding to 0 and 1 in a such way that it is convenient for you to plot the fraction $\frac{4}{7}$. Then, plot it.

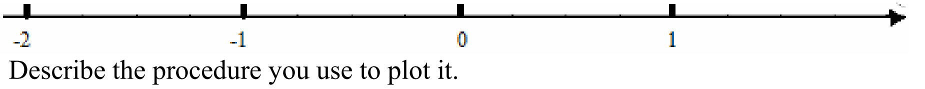
Solution:
The points corresponding to 0 and 1 should be such that the unit between 0 and 1 is divided into 7 equal parts (since the denominator is 7).



Starting from 0, we count 4 such parts to the right.

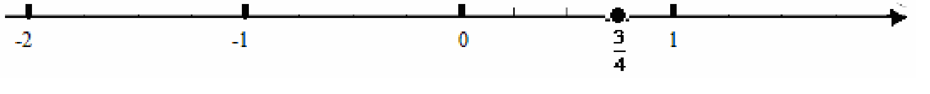


Example 9.2 Plot (as precisely as you can) $\frac{3}{4}$ on the following number line.

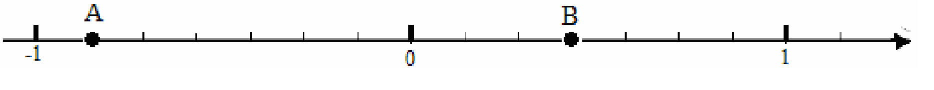


Describe the procedure you use to plot it.

Solution:
Since $\frac{3}{4}$ is between 0 and 1, we divide the unit into 4 equal parts and count, starting from 0, to the right 3 such parts.



Example 9.3 What fraction corresponds to point A and B on the following number line. Assume that all marks on the line are equally spaced.

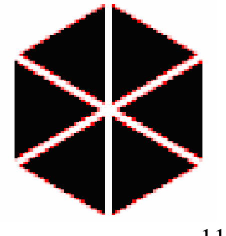


Solution:
To find the fraction representing point B, we count the spaces between 0 and 1. There are 7 of them. This means that the unit was divided into 7 equal parts and thus the denominator is 7. Now, we count the number of spaces between 0 and point B. There are 3 spaces so the numerator is 3. The fraction corresponding to the point B is $\frac{3}{7}$.
Since point A is to the left of 0, the fraction representing it is negative. To find the

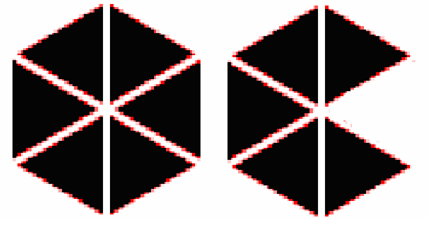
denominator, we count the number of spaces between 0 and -1. To find the numerator, we count the number of spaces between 0 and A. The fraction corresponding to the point A is $-\frac{6}{7}$.

Mixed numbers

Let the following represent one unit.



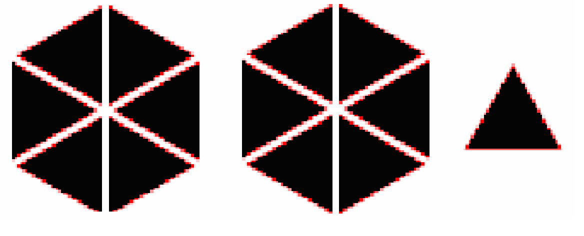
Then the fraction representing the area below is $\frac{11}{6}$.



We can also look at this as one unit plus an additional five sixths parts $1+\frac{5}{6}$. This is written as $1\frac{5}{6}$.

$$\frac{11}{6}=1+\frac{5}{6}=1\frac{5}{6}$$

What fraction represents the area below?



The fraction is $\frac{13}{6}$. But since we have two units and one-sixth $2+\frac{1}{6}$, we can write it as $2\frac{1}{6}$.

The numbers that have both, integer and a fraction with its numerator less than its denominator (proper fraction) are called mixed numbers. Both $1\frac{5}{6}$ and $2\frac{1}{6}$ are examples of mixed numbers.

$$1\frac{5}{6} = 1 + \frac{5}{6}$$

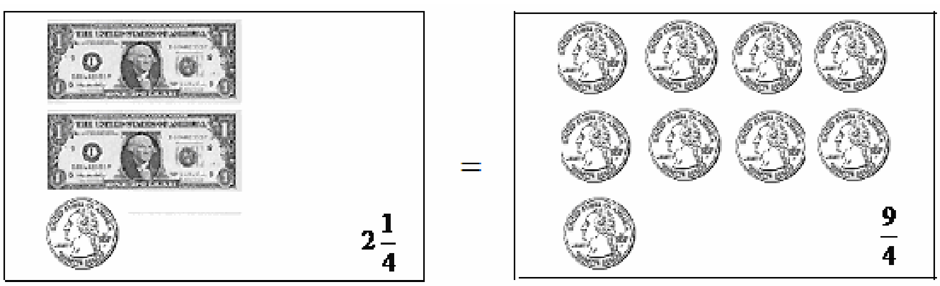
mixed number | integer | proper fraction

The mixed numbers can be also negative. For example, $-2\frac{5}{7}$, $-8\frac{1}{3}$. The meaning of a negative mixed number is $-2\frac{5}{7} = -2 - \frac{5}{7}$,

$-8\frac{1}{3} = -8 - \frac{1}{3}$. (Compare it to $3\frac{2}{7} = 3 + \frac{2}{7}$).

<u>Converting a mixed number to a fraction</u>

Suppose that we have 2 one-dollar bills and 1 quarter, $2\frac{1}{4}$ (a quarter is one fourth of a dollar) and we wish to change it to quarters. Since each dollar bill is worth 4 quarters, 2 dollar bills are equivalent to $2 \times 4 = 8$ quarters. With the additional quarter we had, we now have $2 \times 4 + 1 = 9$ quarters. Thus $2\frac{1}{4} = \frac{9}{4}$.



The above example explains how to convert a mixed number to a fraction.

To convert a positive mixed number to a fraction:

1. **Multiply the denominator of the fraction by the integer part.**
2. **Add the numerator of the fraction to the product found in step 1.**
3. **Write the sum found in step 2 over the denominator of the fraction.**

For example, to convert $2\frac{1}{4}$ to a fraction, multiply $2 \times 4 = 8$. Add 1 to the result $8 + 1 = 9$. We have 9 parts.

$$2\frac{1}{4} = \frac{\overbrace{2\times 4}^{\textit{MULTIPLY THE WHOLE NUMBER BY THE DENOMINATOR}} + 1}{4} = \frac{\overbrace{8+1}^{\textit{ADD THE NUMERATOR TO THEPRODUCT}}}{4} = \frac{9}{4}$$

To convert a negative mixed number to a fraction, keep the minus, and follow the algorithm described above. $-7\frac{3}{5} = -\frac{7\times 5+3}{5} = -\frac{38}{5}$

<u>Example 9.4</u> Convert the following mixed numbers to fractions.

a) $4\frac{5}{8}$ b) $-2\frac{1}{3}$

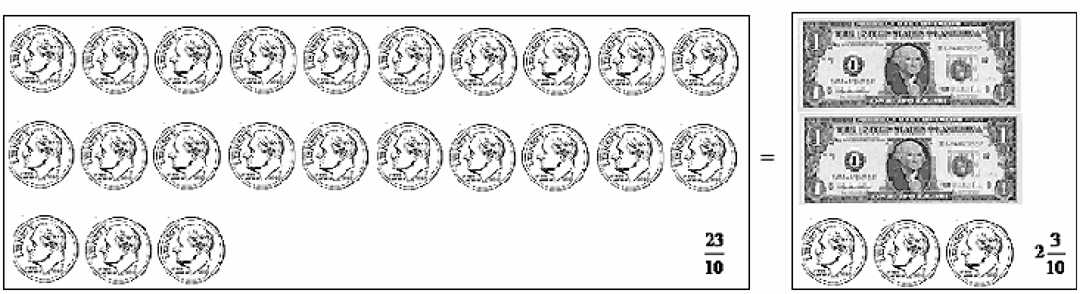
Solution:

a) $4\frac{5}{8} = \frac{4\times 8+5}{8} = \frac{32+5}{8} = \frac{37}{8}$.

b) $-2\frac{1}{3} = -\frac{2\times 3+1}{3} = -\frac{6+1}{3} - \frac{7}{3}$

<u>Converting fractions to mixed numbers</u>

Any time we have a fraction greater than one (that is, such that its numerator is greater than its denominator), we can write it as a mixed number. Consider $\frac{23}{10}$. We will illustrate the process using an example with dimes and dollar bills. The fraction $\frac{23}{10}$ represents 23 dimes and we would like to change it to dollars bills. To find how many dollar bills we can have for 23 dimes, we divide 23 by 10 (by 10, because one dollar is equivalent to 10 dimes). $23 \div 10$ is equal 2 and the reminder is 3. We have 2 one-dollar bills and 3 dimes.



This means $\frac{23}{10} = 2 + \frac{3}{10} = 2\frac{3}{10}$.

To convert a positive improper fraction to a mixed number:

1. Divide the numerator by the denominator to get the integer part.

2. The remainder will be the numerator of the fractional part. The denominator remains unchanged.

For example, to convert $\frac{34}{7}$ to a mixed number, divide the numerator 34 by the denominator 7. The quotient is 4, the remainder is 6. Thus $\frac{34}{7} = 4\frac{6}{7}$.

To convert a negative fraction to a mixed number, keep the minus, and follow the algorithm described above.

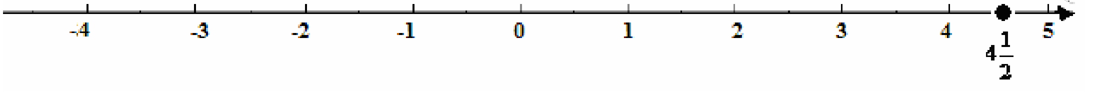
$-\frac{22}{5} = -4\frac{2}{5}$ since 22 divided by 5 is 4, and the remainder 2.

Example 9.5 Convert $\frac{27}{4}$ to a mixed number.

Solution:
Divide the numerator by the denominator $27 \div 4$. The result is 6 and the remainder is 3. Thus $\frac{27}{4} = 6\frac{3}{4}$.

Plotting fractions and mixed numbers

We know how to plot a proper fraction. **To plot an improper fraction, we convert it to a mixed number.** This allows us to find between what consecutive integers the point corresponding to the fraction is located. For example, $\frac{9}{2} = 4\frac{1}{2}$, thus we know that $\frac{9}{2}$ is between 4 and 5. To find its exact location divide the unit between 4 and 5 into 2 equal parts (2, since the denominator is equal to 2), and count one part starting from 4 to the right (1, since the numerator is equal to 1).

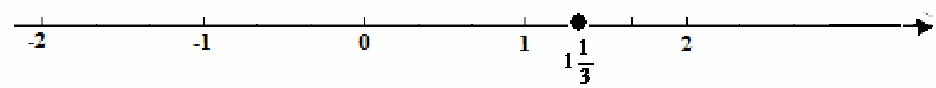


Example 9.6 Plot the following fraction on a number line.

a) $\frac{4}{3}$ b) $-\frac{8}{5}$

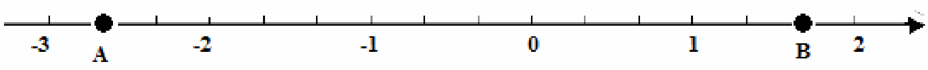
Solution:

a) Convert the fraction to a mixed number $\frac{4}{3} = 1\frac{1}{3}$. The number $1\frac{1}{3}$ is between 1 and 2. Divide the unit between 1 and 2 into 3 equal parts and, starting from 1, count 1 part to the right.



b) $-\frac{8}{5}=-1\frac{3}{5}$. Thus $-\frac{8}{5}$ is between -1 and -2. Divide the unit between -1 and -2 into 5 equal parts and, starting from -1 count 3 such parts to the left.

Example 9.7 Which mixed number best describes the location of point A and B on the number line shown below?



Solution:
The point A is between -3 and -2 The unit between -3 and -2 is divided into 3 equal parts (to find it out count the number of spaces between -3 and -2) and we take 2 of them. So the mixed number representing A is $-2\frac{2}{3}$. The point B is between 1 and 2. The unit is divided into 3 equal parts and we take 2 of such parts. The point B is represented by $1\frac{2}{3}$.

Adding positive mixed numbers

If in the morning you ate 1 whole chocolate bar and 4 out of 9 equal pieces of the second one

one and four-ninths of a chocolate bar

and in the evening you ate 2 whole chocolate bars and 1 out of 9 equal pieces of the third one

two and one-ninth of a chocolate bar

How much of a chocolate bar did you eat? You ate 3 whole chocolate bars and 5 pieces each being one-ninth of the chocolate bar. $1\frac{4}{9} + 2\frac{1}{9} = 3\frac{5}{9}$

.

To add mixed numbers we add the whole numbers together and then the fractions together.

$$1\frac{4}{9} + 2\frac{1}{9} = 1 + 2 + \frac{4}{9} + \frac{1}{9} = 3 + \frac{5}{9} = 3\frac{5}{9}$$

In the above example both fractions had the same denominator. If denominators happen to be different, one has to rewrite them with a common denominator. We will demonstrate this procedure with the help of the following example $3\frac{5}{12} + 8\frac{1}{4}$.

$3\frac{5}{12} + 8\frac{1}{4} =$	In order to add fractions $\frac{5}{12}$ and $\frac{1}{4}$ rewrite them using a common denominator
$3\frac{5}{12} + 8\frac{3}{12} =$	Add the whole numbers together and then the fractions together.
$3 + 8 + \frac{5+3}{12} =$	Write the result as a mixed number.
$11\frac{8}{12} =$	Reduce the fraction. **Always reduce!!!**
$11\frac{2}{3}$	

When adding mixed numbers, it might happen that the sum of the fractions is an improper fraction. In this case we convert it to a mixed number and add the whole numbers together.
We will illustrate this procedure with the following example.

$2\frac{3}{4} + 7\frac{3}{5} =$	Rewrite the fractions using a common denominator 20
$2\frac{15}{20} + 7\frac{12}{20} =$	Add the whole numbers and then the fractions
$2 + 7 + \frac{15+12}{20} =$	Perform the indicated operations
$9 + \frac{27}{20} =$	Since the fraction is improper, convert it to a mixed number
$9 + 1\frac{7}{20} =$	Add 9 and 1 to get the final answer as a mixed number
$10\frac{7}{20}$	

When adding a fraction to a mixed number, there are no whole numbers to add, but the rest of the procedure is the same. For example,

$$3\frac{2}{13}+\frac{5}{26}=3\frac{4}{26}+\frac{5}{26}=3+\frac{4+5}{26}=3\frac{9}{26}$$

Things become even easier when we add an integer to a mixed number.

$$5+2\frac{1}{4}=5+2+\frac{1}{4}=7+\frac{1}{4}=7\frac{1}{4}$$

(Just like if you have 5 dollars in one pocket and 2 dollars and 1 quarter in the other pocket, all together you have $5+2=7$ dollars and 1 quarter.)

Example 9.8 Perform the indicated operations.

a) $10\frac{2}{5}+6\frac{2}{7}$ b) $3\frac{7}{8}+2\frac{3}{4}$ c) $2\frac{4}{7}+3\frac{3}{7}$

d) $4\frac{2}{11}+6$ e) $8\frac{2}{3}+\frac{1}{6}$ f) $2+\frac{4}{9}$

Solution:

a) $10\frac{2}{5}+6\frac{2}{7}=10\frac{2\times 7}{5\times 7}+6\frac{2\times 5}{7\times 5}=10+6+\frac{14}{35}+\frac{10}{35}=16+\frac{24}{35}=16\frac{24}{35}$

b) $3\frac{7}{8}+2\frac{3}{4}=3\frac{7}{8}+2\frac{3\times 2}{4\times 2}=3+2+\frac{7}{8}+\frac{6}{8}=5+\frac{13}{8}=5+1\frac{5}{8}=6\frac{5}{8}$

c) $2\frac{4}{7}+3\frac{3}{7}=2+3+\frac{4+3}{7}=5+\frac{7}{7}=5+1=6$

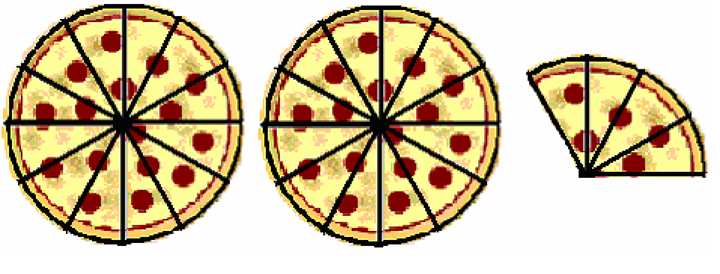
d) $4\frac{2}{11}+6=4+6+\frac{2}{11}=10\frac{2}{11}$ (only the whole numbers must be added)

e) $8\frac{2}{3}+\frac{1}{6}=8\frac{2\times 2}{3\times 2}+\frac{1}{6}=8+\frac{4}{6}+\frac{1}{6}=8\frac{5}{6}$ (only fractions need the addition)

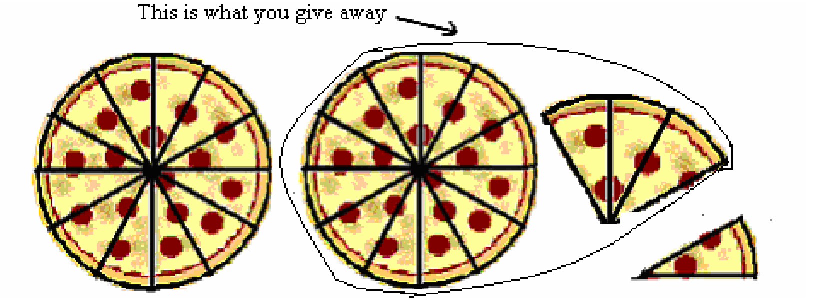
f) $2+\frac{4}{9}=2\frac{4}{9}$

Subtracting positive mixed numbers ("borrowing")

Suppose that in your refrigerator you have 2 whole pizzas and, in addition, 4 slices of pizza that remain from a pizza that originally was cut into 12 equal parts (slices).



Now, suppose that you want to give somebody 1 whole pizza and 3 slices of pizza (each slice of the size you obtain by cutting a pizza into 12 equal parts). What would you do? Most likely you would take out 1 whole pizza together with 3 out of those 4 already cut slices of pizza.



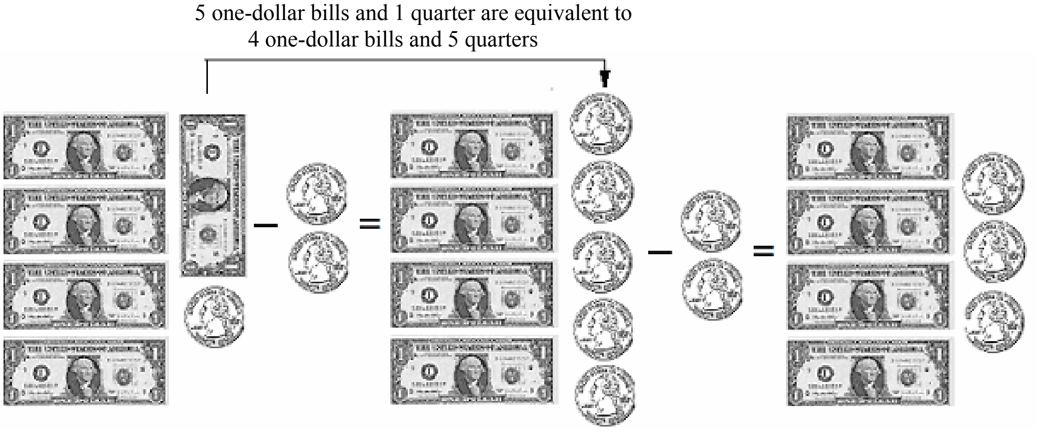
The above operation can be written as $2\frac{4}{12}-1\frac{3}{12}$ and can be done emulating what we did with pizzas: we subtract whole numbers and then fractions to combine the results in the form of a mixed number.

$$2\frac{4}{12}-1\frac{3}{12}=2-1+\frac{4}{12}-\frac{3}{12}=1+\frac{4-3}{12}=1+\frac{1}{12}=1\frac{1}{12}$$

Sometimes, however, subtraction requires an additional step. Suppose, for example, that you had 5 one-dollar bills and 1 quarter and you bought a candy bar for 2 quarters. To find out how much money you will have left after paying for the candy bar, you need to perform the following operation.

$$5\frac{1}{4}-\frac{2}{4}$$

The cashier asked you for the exact change. What do you do? You are 1 quarter short. You take 1 out of your 5 one-dollar bills and change it to quarters.



Just like you would not change all of your 5 one-dollar bills into quarters in order to pay 2 quarters for the candy bar but only 1 bill, we do not have to convert the entire 5 into an improper fraction. Instead, we can take 1 unit out of 5 (leaving 4) and convert only this 1 to a fraction. The operation would look like this:

$$5\frac{1}{4}-\frac{2}{4}=4+1+\frac{1}{4}-\frac{2}{4}=4+\frac{1}{1}+\frac{1}{4}-\frac{2}{4}=4+\frac{4}{4}+\frac{1}{4}-\frac{2}{4}=4+\frac{4+1-2}{4}=4\frac{3}{4}$$

(we wrote 5 as a sum of 4 and 1, and then converted 1 to a fraction with the denominator 4)

This method is called **"borrowing" and is needed when the first fraction is smaller than the one we wish to subtract.**

If the fractions do not have a common denominator, we need to find one and rewrite the mixed numbers so their fractional parts have a common denominator. For example,

$12\frac{1}{5}-8\frac{2}{3}=$	Find the common denominator of fraction parts, 15
$12\frac{1\times 3}{5\times 3}-8\frac{2\times 5}{3\times 5}=$	Perform the indicated operations.
$12\frac{3}{15}-8\frac{10}{15}=$	"Borrow" 1 unit from 12 (since $\frac{10}{15}$ is bigger than $\frac{3}{15}$).
$11+1+\frac{3}{15}-8\frac{10}{15}=$	Convert 1 to a fraction with the denominator 15.
$11+\frac{15}{15}+\frac{3}{15}-8\frac{10}{15}=$	Group the whole numbers and fractions separately.
$11-8+\frac{15+3-10}{15}=$	Perform the indicated operations.
$3+\frac{8}{15}$	Write as a mixed number.
$3\frac{8}{15}$	

The procedure of "borrowing" is also used when subtracting a fraction (or a mixed number) from an integer. For example,

$$31-\frac{2}{7}=30+1-\frac{2}{7}=30+\frac{7}{7}-\frac{2}{7}=30+\frac{7-2}{7}=30+\frac{5}{7}=30\frac{2}{7}$$

(we "borrowed" 1 from 31 and converted it to a fraction with the denominator 7).

When subtracting a fraction from a mixed number, there are no whole numbers to subtract, the remaining steps are the same. For example,

$$3\frac{1}{6}-\frac{7}{8}=3\frac{4}{24}-\frac{21}{24}=2+1+\frac{4}{24}-\frac{21}{24}=2+\frac{24}{24}+\frac{4}{24}-\frac{21}{24}=2+\frac{24+4-21}{24}=2+\frac{7}{24}=2\frac{7}{24}$$

(we "borrowed" 1 from 3 and converted it to a fraction with the denominator 24).

To subtract an integer from a mixed number we simply subtract it from the integer part of the mixed number (just like if you have 13 one-dollar bills and 7 dimes and you are to pay 2 dollars, you use your 13 one-dollar bills to pay the 2 dollars).

$$13\frac{7}{10}-2=13-2+\frac{7}{10}=11+\frac{7}{10}=11\frac{7}{10}$$

<u>Example 9.9</u> Perform the indicated operations.

a) $12\frac{2}{3}-5\frac{4}{9}$ b) $102\frac{1}{4}-1\frac{5}{6}$

c) $187-\frac{5}{13}$ d) $27\frac{4}{7}-\frac{13}{14}$

e) $62-30\frac{3}{8}$ f) $23\frac{5}{9}-3$

Solution:

a) $12\frac{2}{3}-5\frac{4}{9}=12\frac{6}{9}-5\frac{4}{9}=12-5+\frac{6-4}{9}=7\frac{2}{9}$

b) $102\frac{1}{4}-1\frac{5}{6}=102\frac{3}{12}-1\frac{10}{12}=101+1+\frac{3}{12}-1\frac{10}{12}=101+\frac{12}{12}+\frac{3}{12}-1\frac{10}{12}=$
$101-1+\frac{12+3-10}{12}=100\frac{5}{12}$

c) $187-\frac{5}{13}=186+1-\frac{5}{13}=186+\frac{13}{13}-\frac{5}{13}=186\frac{8}{13}$.

d) $27\frac{4}{7}-\frac{13}{14}=27\frac{8}{14}-\frac{13}{14}=26+1+\frac{8}{14}-\frac{13}{14}=26+\frac{14}{14}+\frac{8}{14}-\frac{13}{14}=$
$26+\frac{14+8-13}{14}=26\frac{9}{14}$

e) $62-30\frac{3}{8}=61+1-30\frac{3}{8}=61-30+\frac{8}{8}-\frac{3}{8}=31+\frac{8-3}{8}=31\frac{5}{8}$

f) $23\frac{5}{9}-3=23-3+\frac{5}{9}=20\frac{5}{9}$

<u>Addition/subtraction of mixed numbers</u>

We will combine the previously learned rules for adding/subtracting signed numbers with what we learned about adding/subtracting positive mixed numbers. We illustrate the resulting procedure with two examples.

$-125\frac{2}{3}+\left(-25\frac{3}{4}\right)=$ Change the double signs "+" and "–" to "–"

$-125\frac{2}{3}-25\frac{3}{4}=$ According to the rule that if both numbers have "–" sign in front, then the numbers must be added and the sign of the final answer becomes "–". Place the "–"sign in front of parentheses, while inside indicate the operation of addition that must be performed.

$-\left(125\frac{2}{3}+25\frac{3}{4}\right)=$ Find a common denominator, 12.

$-\left(125\frac{2\times 4}{3\times 4}+25\frac{3\times 3}{4\times 3}\right)=$ Perform the indicated operations

$-\left(125\frac{8}{12}+25\frac{9}{12}\right)=$ Add whole numbers and fractions separately.

$-\left(125+25+\frac{8+9}{12}\right)=$ Perform the indicated addition.

$-\left(150+\frac{17}{12}\right)=$ Convert the fraction $\frac{17}{12}$ to a mixed number.

$-\left(150+1\frac{5}{12}\right)=$ Add whole numbers.

$-\left(150+1+\frac{5}{12}\right)=$ Write it as a mixed number.

$-151\frac{5}{12}$

. The second example,

$-44\frac{5}{14}-\left(-10\frac{3}{7}\right)=$ Change the double signs "–" and "–" to "+"

$-44\frac{5}{14}+10\frac{3}{7}=$ According to the rule that if both numbers have opposite signs in front of them, then the sign of the final answer is the same as the sign of the bigger. The smaller number must be subtracted from the larger one. Place the "–"sign in front of parentheses, while inside indicate the operation of subtraction that must be performed.

$-\left(44\frac{5}{14}-10\frac{3}{7}\right)=$ Find a common denominator 14.

$-\left(44\frac{5}{14}-10\frac{3\times 2}{7\times 2}\right)=$ Perform the indicated operations.

$-\left(44\frac{5}{14}-10\frac{6}{14}\right)=$ Since $\frac{5}{14}$ is less than $\frac{6}{14}$ we have to "borrow" 1 from 44.

$-\left(43+1+\frac{5}{14}-10\frac{6}{14}\right)=$ Change 1 to a fraction with the denominator 14.

$-\left(43+\frac{14}{14}+\frac{5}{14}-10\frac{6}{14}\right)=$ Perform the operations on whole numbers and fractions separately.

$-\left(43-10+\frac{14+5-6}{14}\right)=$ Write the answer as a mixed number

$-33\frac{13}{14}$

We can summarize the above in a table.

HOW TO ADD/SUBTRACT MIXED NUMBERS

Step 1. Replace any adjacent ("double") signs according to the rule:

$$(+)(+) \rightarrow (+) \qquad (-)(+) \rightarrow (-)$$
$$(-)(-) \rightarrow (+) \qquad (+)(-) \rightarrow (-)$$

Step 2. After replacing "double" signs, determine the sign of the final answer and the operation that has to be perform on the numbers as follows:

If in front of **both of the numbers the sign is the same,** the final answer will be that sign (+ or −) and the numbers must be added. In order to perform the addition:

- change fractions to equivalent fractions with a common denominator, if needed
- add whole numbers and fractions separately
- if improper fraction occur, change to the mixed number and add once again add whole numbers together.

For example, $-2\frac{5}{7}+\left(-7\frac{1}{3}\right)=-2\frac{5}{7}-7\frac{1}{3}=-\left(2\frac{5}{7}+7\frac{1}{3}\right)=-\left(2\frac{15}{21}+7\frac{7}{21}\right)=$

$-\left(2+7+\frac{15+7}{21}\right)=-\left(9+\frac{22}{21}\right)=-\left(9+1\frac{1}{21}\right)=-\left(9+1+\frac{1}{21}\right)=-10\frac{1}{21}$

If the signs in front of **the numbers are opposite,** the final answer will be the sign of the larger number the and the smaller number must be subtracted from the bigger one. In order to perform the subtraction:

- change fractions to equivalent fractions with a common denominator, if needed
- if necessary, "borrow" from whole number to subtract fraction
- subtract whole numbers and fractions separately

For example, $-\left(-2\frac{5}{7}\right)-7\frac{1}{3}=2\frac{5}{7}-7\frac{1}{3}=-\left(7\frac{1}{3}-2\frac{5}{7}\right)=-\left(7\frac{7}{21}-2\frac{15}{21}\right)=$

$-\left(6+1+\frac{7}{21}-2\frac{15}{21}\right)=-\left(6+\frac{21}{21}+\frac{7}{21}-2\frac{15}{21}\right)=-\left(6-2+\frac{21+7-15}{21}\right)=-\left(4+\frac{13}{21}\right)=-4\frac{13}{21}$

Example 9.10 Perform the indicated operations.

a) $-7+\left(-\frac{2}{15}\right)$ b) $-1\frac{11}{15}-\left(-99\frac{2}{3}\right)$ c) $-125\frac{1}{6}-\left(-\frac{7}{9}\right)$

Solution:

a) $-7+\left(-\frac{2}{15}\right)=-7-\frac{2}{15}=-\left(7+\frac{2}{15}\right)=-7\frac{2}{15}$

b) $-1\frac{11}{15}-\left(-99\frac{2}{3}\right)=-1\frac{11}{15}+99\frac{2}{3}=99\frac{2}{3}-1\frac{11}{15}=99\frac{10}{15}-1\frac{11}{15}=$

$98+1+\frac{10}{15}-1\frac{11}{15}=98+\frac{15}{15}+\frac{10}{15}-1\frac{11}{15}=98-1+\frac{15+10-11}{15}=97+\frac{14}{15}=97\frac{14}{15}$.

c) $-125\frac{5}{6}-\left(-\frac{1}{9}\right)=-125\frac{5}{6}+\frac{1}{9}=-\left(125\frac{5}{6}-\frac{1}{9}\right)=-\left(125\frac{15}{18}-\frac{2}{18}\right)=$

$-\left(125+\frac{15-2}{18}\right)=-125\frac{13}{18}$

Addition/subtraction of mixed numbers by converting them to fractions

An alternative method for adding/subtracting mixed numbers is to convert them to fractions and then perform the operations according to the rules for fraction. For example,

$3\frac{1}{4}+1\frac{2}{5}=$	Convert both mixed numbers to fractions (to do this, keep the denominators, the numerators are equal to the product of a whole number and the denominator plus the numerator)
$\frac{3\times4+1}{4}+\frac{1\times5+2}{5}=$	Perform the indicated operations in the numerator.
$\frac{13}{4}+\frac{7}{5}=$	Look for a common denominator, in this case it is $4\times5=20$. Rewrite fractions using the new denominator 20.
$\frac{65}{20}+\frac{28}{20}=$	Write as a single fraction.
$\frac{65+28}{20}=$	Add the numbers in the numerator.
$\frac{93}{20}$ (*)	

This method will always work but is computationally more difficult, especially when integer parts are large (in such cases converting a mixed number to a fraction requires multiplication of large numbers).

Multiplication and division of mixed numbers and fractions

In the case of multiplication and division of mixed numbers and fractions, we **must convert the mixed numbers to improper fractions first,** and then perform the operations as usual.

(*) If we perform the same operation by adding whole numbers and fractions separately, we would get $3\frac{1}{4}+1\frac{2}{5}=3\frac{5}{20}+1\frac{8}{20}=3+1+\frac{5+8}{20}=4\frac{13}{20}$ It is important to realize that both answers, the one written as a fraction and the one in the form of a mixed number, are equal $\frac{93}{20}=4\frac{13}{20}$. This is because $4\frac{13}{20}=\frac{4\times20+13}{20}=\frac{93}{20}$.

For example, $1\frac{4}{7}\times 3\frac{1}{5}$.

$1\frac{4}{7}\times 3\frac{1}{5}=$	Convert all mixed numbers to improper fractions.
$\frac{1\times 7+4}{7}\times\frac{3\times 5+1}{5}=$	Perform the indicated operations in the numerators.
$\frac{11}{7}\times\frac{16}{5}=$	Multiply the numerators and denominators.
$\frac{11\times 16}{7\times 5}=$	Perform the indicated multiplications.
$\frac{176}{35}$	

Or, perform the following $-2\frac{3}{4}\div 4\frac{2}{3}$.

$-2\frac{3}{4}\div 4\frac{2}{3}=$	Convert all mixed numbers to improper fractions.
$-\frac{2\times 4+3}{4}\div\frac{4\times 3+2}{3}=$	Perform the indicated operations in the numerators.
$-\frac{11}{4}\div\frac{14}{3}=$	Flip the divisor $\frac{14}{3}$ and replace division with multiplication.
$-\frac{11}{4}\times\frac{3}{14}=$	Multiply the numerators and denominators, keeping the minus sign,
$-\frac{11\times 3}{4\times 14}=$	Perform the indicated multiplications.
$-\frac{33}{56}$	

<u>Example 9.11</u> Perform the indicated operations.

a) $\dfrac{-3\frac{3}{4}}{1\frac{1}{8}}$ b) $\left(1\frac{3}{7}\right)\left(-2\frac{4}{5}\right)$ c) $-4\frac{1}{2}\left(-1\frac{2}{5}\right)\left(-\frac{1}{4}\right)$

Solution:

a) $$\frac{-3\frac{3}{4}}{1\frac{1}{8}} = \left(-3\frac{3}{4}\right) \div \left(1\frac{1}{8}\right) = -\frac{15}{4} \div \frac{9}{8} = -\frac{15}{4} \times \frac{8}{9} = -\frac{15 \times 8}{4 \times 9} =$$

$$-\frac{3 \times 5 \times 2 \times 4}{4 \times 3 \times 3} = -\frac{\overset{1}{\cancel{3}} \times 5 \times 2 \times \overset{1}{\cancel{4}}}{\underset{1}{\cancel{4}} \times \underset{1}{\cancel{3}} \times 3} = -\frac{1 \times 5 \times 2 \times 1}{1 \times 1 \times 3} = -\frac{10}{3}$$

b) $$\left(1\frac{3}{7}\right)\left(-2\frac{4}{5}\right) = \left(\frac{10}{7}\right)\left(-\frac{14}{5}\right) = -\frac{10}{7} \times \frac{14}{5} = -\frac{10 \times 14}{7 \times 5} = -\frac{5 \times 2 \times 7 \times 2}{7 \times 5}$$

$$-\frac{\overset{1}{\cancel{5}} \times 2 \times \overset{1}{\cancel{7}} \times 2}{\underset{1}{\cancel{7}} \times \underset{1}{\cancel{5}}} = -\frac{1 \times 2 \times 1 \times 4}{1 \times 1} = -\frac{4}{1} = -4$$

c) $$-4\frac{1}{2}\left(-1\frac{2}{5}\right)\left(-\frac{1}{4}\right) = -\frac{9}{2}\left(-\frac{7}{5}\right)\left(-\frac{1}{4}\right) = = -\frac{9 \times 7 \times 1}{2 \times 5 \times 4} = -\frac{63}{40}$$

Exponentation of mixed numbers

Since exponentiation is a shortcut for multiplication, the same rules apply to exponentiation as to multiplication. This means that **before we raise any mixed number to a given power, we must convert it to an improper fraction.**
For example,

$$\left(2\frac{1}{4}\right)^2 = \left(\frac{2 \times 4 + 1}{4}\right)^2 = \left(\frac{9}{4}\right)^2 = \frac{9^2}{4^2} = \frac{81}{16}$$

Example 9.12 Perform the indicated operations $\left(-1\frac{2}{3}\right)^3$.

Solution:

$$\left(-1\frac{2}{3}\right)^3 = \left(-\frac{1 \times 3 + 2}{3}\right)^3 = \left(-\frac{5}{3}\right)^3 = -\frac{5^3}{3^3} = -\frac{125}{27}.$$

The sign of the final result is negative, because the exponent is odd.

All operations combined on mixed numbers and fractions

As always, if more than one operation is involved, we follow the order of operations.

$-1\frac{2}{3}-4\frac{1}{3}\times\frac{1}{26}=$ According to the order of operations we perform the multiplication first. To this end, we change the mixed number to an improper fraction.

$-1\frac{2}{3}-\frac{4\times3+1}{3}\times\frac{1}{26}=$ Perform the operations indicated in the numerator.

$-1\frac{2}{3}-\frac{13}{3}\times\frac{1}{26}=$ Cancel 13 and 26

$-1\frac{2}{3}-\frac{1}{3}\times\frac{1}{2}=$ Multiply numerators and denominators of $\frac{1}{3}$ and $\frac{1}{2}$.

$-1\frac{2}{3}-\frac{1}{6}=$ Since both signs are negative, place minus in front of parentheses and set up the addition.

$-\left(1\frac{2}{3}+\frac{1}{6}\right)=$ Rewrite the fractions with a common denominator 6.

$-\left(1\frac{4}{6}+\frac{1}{6}\right)=$ Add fractions together

$-\left(1+\frac{4+1}{6}\right)=$ Write the final answer as a mixed number.

$-1\frac{5}{6}=$

Example 9.13 Perform the indicated operations.

a) $-10\times\left(-2\frac{1}{3}\right)^2$ b) $\left(1-\frac{4}{9}\right)\left(-\frac{3}{4}+2\frac{1}{4}\right)$ c) $\dfrac{1+1\frac{2}{3}}{3\frac{1}{2}\times\frac{1}{4}}$

Solution:

a) We first exponentiate and then multiply the result by -10. To this end we convert the mixed number to an improper fraction.

$$-10\times\left(-2\frac{1}{3}\right)^2=-10\times\left(-\frac{7}{3}\right)^2=-10\times\left(-\frac{7}{3}\right)\times\left(-\frac{7}{3}\right)=-10\times\frac{7^2}{3^2}=-10\times\frac{49}{9}=$$
$$-\frac{10\times49}{9}=-\frac{490}{9}$$

b) We first perform the operations in parentheses. Since the integer parts are small and to perform the multiplication we will need to convert all mixed numbers to fractions anyway, we will do it right from the beginning.

$$\left(1-\frac{4}{9}\right)\left(-\frac{3}{4}+2\frac{1}{4}\right)=\left(\frac{9-4}{9}\right)\left(-\frac{3}{4}+\frac{2\times4+1}{4}\right)=\left(\frac{5}{9}\right)\left(-\frac{3}{4}+\frac{9}{4}\right)=\frac{5}{9}\times\frac{-3+9}{4}=$$

$$\frac{5}{9}\times\frac{6}{4}=\frac{5\times6}{9\times4}=\frac{5\times2\times3}{3\times3\times2\times2}=\frac{5\times\overset{1}{\cancel{2}}\times\overset{1}{\cancel{3}}}{\underset{1}{\cancel{3}}\times3\times2\times\underset{1}{\cancel{2}}}=\frac{5\times1\times1}{1\times3\times2\times1}=\frac{5}{6}$$

c) We perform the indicated operations in the numerator and denominator separately, and then divide the results. To perform multiplication, we convert mixed numbers to improper fractions.

$$\frac{1+1\frac{2}{3}}{3\frac{1}{2}\times\frac{1}{4}}=\frac{1+1+\frac{2}{3}}{\frac{7}{2}\times\frac{1}{4}}=\frac{2+\frac{2}{3}}{\frac{7}{2}\times\frac{1}{4}}=\frac{2\frac{2}{3}}{\frac{7\times1}{2\times4}}=\frac{2\frac{2}{3}}{\frac{7}{8}}=2\frac{2}{3}\div\frac{7}{8}=\frac{8}{3}\div\frac{7}{8}=\frac{8}{3}\times\frac{8}{7}=\frac{8\times8}{3\times7}=\frac{64}{21}$$

Common mistakes and misconceptions

Mistake 1.1

When multiplying mixed numbers, always remember about converting them into improper fractions first. $3\times5\frac{2}{3}\neq15\frac{2}{3}$.

Mistake 1.2

$2\times\frac{3}{7}=\frac{6}{7}$ but $2\frac{3}{7}=\frac{24}{7}$. Be careful not to interpret $2\times\frac{3}{7}$ as $2\frac{3}{7}$ ($2\frac{3}{7}$ means $2+\frac{3}{7}$ or $\frac{24}{7}$).

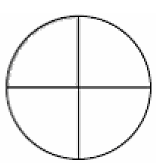
Exercises with Answers (For answers see Appendix A)

Ex.1 If somebody asks you to count by 2 starting from 0 up to 10, you start with 0 and in each consecutive step you add 2 to the previous number. Once you reach number 10, you are done: 0, 2, 4, 6, 8, 10.

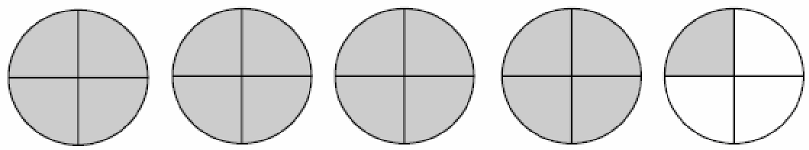
a) Count by $\frac{1}{2}$ starting from 1 up to 6. Express the answer as mixed numbers or integers: 1, $1\frac{1}{2}$, 2, etc.

b) Count by $\frac{1}{5}$ starting from 1 up to 4. Express the answer as mixed numbers or integers

Ex.2 If this is one unit,



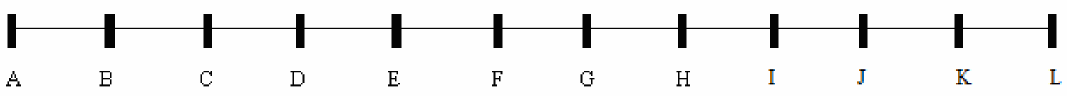
which mixed number shows how much is shaded?



Ex.3 If this is one unit,

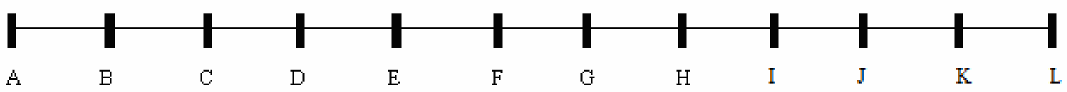
which mixed number shows how much is shaded?

Ex.4 All marks are equally spaced. Assume that the length of the segment AE represents one unit.



a) Which mixed number represents the length of the segment AL?
b) Which mixed number represents the length of the segment AH?
c) Which mixed number represents the length of the segment AF?

Ex.5 All marks are equally spaced. Assume that the length of the segment AC represents one unit.



a) Which mixed number represents the length of the segment AL?
b) Which mixed number represents the length of the segment AD?
c) Which mixed number represents the length of the segment AJ?

Ex.6 Explain the difference between $7\frac{1}{5}$ and $7\left(\frac{1}{5}\right)$.

Ex.7 Convert the following mixed numbers to fractions.

a) $3\frac{5}{6}$ b) $-100\frac{2}{3}$ c) $4\frac{2}{5}$

d) $-2\frac{7}{100}$ e) $-7\frac{2}{3}$ f) $1\frac{9}{345}$

Ex.8 Find the missing numerator.

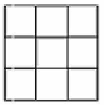
a) $2\frac{3}{8}=\frac{?}{8}$ b) $8\frac{5}{9}=\frac{?}{9}$ c) $-2\frac{7}{10}=-\frac{?}{10}$

Ex.9 Convert the following improper fractions to a mixed number.

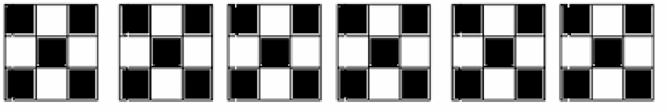
a) $\frac{46}{7}$ b) $-\frac{106}{10}$ c) $-\frac{27}{5}$

d) $\frac{33}{9}$ e) $\frac{19}{6}$ f) $-\frac{23}{4}$

Ex.10 If this is one unit,



what fraction represents the shaded area. Convert the fraction to a mixed number.



Ex.11 For a party, all the pies were cut into sixths. After the party, 26 pieces were left. Which mixed number represents the total number of pies left?

Ex.12 A baker sells whole loaves of bread and half loaves of bread. If at the end of the week 9 halves of bread are left, which mixed number represents the total number of breads left?

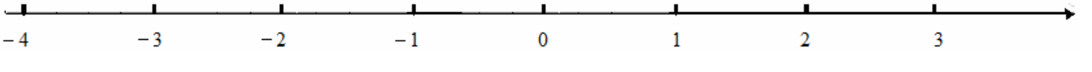
Ex.13 For each of the following fractions determine between which two integers on the number line the fraction is located.

a) $\frac{4}{9}$ b) $-\frac{12}{5}$ c) $-\frac{39}{5}$

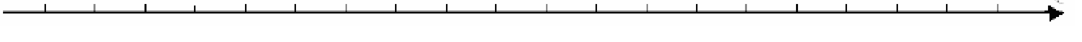
d) $-\frac{52}{7}$ e) $\frac{17}{8}$ d) $-\frac{99}{100}$

Ex.14 Plot (as precisely as you can) the following numbers. Describe the procedure you used plotting.

a) $-\frac{3}{2}$ b) $\frac{11}{4}$ c) $-\frac{11}{3}$



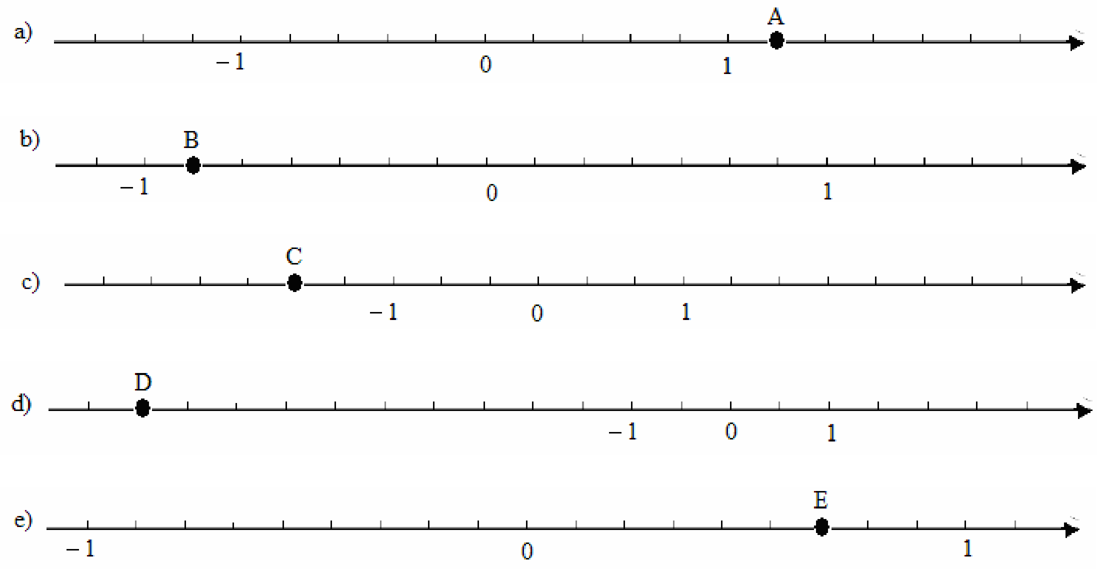
Ex.15 Below is the line with equally spaced marks on it.



Choose the point corresponding to 0 and 1 in such way that it is convenient for you to plot the following fraction (for each point use different number line).

a) $\frac{5}{8}$ b) $-\frac{6}{7}$ c) $-\frac{5}{2}$

Ex.16 What fraction corresponds to point A, B, C, D, and E on the following number lines. Assume that all marks on the line are equally spaced. If a fraction is an improper fraction, write it also as a mixed number.



Ex.17 Perform the indicated operations.

a) $32\frac{1}{7}+12\frac{2}{7}$ b) $101\frac{4}{15}+7\frac{1}{15}$ c) $2\frac{4}{9}+122\frac{2}{9}$ d) $3245\frac{3}{5}+\frac{1}{5}$

e) $271\frac{1}{5}+1$ f) $\frac{14}{17}+121\frac{2}{17}$ g) $44\frac{2}{3}+6$ h) $12+\frac{14}{27}$

Ex.18 Perform the indicated operations.

a) $22\frac{3}{8}+42\frac{1}{3}$ b) $2000\frac{4}{7}+4\frac{3}{21}$ c) $\frac{2}{5}+767\frac{1}{2}$

d) $13\frac{1}{6}+7\frac{3}{8}$ e) $45\frac{2}{9}+\frac{7}{18}$ f) $25\frac{1}{5}+25\frac{7}{10}$

Ex.19 Perform the indicated operations.

a) $38\frac{6}{7}+1\frac{4}{5}$ b) $349\frac{7}{8}+\frac{5}{6}$ c) $99\frac{3}{4}+100\frac{1}{2}$

d) $\frac{11}{12}+45\frac{1}{12}$ e) $9\frac{3}{4}+11\frac{4}{5}$ f) $15\frac{17}{18}+42\frac{5}{9}$

Ex.20 Perform the indicated operations.

a) $88\frac{7}{9}-8\frac{4}{9}$ b) $2004\frac{17}{20}-4$ c) $199\frac{7}{12}-\frac{1}{12}$

d) $25\frac{2}{3}-5\frac{2}{3}$ e) $78\frac{3}{24}-70\frac{1}{24}$ f) $39\frac{4}{11}-39\frac{2}{11}$

Ex.21 Replace x with a number to make the following statement true.

a) $3+x=3\frac{2}{7}$ b) $x+5\frac{1}{3}=7\frac{1}{3}$ c) $5\frac{2}{7}+x=5\frac{4}{7}$ d) $2\frac{5}{9}-x=2$

e) $3\frac{8}{9}-x=\frac{8}{9}$ f) $x-3\frac{2}{7}=1$ g) $4\frac{1}{2}+x=5$ h) $1-x=\frac{1}{3}$

Ex.22 Perform the indicated operations.

a) $45\frac{6}{13}-5\frac{1}{26}$ b) $562\frac{4}{7}-\frac{2}{5}$ c) $67\frac{5}{8}-5\frac{1}{3}$

d) $203\frac{7}{8}-3\frac{3}{4}$ e) $400\frac{3}{7}-100\frac{2}{9}$ f) $67\frac{2}{3}-1\frac{3}{11}$

Ex.23 Perform the indicated operations.

a) $15\frac{1}{7}-4\frac{5}{6}$ b) $79\frac{3}{8}-\frac{5}{7}$ c) $58-\frac{4}{9}$ d) $37-6\frac{2}{9}$

e) $66-\frac{4}{29}$ f) $29\frac{3}{10}-19\frac{7}{15}$ g) $43\frac{1}{2}-\frac{5}{8}$ h) $12-6\frac{7}{27}$

Ex.24 Perform the indicated operations.

a) $-10\frac{14}{25}-20$ b) $-34\frac{1}{3}+12\frac{2}{7}$ c) $-22\frac{2}{3}-5\frac{1}{2}$

d) $222\frac{4}{15}-\frac{4}{5}$ e) $2\frac{6}{7}-203\frac{3}{5}$ f) $-1+\frac{1}{112}$

g) $-1\frac{4}{5}+2\frac{1}{2}$ h) $-14\frac{45}{50}-86\frac{11}{100}$ i) $-3\frac{3}{7}+44$

Ex.25 Perform the indicated operations.

a) $-2\frac{4}{7}-\left(-23\frac{1}{14}\right)$ b) $-\left(-41\frac{2}{5}\right)-7\frac{2}{3}$ c) $-8-\left(-\frac{8}{11}\right)$

d) $-2\frac{3}{7}+\left(-37\frac{3}{4}\right)$ e) $-25\frac{3}{7}-\left(-1\frac{2}{3}\right)$ f) $-4\frac{5}{19}-(-15)$

Ex.26 Perform the indicated operations.

a) $\dfrac{1\frac{3}{4}}{-2\frac{1}{2}}$ b) $-3\frac{4}{7}\times 1\frac{2}{5}$ c) $4\frac{2}{3}\times 1\frac{2}{7}$

d) $3\frac{2}{4}\div 1\frac{3}{28}$ e) $1\frac{3}{17}\times\left(-2\frac{1}{5}\right)$ f) $-2\frac{2}{3}\left(-1\frac{4}{5}\right)$

g) $1\frac{3}{10} \div \left(-2\frac{4}{5}\right)$ h) $-3\frac{1}{4} \times 1\frac{7}{13}$ i) $1\frac{7}{9} \div \left(-4\frac{2}{3}\right)$ j) $\dfrac{-2\frac{3}{4}}{4\frac{1}{2}}$

Ex.27 Perform the indicated operation.

a) $\left(-1\frac{1}{4}\right)^3$ b) $\left(-1\frac{1}{2}\right)^4$ c) $-\left(1\frac{1}{2}\right)^4$ d) $-\left(-2\frac{3}{4}\right)^2$

Ex.28 Perform the indicated operations.

a) $4\frac{4}{7} + 2\frac{1}{6} \div 1\frac{4}{9}$

b) $\left(20\frac{3}{22} - 18\frac{7}{11}\right) \times 4\frac{2}{5}$

c) $5\frac{2}{3} \div 1\frac{5}{12} \times \left(-2\frac{5}{8}\right)$

d) $\dfrac{-98\frac{2}{7} - 1\frac{5}{7}}{3\frac{1}{8}}$

e) $\left(1\frac{1}{3} - 2\frac{5}{9}\right) \div \left(-4\frac{5}{6} + 2\frac{7}{12}\right)$

f) $-\left(3\frac{1}{4}\right)^2 \left(-1\frac{3}{13}\right)$

g) $-1\frac{3}{8} \times 2\frac{1}{11} + 2\frac{3}{8}$

h) $\left(-1\frac{2}{3} + 1\frac{5}{6}\right)^2$

i) $2\frac{1}{2} - \left(-3\frac{2}{7}\right) \div \left(-2\frac{7}{8}\right)$

j) $-2\frac{1}{8} \times \left(2\frac{2}{3}\right) + 5\frac{4}{7}$

k) $3\frac{3}{8} \div \left(2\frac{4}{5} - 5\frac{1}{2}\right)$

l) $\dfrac{2\frac{2}{3} + 1}{-4\frac{1}{2} \times 2\frac{2}{3}}$

m) $3\frac{4}{5} \times 1\frac{1}{38} - 1$

n) $4\frac{2}{7} \div 2\frac{8}{11} - (-1)^5$

o) $2\frac{3}{5} \div (-26) - 1\frac{1}{20}$

p) $\dfrac{-1\frac{1}{2} + 2\frac{1}{2}}{-4\frac{7}{9}}$

q) $-2\frac{5}{6} + 6 \times \left(-8\frac{1}{12}\right)$

r) $-3\left(-1\frac{4}{5}\right) - 10\frac{1}{4}$

s) $-5\frac{3}{100} - \left(-1\frac{1}{10}\right)^2$

t) $-3 + \left(-4\frac{2}{3}\right) - 6 - \left(-1\frac{1}{3}\right)$

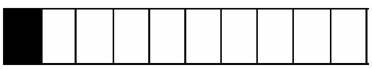
Lesson 10

Topics: Decimal notation; Converting decimals to fractions and fractions to decimals; Multiplication and division of decimals by powers of 10; Comparing decimals.

Among all fractions there is a set of special fractions called *decimal fractions*. Decimal fractions are fractions where the denominator is a power of ten. There is an alternative way of writing a decimal fraction that does not use a fraction bar, nor is a denominator shown. This notation is called decimal notation.

<u>*Decimal notation*</u>

The following figure is divided into 10 equal parts.

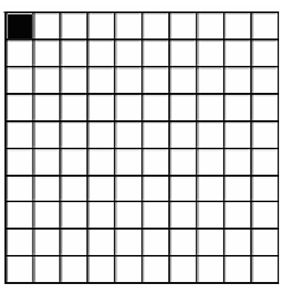


One tenth of the figure is shaded. We can use the fraction notation to describe the amount of the figure, $\frac{1}{10}$ or the decimal notation, 0.1

$$\frac{1}{10} = 0.1$$

We read it as "one tenth". Notice that we use the same name regardless of notation.

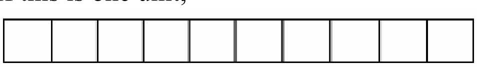
The figure below is cut into 100 equal parts.



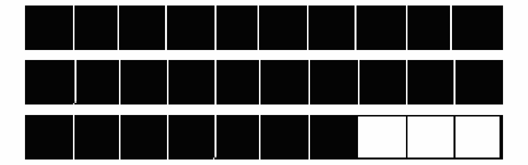
One hundredth of the figure is shaded. $\frac{1}{100} = 0.01$

We read it as "one hundredth".

If this is one unit,



then the area shaded below is represented by the mixed number $2\frac{7}{10}$ or, in the decimal notation, 2.7 Both are read “two and seven tenths”.



Any number written in decimal notation consists of the row of digits (possibly with the minus in front, if the number is negative) separated by the decimal point into two parts, an integer part and a fractional part. For example, the integer part of 2.7 is 2, and its fractional part is .7

The part to the left of the decimal point is called **integer part** **67**	**6 7 . 2 3** ↑ **decimal point**	The part to the right of the decimal point is called **fractional part** **.23**

<u>Example 10.1</u> Identify the integer part and the fractional part of each of the following decimals.

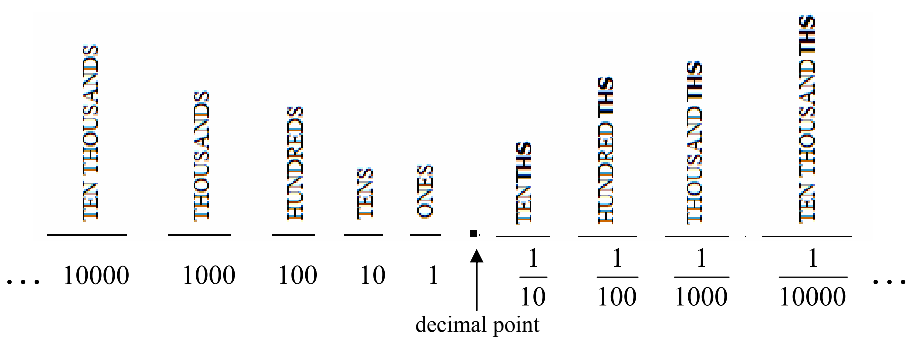
a) 7.35 b) -81.0346 c) 0.8

Solution:

a) The integer part: 7; fractional part: .35
b) The integer part: -81; fractional part: .0346
c) The integer part: 0; fractional part: .8

The value of each digit in a decimal depends on its place in the decimal. The first digit to the left of the decimal point represents units, the next, more to the left, tens, then hundreds, thousands and so on. The first digit after the decimal point represents tenths and tells how many tenths are in the number. The second to the right represents hundredths and tells how many hundredths there are in the number. The third digit is the thousandths place, ten thousandths and so on.

Decimal Place Value Chart



For example, in the decimal 0.7 the digit 7 represents tenths and tells us that there are seven tenths in 0.7. All other digits are zeros, so $0.7 = \frac{7}{10}$. The digit 7 in the decimal 0.07 is in the hundredths place, thus representing seven hundredths, $0.07 = \frac{7}{100}$. The value of any digit depends on its **place value**. In the number 43.629 there are 4 tens, 3 ones, 6 tenths, 2 hundredths and 9 thousandths (*).

Example 10.2 For each of the following decimals count the number of decimal places to the right of the decimal point and then name the last digit (most to the right column) in the fractional part and its place value.

a) 25.8 b) -4.0036

Solution:

a) There is one decimal place after the decimal point. The last digit is 8 and its place value is tenths.

b) There are four decimal places after the decimal point. The last digit is 6 and its place value is ten thousandths.

Reading and writing decimals

To read a decimal (**):

1. Look to the left of the decimal point and say the name of the integer.
2. The decimal point is read as "and"
3. Say a fractional part of the decimal as a whole number followed by the name of the place value of the digit that is farthest to the right.

For example, we read

3.46 as "three and forty six hundredths

5.7 as " five and seven tenths"

-0.005 as "minus five thousandths".

Example 10.3 Write each decimal in words.

a) 456.0003 b) -4.76

Solution:

a) four hundred fifty-six and three ten thousandths.

b) minus four and seventy six hundredths.

(*) The number 43.629 (as any other decimal) is the sum of its place values.

$$43.629 = 4 \times 10 + 3 \times 1 + 6 \times \frac{1}{10} + 2 \times \frac{1}{100} + 9 \times \frac{1}{1000}$$

(**) Decimals are often read in a more "straightforward" way. The decimal 0.46 can be read as "zero point forty six", 5.7 as "five point seven". Although this pronunciation is commonly used and seems easier than the one presented, in order to be able to communicate with somebody who uses the other notation, you must know both pronunciations.

Example 10.4 Write the following numbers in decimal notation.

a) two and twenty seven hundredths
b) minus five and two thousandths
c) twenty-three hundredths

Solution:

a) 2.27
b) -5.002 Notice the position of 2.
c) 0.23

Equivalent forms of decimals

If the integer part of a fraction is zero, if desired, we might not write the zero (*).

$$0.6 = .6$$
$$0.762 = .762$$
or
$$-0.02 = -.02$$

Any integer can be written as a decimal by placing a decimal point directly after the integer, and entering a zero, or zeros, to the right of it. **Every integer has an unwritten decimal point to the right.**

$$9 = 9.0$$
$$873 = 873.00$$
or
$$-96 = -96.0$$

Writing additional zeros *to the right of the decimal point* following the last digit does not change the value of the number (**).

$$0.3 = 0.30 = 0.300 \ \ldots$$
$$-4.712 = -4.7120 = -4.71200 \ \ldots$$
or
$$92.0 = 92.00 = 92.000 \ \ldots$$

To have a complete picture, writing additional zeros *to the left of the decimal* point preceding the first digit does not change the value of the number.

$$3.7 = 03.7 = 003.7$$
$$-56.4 = -056.4 = -0056.4$$
or
$$761.0 = 0761.0 = 00761.0$$

Although placing extra zeros before the first digit to the left of the decimal point or extra zeros after the last digit to the right of the decimal point does not change the value of the decimal, when giving the final answer, we should remove all unnecessary zeros.

(*) In these materials, we will always use the notation with 0 in front of the decimal point, since it is less likely to cause misreading the decimal.

(**) This is true because $0.3 = \frac{3}{10} = \frac{3 \times 10}{10 \times 10} = \frac{30}{100} = 0.30$. Similarly, $0.3 = \frac{3}{10} = \frac{3 \times 100}{10 \times 100} = \frac{300}{1000} = 0.300$

Example 10.5 Fill in the blanks with the proper symbol "$\neq$" or "=".

a) 0.7 ___ 0.07 b) 45 ____ 45.0000 c) -0.60000 _____ $-.6$

Solution:

a) $0.7 \neq 0.07$
b) $45 = 45.0000$
c) $-0.60000 = -.6$

Example 10.6 Rewrite the fraction 0.27 in its equivalent form, in such a way that there are five digits after the decimal point.

Solution:
0.27000

Example 10.7 Write 8 as a decimal.

Solution:
$8 = 8.0$

Writing decimals as a fraction or a mixed number

Since the name of a fraction is the same regardless of notation, we read the name of the decimal and write it as a fraction.

For example, 0.7 is read "seven tenths" and this means $\frac{7}{10}$.

2. 65 is read "two and sixty five hundredths", so $2.65 = 2\frac{65}{100}$.

It is a good practice to always reduce the final answer, so $2.65 = 2\frac{65}{100} = 2\frac{5\times 13}{5\times 20} = 2\frac{13}{20}$. Also, if you wish to have your answer written as an improper fraction, convert the mixed number $2\frac{13}{20}$ to a fraction $2\frac{13}{20} = \frac{2\times 20+13}{20} = \frac{53}{20}$. Thus, $2.65 = \frac{53}{20}$.

Multiplication of decimals by powers of ten

As you might recall from Lesson 1, the numbers 10, 100, 1000, … are called powers of 10. By now, it should be clear, where this name comes from. They are called powers of 10, because

$$10 = 10^1$$
$$100 = 10^2$$
$$1000 = 10^3$$

One of the advantages of using decimal notation is that multiplication and division of decimals by any power of 10 can be performed almost instantaneously. To develop the rule for such multiplication, let us multiply 0.345 by three different powers of 10.

$$0.345 \times 10 = \frac{345}{1000} \times 10 = \frac{345 \times 10}{1000} = \frac{345 \times 10}{100 \times 10} = \frac{345}{100} = 3\frac{45}{100} = 3.45$$

$$0.345 \times 100 = \frac{345}{1000} \times 100 = \frac{345 \times 100}{1000} = \frac{345 \times 100}{10 \times 100} = \frac{345}{10} = 34\frac{5}{10} = 34.5$$

$$0.345 \times 1000 = \frac{345}{1000} \times 1000 = \frac{345 \times 1000}{1 \times 1000} = \frac{345}{1} = 345$$

Notice that each time we multiply 0.345 by a given power of 10, we move the decimal point to the right by the number of places equal to the number of zeros following the number 1 (which is also equal to the power of 10).

> To multiply a decimal by the power of 10, such as 10, 100, 1000, …
>
> 1. Count the number of zeros: $1\underbrace{0\,0}_{\text{2 zeros}} \times\ 3.472$
> 2. Move the decimal point that many places to the right: $1\,0\,0 \times 3.4\,7\,2 = 347.2$
>
> decimal point moved
> 2 places to the right

Sometimes, in order to be able to place the decimal point in the answer, it is necessarily to add extra zeros to the right of the last digit. This idea is illustrated in the next two examples.

Multiply 0.57×1000.
Since the number 1000 has three zeros, we must move the decimal point three places to the right. The number 0.57 displays only two decimal places, so the extra zero is needed. We use the fact that $0.57 = 0.570$.

$$0.5\,7\,0 \longrightarrow 0\,5\,7\,0. = 5\,7\,0$$

3 places to the right

$0.57 \times 1000 = 570$

Multiply 100×36 (*).
The number 100 has two zeros, we must move the decimal point two places to the right. We use the

(*) Recall that to multiply an integer by a power of 10, we can simply rewrite the integer and add the number of zeros equal to the power of ten at the end of the number. Notice that adding a given number of zeros at the end of a number is equivalent to moving the decimal point to the right the same number of places. The procedure of multiplying by the power of 10 by moving a decimal point can be viewed as an extension of the procedure of adding zeros at the end. Simply rewriting the integer and adding the appropriate number of zeros *only* applies to integers. The new rule of moving the decimal point the appropriate number of places applies to all decimals, including integers, since any integer can be written as a decimal.

fact that $36 = 36.00$

$$3\,6\,.\,0\,0 \longrightarrow 3\,6\,0\,0\,. = 3600$$

2 places to the right

$100 \times 36 = 3600$

Example 10.8 Perform the following multiplications.

a) 10×9.46 b) 0.7×1000

Solution:

a) We move the decimal point one place to the right $10 \times 9.46 = 94.6$

b) We move the decimal point three places to the right and we use the fact that $0.7 = 0.700$:

$0.7 \times 1000 = 700. = 700$ (It is not necessary to write the decimal point after the integer part; it is understood).

Example 10.9 What is the smallest power of 10 we can use to multiply by 0.03 in order to get an integer?

Solution:

We count the number of decimal places (or, equivalently, number of digits) after the decimal point. There are 2 places (2 digits: 0 and 3), thus we have to multiply 0.03 by 100 (second power of 10; two zeros after 1)

$0.03 \times 100 = 3$.

Division of decimals by powers of ten

Division is the opposite operation to multiplication which means that we can "undo" the operation of multiplication by a given number by performing division by the same number. Hence, if the multiplication of a decimal by a power of 10 means moving the decimal point *to the right* a given number of places, the division must be moving the decimal point *to the left* the same number of places.

To divide a decimal by the power of 10, such as 10, 100, 1000, …

1. Count the number of zeros: $347.2 \div 1\underbrace{00}_{\text{2 zeros}}$
2. Move the decimal point that many places to the right: $347.2 \div 100 = 3.472$

decimal point moved
2 places to the left

In the multiplication by power of 10 we often need to place additional zeros *to the right* of the last digit after the decimal point. In division, we often need to place additional zeros *to the left* of the first digit before the decimal point.

Divide $0.6 \div 100$.
The number 100 has two zeros, so we must move the decimal point two places to the left. We use the fact that $0.6 = 000.6$

$$0\ \ 0\ \ 0 \bullet 6 \ \longrightarrow\ 0 \bullet 0\ 0\ 6$$

2 places to the left

$0.6 \div 100 = 0.006$

Divide $7 \div 10000$
The number 10000 has four zeros, so we must move the decimal point four places to the left. We use the fact that $7 = 00007.0$

$$0\ \ 0\ \ 0\ \ 0\ \ 7 \bullet 0 \ \longrightarrow\ 0 \bullet 0\ 0\ 0\ 7$$

4 places to the left

$7 \div 10000 = 0.0007$

<u>Example 10.10</u> Perform the following divisions.

a) $296.34 \div 100$ b) $0.3 \div 1000$

Solution:

a) Since the number 100 has two zeros, we move the decimal point two places to the left. $296.34 \div 100 = 2.9634$

b) Since the number 1000 has three zeros, we move the decimal point three places to the left and use the fact that $0.3 = 0000.3$

$0.3 \div 1000 = 0.0003$

<u>*Writing fractions and mixed numbers as decimals*</u>

Suppose that we would like to write $\frac{11}{4}$ as a decimal. To this end, we will interpret a fraction as implied division: $\frac{11}{4} = 11 \div 4$. We use the algorithm to perform this division that is very similar to the long division algorithm.

Set up the division as you set up long division.

$$4\overline{)11}$$

The first step is exactly the same as it would be in long division. Divide $11 \div 4$. Place the whole number result, 2, at the top. Do not worry about the remainder at this moment. Instead, proceed to multiply the 2 by the devisor 4.

$$4\overline{)11}^{\;2}$$

Multiply $4 \times 2 = 8$. Place the result, 8, under the number that was divided, 11. Subtract the bottom number from the top one to get the remainder: $11 - 8 = 3$.

$$\begin{array}{rl} 2 & \\ 4\overline{)11} & \\ \underline{-8} & \\ 3 & \quad \leftarrow \textit{remainder} \end{array}$$

We no longer can use the algorithm for long division, because there are no more digits to bring down. We are going to **extend the algorithm**.
Place the decimal point on the top (in this example, next to 2). Remembering that $11 = 11.0$, **bring down a zero from 11.0 and continue**.

$$\begin{array}{rl} \textit{Decimal point was placed here } \downarrow & \\ 2. & \\ 4\overline{)11\;\mathbf{.0}} & \quad \leftarrow \textit{This zero was brought down.} \\ -8 \downarrow & \\ \hline 3\;\mathbf{0} & \quad \leftarrow \textit{The zero that was brought down. It changes 3 to 30.} \end{array}$$

Again, follow the steps of long division. Divide 30 by 4. Place the whole number result, 7, at the top, to the right of the decimal point.

$$\begin{array}{r} 2.7 \\ 4\overline{)11.0} \\ -8 \\ \hline 30 \end{array}$$

Multiply $4 \times 7 = 28$. Place the result, 28, under the number that was divided, 30. Subtract the bottom number from the top one to get the remainder, $30 - 28 = 2$.

$$\begin{array}{rl} 2.7 & \\ 4\overline{)11.0} & \\ -8 & \\ \hline 30 & \\ -28 & \\ \hline 2 & \quad \leftarrow \textit{remainder} \end{array}$$

Once again bring down a zero $(11.0 = 11.00)$.

$$\begin{array}{r} 2.7 \\ 4\overline{)11.0\,\mathbf{0}} \\ -8 \\ \hline 30 \\ 28 \\ \hline 2\,\mathbf{0} \end{array}$$

← *Bring down this zero.*

← *This is the zero that was brought down.*

Divide $20 \div 4 = 5$. Place the whole number result, 5, at the top, next to 7.

$$\begin{array}{r} 2.7 \\ 4\overline{)11.0\,0} \\ -8 \\ \hline 30 \\ 28 \\ \hline 20 \end{array}$$

Multiply $4 \times 5 = 20$. Place the result, 20, under the number that was divided, 20. Subtract the bottom number from the top one to get the remainder, $20 - 20 = 0$.

$$\begin{array}{r} 2.75 \\ 4\overline{)11.0\,0} \\ -8 \\ \hline 30 \\ 28 \\ \hline 20 \\ -20 \\ \hline 0 \end{array}$$

Since the remainder is equal to 0, we are done with the division. As a result we have $\frac{11}{4} = 2.75$

The above algorithm, called **the extended long division algorithm**, will always work, but in certain cases we can avoid using it and, instead, use other techniques such as those demonstrated in the next two examples below.

- If the denominator of the fraction we are converting is equal to a power of 10, apply the algorithm for division by the power of 10 (move the decimal point to the left a number of spaces equal to the power of 10).
  For example, $\frac{7}{100} = 7 \div 100 = 7.0 \div 100 = 0.07$ or $\frac{234}{10} = 234 \div 10 = 234.0 \div 10 = 23.4$

- If the denominator of a fraction is such that it is easy to find an equivalent fraction with the new denominator being a power of 10, rewrite it in its equivalent form with the denominator equal to a power of 10, then apply the algorithm for division by the power of 10.
  For example, $\frac{2}{5} = \frac{2 \times 2}{5 \times 2} = \frac{4}{10} = 4 \div 10 = 0.4$ or $\frac{3}{200} = \frac{3 \times 5}{200 \times 5} = \frac{15}{1000} = 15 \div 1000 = 0.015$

To change a mixed number to a decimal, separate its integer part from the fraction. Change the fraction to a decimal, and use the integer as the integer part of the result.

For example, let us write $3\frac{4}{25}$ in decimal notation. We ignore the integer part, 3. We change the fraction to a decimal.

$$\frac{4}{25} = \frac{4 \times 4}{25 \times 4} = \frac{16}{100} = 16 \div 100 = 0.16$$

and, remembering about the integer part, we write $3\frac{4}{25} = 3.16$

Negative fractions or mixed numbers stay negative when written in decimal notation.

Example 10.11 Change the following fractions (or mixed numbers) to a decimal.

a) $\frac{234}{100}$ b) $-3\frac{1}{200}$ c) $\frac{5}{8}$

Solution:

a) $\frac{234}{100} = 234 \div 100 = 234.0 \div 100 = 2.34$

b) $-3\frac{1}{200} = -3\frac{1 \times 5}{200 \times 5} = -3\frac{5}{1000} = -3.005$

c)
$$\begin{array}{r} 0.625 \\ 8\overline{)5.000} \\ 50 \\ -48 \\ \hline 20 \\ -16 \\ \hline 40 \\ -40 \\ \hline 0 \end{array}$$

Thus $\frac{5}{8} = 0.625$

Comparing decimals

Decimals are compared in exactly the same way as other numbers: by comparing the different place values from left to right.
To compare any two positive decimals, compare the integer part of each decimal. The one with bigger integer part is bigger. If the integer parts are the same, start with the tenths place. If one decimal has a higher number in the tenths place then it is larger than a decimal with fewer tenths. If the tenths are equal, compare the hundredths, then the thousandths etc. until one decimal is larger or there are no more places to compare. If each decimal place value is the same then the decimals are equal.

For example, compare 0.263 and 0.268.

Both numbers have integer parts equal to zero.
We compare the tenths place value digits.

0.263   0.268

The digits in the tenths place are the same, both are 2.
We compare the digits in the hundredths place.

0.263   0.268

The digits in the hundredths place are the same, both are 6.
We compare the digits in the thousandths place.

0.263   0.268

The digits in the thousandths place differ.
Since $3 < 8$, we conclude that
$0.263 < 0.268$.

You might find performing the comparison easier if you write decimals vertically, lining up the decimal points properly and only then comparing the digits.

0.263
0.268   The first digits that are not the same are 3 and 8, thus $0.263 < 0.268$.

If one decimal has more digits after the decimal point, we can always add zeros at the end without changing the value of the decimal. Thus, any time we compare a digit with "an empty decimal place", we know that "the empty decimal place" stands for 0.

Compare 2.3 and 2.34.

2.30	The integer part is the same. It is 2.
2.34	The tenths digit is the same. It is 3.
	The hundredths digit in 2.34 is 4, and in 2.3 is 0. Since $4 > 0$, we get $2.34 > 2.3$.

To compare two negative decimals, compare them first ignoring the minus signs, and then reverse the inequality sign.

Compare -0.7 and -0.8.

We first compare 0.7 and 0.8: $0.7 < 0.8$ (since $7 < 8$)

and then we reverse the inequality sign: $-0.7 > -0.8$.

<u>Example 10.12</u> Fill in the blank with one of the symbols <, =, or >.

a) 21.99 _________ 22.1

b) -4.82 __________ -4.823

c) 3.9 ________ 3.90

Solution:

a) The integer part 22 is greater than 21, hence $21.99 < 22.1$.

b) We first compare 4.82 with 4.823. Remembering that $4.82 = 4.820$, we get $4.82 < 4.823$ After reversing the inequality sign, the final answer is $-4.82 > -4.823$.

c) $3.9 = 3.90$.

<u>Example 10.13</u> Replace X with any digit to make the statement true.

$0.2X > 0.27$

Solution:

0.2X will be greater than 0.27 if we replace X with any digit greater than 7. There are two possibilities $X = 8$ or $X = 9$. Either choice gives the correct answer.

Exercises with Answers (For answers see Appendix A)

Ex.1 Fill in the blank with the word "right" or "left" to make the following statements true.

Every integer has an unwritten decimal point to the __________.

To multiply a decimal by a power of 10, move the decimal point to the _________ the same number of decimal places as the number of zeros in the power of ten.

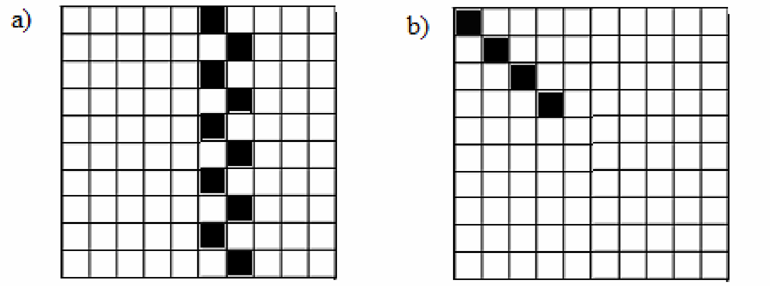
Fractional part of a decimal is to the _________ of the decimal point.

To divide a decimal by a power of 10, move the decimal point to the __________ the same number of decimal places as the number of zeros in the power of ten.

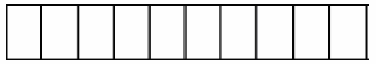
The place value of hundredths is to the _________ of the place value of tenths.

Place value of tens is to the _________ of the decimal point.

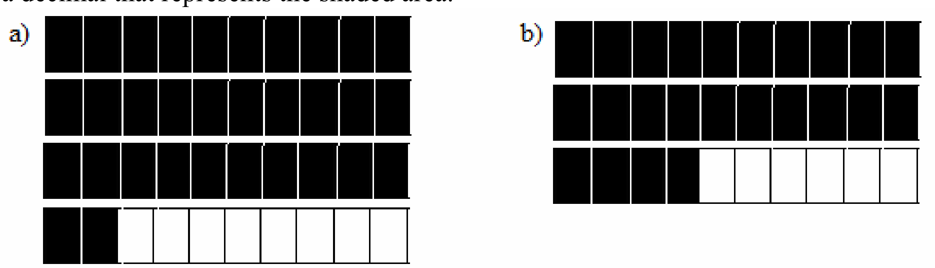
Ex.2 Write a decimal that represents the shaded area.



Ex.3 This is a unit.



Write a decimal that represents the shaded area.



Ex.4 Identify the integer part and the fractional part of each of the following decimals.
a) 4.06 b) -2.986 c) 0.901

Ex.5 In the number 45.0129 find the following place values.
a) hundredths
b) tens
c) ten thousandths

Ex.6 In the number -345.678 find the following place values.
a) hundreds
b) thousandths
c) tenths

Ex.7 Construct a decimal with the following place values: 3 in the tenths decimal place, 0 in the ones decimal place, 8 in the tens decimal place, 7 in the thousandths decimal place, and 5 in the hundredths decimal place (be careful, since the place values have been given out of order).
Ex.8 Construct a *negative* decimal with the following place values: 6 in the hundreds decimal place, 7 in the ones decimal place, 2 in the thousands decimal place, 4 in the tenths decimal place, 0 in the tens decimal place, and 9 in the hundredths decimal place.

Ex.9 For each of the following decimals, count the number of decimal places to the right of the decimal point. Name the last digit (most to the right) and name its place value.

a) 78.4 b) 3.672

c) -0.48 d) -0.0001

Ex.10 What is the name of the decimal place (ones, tens, hundreds, tenths, hundredths etc.) where the digit 6 is placed in each of the following number.

a) -264.78 b) 0.456

c) 27.643 d) -6.4

Ex.11 Write each decimal in words.

a) -5.07 b) 0.234

c) 124.0001 d) -0.4

Ex.12 Write the following fraction in a decimal notation.

a) three and fifty seven hundredths

b) thirty five hundredths

c) minus two hundred seven and ninety five ten thousandths

d) minus forty four and two thousandths

Ex.13 Fill in the blanks with the proper symbol $\neq$ or $=$.

a) 0.8_____0.800 b) 95_____95.0

c) 005_____00.5 d) 0.789_____.789

e) -0.230_____$-.23$ f) -45_____-450

g) 0000.36_____0.360 h) -7.002_____-7.0002

i) $-.4$_____-0.4 j) 00071_____71

Ex.14 Circle all decimals that are equal to 23.7

0.237 23.7000 00023.7 237.00 23.70 0.237

Ex.15 Write the decimal 0.7 in its equivalent way such that there are

a) 2 digits after the decimal point

b) 5 digits after the decimal point

Ex.16 Write a) 8

b) -248

as a decimal.

Ex.17 Write each decimal as a fraction.

a) 3.4 b) -0.7

c) 2.005 d) 13.7

e) -0.003 f) 0.2060

Ex.18 Perform the following multiplications.

a) 1000×0.17 b) 10×0.46

c) 34.7×10000 d) 7×100

e) 10×0.7629 f) 100×3.4

Ex.19 Perform the following divisions.

a) $23.1 \div 10$ b) $678.2 \div 100$
c) $0.0067 \div 1000$ d) $421 \div 100$
e) $4 \div 10000$ f) $0.40 \div 1000$

Ex.20 The decimal point of a given number was

a) moved 3 places to the right
b) moved 2 places to the left
c) moved 1 place to the left

Determine what kind of operation was performed on the number i.e. was the number multiplied or divided and by what power of 10.

Ex.21 Perform the following operations.

a) 0.2×100 b) $0.7 \div 100$
c) $34 \div 10$ d) 1.2×1000
e) 0.2×10 f) $54.90700 \div 1000$
g) $9.004 \div 100$ h) 3400×100
i) 2.345×100 j) $72 \div 1000$
k) 100×35 l) $567.8 \div 100$

Ex.22 What is the smallest power of 10 we can use to multiply each of the following decimals in order to get an integer?

a) 0.78
b) -23.4
c) $-.891$
d) 2.3701
e) -0.005
f) 2.100

Ex.23 Replace x with a number to make the following statement true.

a) $3.7 \cdot x = 370$ b) $461 \div x = 0.461$ c) $x \div 10 = 2.5$
d) $0.64 \cdot x = 6.4$ e) $x \div 100 = 0.02$ f) $x \cdot 1000 = 3715.2$
g) $12 \cdot x = 12000$ h) $893.2 \div x = 8.932$ i) $9 \div x = 0.9$

Ex.24 Change each of the following fractions (or mixed numbers) to a decimal.

a) $\frac{7}{25}$ b) $45\frac{1}{2}$
c) $-\frac{5}{1000}$ d) $\frac{31}{40}$
e) $4\frac{2}{5}$ f) $-2\frac{7}{16}$
g) $-\frac{7}{8}$ h) $-\frac{3}{50}$

i) $\frac{43}{8}$

j) $-\frac{23917}{100}$

k) $-4\frac{102}{10000}$

l) $3\frac{1}{200}$

Ex.25 Fill in the blank with one of the symbols <, =, or >.

a) 4.3 __________ 4.003

b) -23.1 __________ -25.6

c) -4.0 __________ 0.01

d) 0.2345 __________ 0.2635

e) -2.87 __________ -42.874

f) -0.1234 __________ -0.1824

g) 4.56 __________ 4.650

Ex.26 Between what two consecutive integers would be the following decimals located on a number line?

a) 0.6

b) -2.39999

c) 4.0001

d) -0.78

Ex.27 Find all decimals that are less than 0.347

0.348 0.3 -0.1 0.3471 0.247 1.347 0.34

Ex.28 Write the following numbers in order from the smallest to the largest.

-0.4 2.34 -8.7 -0.09 3.1 5 2.36 -0.5

Ex.29 Replace X with any digit to make the statement true.

a) $34.6X > 34.67$

b) $X8.3 < 28.1$

c) $-0.5 < -0.X$

d) $0.27 = 0.27X$

Lesson 11

Topics: Addition, subtraction, multiplication, division, exponentiation and all operations combined on decimals.

Addition/subtraction of decimals

Recall that to add objects they must be of the same type. Hence, **to add decimals, write them vertically to line up their decimal points and columns,** then add corresponding place values: ones to ones, tenths to tenths, hundredths to hundredths and so on. **Perform the addition just as you would add natural numbers, starting with digits from the right. The decimal point in the result is written directly below the decimal point in the problem.**

For example

$$\begin{array}{rcccccc} & 1 & 4 & . & 5 & 3 \\ + & 2 & 2 & . & 7 & 1 \\ \hline \end{array}$$

Line up the decimal point and all columns.

$$\begin{array}{rcccccc} & 1 & 4 & . & 5 & 3 \\ + & 2 & 2 & . & 7 & 1 \\ \hline & & & & & 4 \end{array}$$

Start addition with the column on the far right.
In this example, add hundredths: $3+1=4$

$$\begin{array}{rcccccc} & & 1 & & & \\ & 1 & 4 & . & 5 & 3 \\ + & 2 & 2 & . & 7 & 1 \\ \hline & & & & 2 & 4 \end{array}$$

Add tenths: $5+7=12$.
We get 12 tenths $=$ 1 one~~s~~ $+$ 2 tenth~~s~~, so we carry 1 to the ones column.

$$\begin{array}{rcccccc} & & 1 & & & \\ & 1 & 4 & . & 5 & 3 \\ + & 2 & 2 & . & 7 & 1 \\ \hline & & 7 & . & 2 & 4 \end{array}$$

Add ones: $1+4+2=7$

$$\begin{array}{rcccccc} & & 1 & & & \\ & 1 & 4 & . & 5 & 3 \\ + & 2 & 2 & . & 7 & 1 \\ \hline & 3 & 7 & . & 2 & 4 \end{array}$$

Add tens: $1+2=3$
Bring down the decimal point.

Thus $14.53+22.71=37.24$

Subtraction is performed in the same manner. First line up the columns and the decimal points. Then subtract just as you would subtract natural numbers, starting with digits from the right, ignoring the decimal points, borrowing if necessary. The decimal point in the result is written directly below the decimal point in the problem.

$$\begin{array}{rcccccc} & 1 & 8 & . & 6 & \\ - & 1 & 2 & . & 4 & 1 \\ \hline \end{array}$$

Line up the columns and the decimal points. Notice that there is no digit directly above 1 (in hundredths column). This always means that there is an “unwritten” 0 there. Many find it helpful to write in the zero (replace 18.6 with 18.60).

$$\begin{array}{rcccccc} & & & & 5 & 10 \\ & 1 & 8 & . & \cancel{6} & 0 \\ - & 1 & 2 & . & 4 & 1 \\ \hline & & & & & 9 \end{array}$$

Borrow from tenths in order to subtract hundredths : $10-1=9$.

$$\begin{array}{rcccccc} & & & & 5 & 10 \\ & 1 & 8 & . & \cancel{6} & 0 \\ - & 1 & 2 & . & 4 & 1 \\ \hline & & & & 1 & 9 \end{array}$$

Subtract tenths: $5-4=1$

$$\begin{array}{rcccccc} & & & & 5 & 10 \\ & 1 & 8 & . & \cancel{6} & 0 \\ - & 1 & 2 & . & 4 & 1 \\ \hline & & 6 & & 1 & 9 \end{array}$$

Subtract ones: $8-2=6$

$$\begin{array}{rcccccc} & & & & 5 & 10 \\ & 1 & 8 & . & \cancel{6} & 0 \\ - & 1 & 2 & . & 4 & 1 \\ \hline & & 6 & . & 1 & 9 \end{array}$$

Subtract tens. $1-1=0$. Since the zero would be the first digit, we do not write it. Bring down the decimal point.

Thus, $18.6-12.41=6.19$.

When adding or subtracting involves natural numbers, keep in mind that there is an “unwritten” decimal point that can be written in, if desired. For example, let us subtract $9-0.3$ by writing 9 as 9.0.

$$\begin{array}{rccc} & 8 & & 10 \\ & \cancel{9} & . & 0 \\ - & 0 & . & 3 \\ \hline & 8 & . & 7 \end{array}$$

$9-0.3=8.7$

To add/subtract decimals, we replace any ‘double signs’ in the expression with one sign following the rules $(+)(+)\ \rightarrow(+)$ $(-)(+)\ \rightarrow(-)$
$(-)(-)\ \rightarrow(+)$ $(+)(-)\ \rightarrow(-)$
and perform the operations according to the rules for decimals.

For example,

$0.2-(-3.4)=$	Replace the double minus signs by plus.	
$0.2+3.4=$	If you do not want to perform the addition	0.2
3.6	mentally, you can do it off to the side	$+\ 3.4$
	and then write your result in the actual problem.	3.6

Example 11.1 Perform the indicated operations.

a) $3+4.07$ b) $-0.7+0.63$ c) $5.1-(-0.71)+(-2.003)$

Solution:

a) We set up the addition remembering that $3=3.00$.

$$\begin{array}{r} 3.00 \\ +\ \ 4.07 \\ \hline 7.07 \end{array}$$

Thus, $3+4.07$ =7.07.

b) The numbers are of the opposite signs, and 0.7 > 0.63, so the sign of the result is negative and we need to subtract 0.63 from 0.7

$$\begin{array}{r} {\scriptstyle 6\ 10} \\ 0.\cancel{7}\cancel{0} \\ +\ \ 0.6\,3 \\ \hline 0.0\,7 \end{array}$$

Thus, $-0.7+0.63=-0.07$.

c) We replace all "double signs" with one sign according to rules.

$5.1-(-0.71)+(-2.003)=5.1+0.71-2.003$.

$$\begin{array}{r} 5.1\,0 \\ +\ \ 0.7\,1 \\ \hline 5.8\,1 \end{array} \qquad \begin{array}{r} {\scriptstyle 0\ 10} \\ 5.\,8\,\cancel{1}\,\cancel{0} \\ -\ \ 2.\,0\,0\,3 \\ \hline 3.\,8\,0\,7 \end{array}$$

Thus $5.1-(-0.71)+(-2.003)=5.1+0.71-2.003=3.807$

Notice, that in both operations we wrote in additional zeros. We wrote 5.10 instead of 5.1 and 5.810 instead of 5.81. As mentioned before it is not a necessary step, but is recommended, particularly for subtraction.

Multiplication of decimals

Let us multiply 0.2 and 0.08.

$0.2\times 0.08=$	Change to fractions.
$\frac{2}{10}\times\frac{8}{100}=$	Multiply numerators and multiply denominators.
$\frac{2\times 8}{10\times 100}=$	Perform the indicated operations.
$\frac{16}{1000}=$	Write the answer as a decimal.
0.016	

Notice that: The digits in the answer came from multiplying $2\times 8=16$.

Since the denominator is equal to $10\times 100=1000$ and division by 1000 (fraction bar indicates division) means "moving the decimal point 3 places to the left", we get 0.016

Based on the above example, the multiplication of 0.2×0.08 done without converting decimals to fractions, should be performed as follows.

Step 1 Mentally delete the decimal points in both numbers: 0.2 becomes $02 = 2$ and 0.08 becomes $008 = 8$

Step 2 Perform the multiplication of the resulting integers: $2 \times 8 = 16$

Step 3 Count the sum of numbers of decimal places in both decimals involved in the operation: 0.2 has 1 decimal place; 0.08 has 2 decimal places; together, there are $1 + 2 = 3$ decimal places in both numbers

Step 4 Write the integer obtained in step 2 as a decimal and move its decimal point to the left as many places as the number obtained in step 4 indicates. If needed, before moving it, add extra zeros at the beginning of the number: write 16 as 0016.0 and move the decimal point 3 places to the left.

$$0\ 0\ 1\ 6\ .\ 0 \rightarrow \quad 0\,.\,0\ 1\ 6\ 0$$

Step 5 Write the final answer without unnecessary zeros: $0.0160 = 0.016$

As a result, $0.2 \times 0.08 = 0.016$

If negative decimals are involved in the multiplications, we follow the rules that we have always used for determining the sign in multiplication. For example, the result of $-0.3 \times 0.2 = -0.06$ is negative because we multiply numbers of opposite signs.

We summarize the above procedures in the table below.

HOW TO MULTIPLY DECIMALS
Step 1. Determine the sign of the result of the multiplication of decimals following the same rules as in determining the sign of the product of multiplication. **(+)(+) → (+)** **(−)(+) → (−)** **(−)(−) → (+)** **(+)(−) → (−)** Step 2. Multiply the decimals as if they were whole numbers. Step 3. Find the total number of decimal places in both factors. Step 4. Place the decimal point in the result so that the answer has the same number of decimal places as the total found in Step 3. For example, $0.2 \times (-1.01) = -0.2 \times 1.01 = -0.2 \times 1.01 = -0.202$ (Multiply $2 \times 101 = 202$; Total number of decimal places is $1 + 2 = 3$; Move the decimal point in 202=202.0 three places to the left)
Multiplying more than two decimals: If the number of negative factors in the product is an even number, the result is positive. If the number of negative factors in the product is odd, the result is negative. Count the total number of decimal places in *all* factors. For example, $0.3 \times (-0.1) \times 0.5 = -0.015$ or $0.3 \times (-0.1) \times (-0.5) = 0.015$

Example 11.2 Perform the indicated operations.

a) 0.05×0.4 b) $0.34 \times (-0.00001)$

c) $-4 \times (-0.2)$ d) $-50 \times 0.1 \times (-0.02) \times (-0.01)$

Solution:

a) $0.05 \times 0.4 = 0.020 = 0.02$

($5 \times 4 = 20$, move decimal point 3 places to the left; although initially we get 0.020, we should rewrite it as 0.02).

b) $0.34 \times (-0.00001) = -0.0000034$ ($1 \times 34 = 34$, 7 places to the left).

c) $-4 \times (-0.2) = 0.8$

($4 \times 2 = 8$, 1 place to the left since 4 has no decimal places and 0.2 only has one)

d) $-50 \times 0.1 \times (-0.02) \times (-0.01) = -0.00100 = -0.001$

($50 \times 1 \times 2 \times 1 = 100$, the sign is negative since there are an odd number of negative factors; 5 places to the left; eliminate unnecessary 0's in the final answer) .

Exponentiation of decimals

As always, once we know how to perform the multiplication operation, we know how to exponentiate. The only additional convention we need to remember is that to avoid confusion, we should **place the decimal that is being raised to a given power in parentheses.**

$$(0.2)^3 = 0.2 \times 0.2 \times 0.2 = 0.008$$

Finding a decimal of a given quantity

Recall that to find a fraction of a give number we multiply the fraction by the number. And thus, **to find a decimal of a given number, we multiply the decimal by the given number.**

For example, if we know that on a given year, out of 58000 students attending Washington D.C. high school, 0.2 of them passed a proficiency exam, how many students passed the exam?

We perform the multiplication: 0.2×58000=10600.

Answer: 10600 students passed the proficiency exam.

Division of decimals

We often solve problems by reducing them to a situation we know how to handle. We will apply this strategy to the division of decimals. That is, we will perform certain operations so that instead of dividing decimals, we will be able to divide integers, which should be easier. The example below explains how to do that. Let us divide $0.3 \div 0.05$

$0.3 \div 0.05 =$	Write the division using the fraction bar.
$\frac{0.3}{0.05} =$	We know that if we multiply the numerator and the denominator by the same non-zero number, the resulting fraction will be equal to the original one. Let us multiply the numerator and denominator by 100. We want to do that because after performing the multiplication, both the numerator and denominator are no longer decimals.

$\dfrac{0.3\times 100}{0.05\times 100}=$	Perform the indicated operations.
$\dfrac{30}{5}=$	Reduce. The obtained fraction (or integer) is the answer.
6	

The missing part of the algorithm is how to find a number by which we should multiply the numerator and denominator in order to eliminate decimals (in the above case 100). To find the number with this property we do the following.

Step 1 Count the number of decimal places in the numerator: the numerator 0.3 has decimal place.
Step 2 Count the number of decimal places in the denominator: the denominator 0.05 has 2 decimal places.
Step 3 Take the bigger number of the two found in previous steps: 2 is the bigger number
Step 4 Raise 10 to the number found in step 3 to get the number we are looking for: $10^2=100$

Instead of memorizing the above algorithm, let us understand how and why it works. Our goal is to eliminate the decimal point from the numerator and denominator by multiplying both by the *same* number equal to a power of 10. If we try to use 10, we can see that although it will work for the numerator $0.3\times 10=3$ (no longer a decimal), it will not work for the denominator since $0.05\times 10=0.5$ (still a decimal). Since the denominator has 2 decimal places, we need to use 100. The number 100 will work for both:

$$\frac{0.3}{0.05}=\frac{0.3\times 100}{0.05\times 100}=\frac{30}{5}=6$$

The above algorithm will give us the answer in the form of a fraction or an integer. If, for some reason, we would like to have the answer in decimal form, we need to change the fraction (unless it is an integer) to a decimal applying the extended long division algorithm or, if possible, some other techniques.
For example,

$$0.1\div(-1.6)=-\frac{0.1}{1.6}=-\frac{0.1\times 10}{1.6\times 10}=-\frac{1}{16}.$$

To have an answer in the form of a decimal, we must perform the extended long division.

$$\begin{array}{r} 0.0625 \\ 16\overline{)1.0000} \\ 1\,00 \\ \underline{96} \\ 40 \\ \underline{32} \\ 80 \\ \underline{80} \\ 0 \end{array}$$

Thus, $0.1 \div (-1.6) = -\frac{0.1}{1.6} = -\frac{0.1 \times 10}{1.6 \times 10} = -\frac{1}{16} = -0.0625$

We summarize the above procedures in the table below.

HOW TO DIVIDE DECIMALS
Step 1. Determine the sign of the result following the same rules as in determining the sign of the quotient of integers. $(+)(+) \to (+)$ $(-)(+) \to (-)$ $(-)(-) \to (+)$ $(+)(-) \to (-)$ Step 2. Count the number of the decimal places in the numerator and the denominator and take the bigger number.   Step 3. Move the decimal point in the numerator and denominator to the right by the number found in step 2.   Step 4. Reduce the resulting fraction and if asked for the answer in the form of a decimal, change The fraction to a decimal. If required, apply the extended long division algorithm.   For example, $0.004 \div (-0.2) = -\frac{0.004}{0.2} = -\frac{0.004}{0.2} = -\frac{4}{200} = -\frac{2}{100} = -0.02$ (Number of decimal places in 0.004 is 3, in 0.2 is 1; The bigger number is 3; We move the decimal points by 3 places to the right).

Example 11.3 Perform the indicated operation.

a) $\frac{0.24}{0.6}$. b) $-2.2 \div 0.011$

Solution:

a) $\frac{0.24}{0.6}. = \frac{0.24 \times 100}{0.6 \times 100} = \frac{24}{60} = \frac{12 \times 2}{12 \times 5} = \frac{2}{5}$ The decimal point was moved in the numerator and denominator 2 places to the right.

b) $-2.2 \div 0.011 = -\frac{2.2}{0.011} = -\frac{2200}{11} = -200$ The decimal point was moved in the numerator and denominator 3 places to the right.

Example 11.4 Perform the following operation $0.4 \div (-8)$. Give the answer as an integer or as a decimal.

Solution:

$0.4 \div (-8) = -\frac{0.4}{8} = -\frac{4}{80} = -\frac{1}{20}$. If we were not asked to give the answer in the form of a decimal, $-\frac{1}{20}$ would be a correct answer. Since the directions asked for an integer or a decimal, we must convert $-\frac{1}{20}$ to a decimal. The easiest way is to multiply the numerator and the denominator by 5.

$$-\frac{1}{20} = -\frac{1 \times 5}{20 \times 5} = -\frac{5}{100} = -0.05$$

<u>All operations on decimals combined</u>

As always, if more than two operations are involved in evaluating an expression, we perform them according to the order of operations.

<u>Example 11.5</u> Perform the indicated operations.

a) $-(0.3 \times 0.2)^2$ b) $\frac{-2.1-(-1.9)}{0.04}$

c) $-0.6 + 0.7 \times 0.02$ d) $-0.9 \div (-0.3) \times (-0.002)$

Solution:

a) $-(0.3 \times 0.2)^2 = -(0.06)^2 = -0.06 \times 0.06 = -0.0036$

b) $\frac{-2.1-(-1.9)}{0.04} = \frac{-2.1+1.9}{0.04} = \frac{-0.2}{0.04} = \frac{-0.2 \times 100}{0.04 \times 100} = \frac{-20}{4} = -5$

c) $-0.6 + 0.7 \times 0.02 = -0.6 + 0.014 = -0.586$

d) $-0.9 \div (-0.3) \times (-0.002) = \frac{-0.9}{-0.3} \times (-0.002) = \frac{0.9 \times 10}{0.3 \times 10} \times (-0.002) =$

$\frac{9}{3} \times (-0.002) = 3 \times (-0.002) = -0.006$

Exercises with Answers (For answers see Appendix A)

Ex.1 Perform the indicated operations.

a) $4.71 + 12.14$ b) $0.77 + 5.644$ c) $8 + 0.004$

d) $2.88 - 0.41$ e) $3.9 - 0.02$ f) $44 - 0.003$

Ex.2 Perform the indicated operations.

a) $-0.23 + 1.299$ b) $-2.3 - 0.8$ c) $3.9 - 4.67$

d) $-0.93 + 0.8$ e) $-0.5 - 0.79$ f) $0.2 - 2$

Ex.3 Perform the indicated operations.

a) $3.2-(-1.4)$
b) $-0.8-(-0.04)$
c) $-0.9-0.2-0.4$
d) $7.22+(-0.002)$
e) $0.71-(-5.1)-3.4$
f) $-(-0.8)+(-0.2)$
g) $-2.3-0.5-(-1.1)$
h) $-2.4+0.3+0.2-0.7$
i) $0.3-(+0.55)-(-0.65)$
j) $0.2+2.345-0.6+0.4-2.345$

Ex.4 Perform the indicated operations.

a) 0.03×0.4
b) $0.4\times(-0.01)$
c) $-0.8\times(-0.2)$
d) -0.3×2.01
e) $-3.5\times(-2)$
f) $0.0009\times(-0.2)$
g) $4\times(-0.005)$
h) -79.3×0.001

Ex.5 Perform the indicated operations.

a) $-0.2\times(-0.3)\times(-0.4)$
b) $1.5\times(-0.2)\times 0.03$
c) $(-0.5)\times 1.1\times 0.001$
d) $0.2\times(-0.2)\times(-0.5)$
e) $50.2\times 0.1\times(-0.05)$
f) $-2\times(-0.4)\times(-0.2)\times(-0.001)$
g) $40\times(-0.5)\times(-0.3)\times 2$
h) $-0.1\times 0.2\times(-0.3)\times 0.5\times(-0.3)$
i) $0.02\times(-0.3)\times(-7)\times(-1)\times(-0.02)$
j) $4\times 5\times(-0.1)\times 100\times 0.5$

Ex. 6 If 0.7 of my 20 dresses are long, how many long dresses do I have?

Ex. 7 Tim ate 0.4 of 2 pounds of salad during one day. How many pounds of salad did Tim eat?

Ex.8 Out of 2000 applicants, 0.95 of them obtained a job. How many applicants obtained a job?

Ex.9 What is 3.1 of 20?

Ex.10 Write the following statement using exponential notation and determine the sign of the result. Remember about the proper use of parentheses. *Do not evaluate.*

a) 1.89 raised to the seventh power
b) -3.4504 raised to the eleventh power.
c) -0.6402 raised to the sixteenth power.

Ex.11 Perform the indicated operations.

a) $(0.9)^2$
b) $(-0.4)^3$
c) $-(1.1)^2$
d) $(-0.06)^2$
e) $-(-0.01)^4$
f) $(-0.1)^7$

Ex.12 Perform the indicated operations.

a) $0.8\div(-0.33)$
b) $\dfrac{1.2}{-0.018}$
c) $\dfrac{-49}{0.21}$
d) $-0.5\div(-0.0001)$

e) $\frac{-0.26}{-4}$ f) $2.46 \div 6.8$

g) $0.7 \div (-0.8)$ h) $-0.02 \div 0.3$

Ex.13 Perform the indicated operations. Give your answer in the form of a decimal or an integer.

a) $-0.3 \div 0.6$ b) $-0.03 \div (-0.8)$ c) $\frac{0.01}{-0.025}$

d) $\frac{0.0073}{0.1}$ e) $0.006 \div (-0.8)$ f) $\frac{-0.16}{-4}$

Ex.14 Perform the indicated operations.

a) $-7.4-4.8$ b) $-6.7 \div 100$

c) $-4-(-0.9)$ d) $-(-0.9)^2$

e) $-0.03 \times (-5)$ f) $2 \div (-0.004)$

g) $0.15 \times (-0.04)$ h) $-0.7+0.4-0.5-0.6$

i) $0.3-(-2.4)+(-0.08)$ j) 2.234×100

k) $-0.7 \times (-0.002) \times 10$ l) $0.05 \div (-0.025)$

m) $\frac{-5}{0.002}$ n) $0.7 \times (-0.2) \times (-0.02)$

o) $(-0.01)^4$ p) $-5.7-(-67.2)$

Ex.15 Perform the indicated operations if possible. Otherwise, write "undefined".

a) $0.3-(-0.2+0.7)$ b) $-2+0.7 \times 0.1$

c) $\frac{-4.1+4.2-0.1}{0.2}$ d) $(-0.2-0.9)(0.01-0.2)$

e) $-(2.3-2.5)^3$ f) $-0.05 \div 0.5 \times (-100)$

g) $-(-0.2 \times 0.5)^5$ h) $0.7(-0.05+0.06)$

i) $-4.2-(3.9-4.6)$ j) $-2 \times (0.7)^2$

k) $\frac{9.9}{0.1 \times (-0.2)+0.02}$ l) $\frac{-0.2}{-0.01}-20.1$

m) $0.9 \div 1.8 \times 10$ n) $(-0.2)(-5)+(-0.2)(0.3)$

o) $(-0.7+0.3)(-0.02)$ p) $-0.7+(0.3)(-0.02)$

q) $-(-2.3)+(-0.1)(4.05)$ r) $(-3.4+3.6) \div (-0.002)$

Ex.16 Write the following statement as a single numerical expression and then evaluate them.

a) Add −0.5 and 0.2 and then multiply the result by 10
b) Raise −0.3 to the third power and then divide the result by 0.09.
c) Subtract 3 from −4.1 and then add the result to 7.2.
d) Multiply −0.03 and −0.05 and then subtract the result from 2.
e) Divide 0.4 by 0.02 and multiply the result by −0.01.
f) Add −0.6 to −0.4 and then raise the result to the seventeen power.
g) Find the product of −0.2, −0.06 and 1000.
h) Divide −2 by 0.2 and then subtract the result from −0.3.
i) Add −2.1 and −3.9 and then multiply the result by 0.0001.

Ex.17 Knowing that $0.23 \times 3.1 = 0.173$ evaluate.

a) $0.23 \times (-3.1)$

b) $-0.23 \times 3.1 + 0.173$

c) $0.23 \times 3.1 \div 10$

d) $10 + 0.23 \times 3.1$

e) $10 \times 0.23 \times 3.1$

f) $(0.23 \times 3.1 - 0.073)^3$

Ex.18 Replace x with a number to make the statement true.

a) $0.2^x = 0.04$

b) $0.2 - x = 0.1$

c) $0.3 + x = 1$

d) $-0.2 \div x = 1$

e) $x^3 = 0.001$

f) $x - 0.3 = 2$

g) $2.35 - x = 2$

h) $x \cdot 0.3 = 0.6$

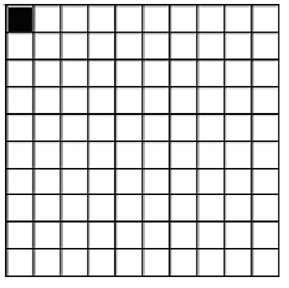
Lesson 12

Topics: Percent; Changing percents to decimals and decimals to percent; Solving percent problems: finding the percent and finding the amount.

Percents are a popular way of presenting numerical information. We will now study percent notation.

Percents

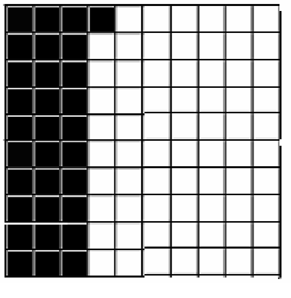
The figure is cut into 100 equal parts and 1 of them is shaded.



The shaded portion is $\frac{1}{100}$ = 0.01 of the whole figure. But one can also say that the shaded portion is 1% , one part out of one hundred parts.

Percent means "parts per one hundredth": $1\% = \frac{1}{100}$

Example 12.1 Using percent notation, write the number representing
a) the portion of the area that is shaded.
b) the portion of the area that is NOT shaded.



Solution:

a) 31 out of 100 parts are shaded, thus the number representing the portion shaded is 31%.
b) 69 out of 100 parts are not shaded, thus the number representing the portion that is not shaded is 69%.

Notice that 100% always represents one unit and thus anything less than 100% represents less than one unit and anything over one unit will always have to be greater than 100%.

If today's gas price is 90% of yesterday's price, it means the price went down. If the today's price is (let us say) 120% of yesterday's, the price went up.

Changing percents to decimals

To change a percent to a decimal, we use the definition of percent, i.e. $1\% = \frac{1}{100}$.

For example,

$$7\% = 7 \times \frac{1}{100} = \frac{7}{100} = 0.07$$

or

$$34.5\% = 34.5 \times \frac{1}{100} = \frac{34.5}{100} = 0.345$$

Notice, that each time we divided the number of percents by 100, and so **to change the percent to a decimal, drop the % symbol and divide it by 100 by moving the decimal point 2 places to the left.**

Example 12.2 Write the following in decimal notation.
a) 16% b) 125% c) 0.05%

Solution:
In each case we move the decimal point two places to the left and drop % symbol.
a) 16% = 16.0% = 0.16
b) 125% = 125.0% = 1.25
c) 0.05% = 000.05% = 0.0005.

Changing decimals to percent

Writing decimals as percents must be done by reversing the operation of changing percents to decimals. So, **to change a decimal to a percent, multiply the decimal by 100 by moving the decimal point 2 places to the right, and then insert a % symbol.**

Example 12.3 Write the following decimals as percents.
a) 0.06 b) 4 c) 0.327

Solution:
In each case we move the decimal point two places to the right and insert the % symbol.
a) 0.06 = 6%

b) 4 = 4.00 = 400%
c) 0.327 =32.7%

Solving percent problems: finding the percent

Recall the following type of questions (you can find them in Lesson 5).

There are 100 senators in the US Senate. If there is one African-American senator appointed to the Senate right now, what ***fraction*** *of all senators is African- American?*

The same question could be phrased in terms of percents.

There are 100 senators in the US Senate. If there is one African-American senator appointed to the Senate right now, what ***percent*** *of all senators is African- American?*

The first question asks us to give the answer as a fraction, while the second one requires us to give it as a percentage. To find what percentage of all senators is African-American we first find the answer in terms of a fraction, and then **change the fraction to percent by first changing it to a decimal and then changing the decimal to percent.**

We find the fraction of senators who are African-American: $\frac{1}{100}$.

We change the fraction to a decimal: $\frac{1}{100} = 0.01$

We change the decimal to percent: $0.01 = 1\%$.

Answer: 1% of all senators is African-American.

Example 12.4 There are 4 oceans on the Earth, and two of them, the Atlantic and Pacific Oceans, border the USA. What percent of all oceans borders the USA?

Solution:

The fraction of oceans that borders the USA is $\frac{2}{4} = \frac{1}{2}$. We change it to a decimal $\frac{1}{2} = \frac{1 \times 5}{2 \times 5} = \frac{5}{10} = 0.5$. We change the decimal to a percent 0.5 = 0.50 =50%.

Answer: 50% of all oceans borders the USA.

Example 12.5 Three out of the 8 biggest countries in the world are English speaking. What percent of the 8 biggest countries are English speaking?

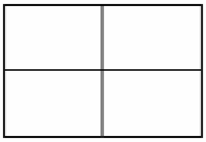
Solution:

We first find the fraction of English speaking countries among the 8 biggest countries. It is $\frac{3}{8}$. As a next step, we change the fraction to a decimal. Since it does not look like it is easy to rewrite this fraction with the denominator equal to a power of 10, we perform long division.

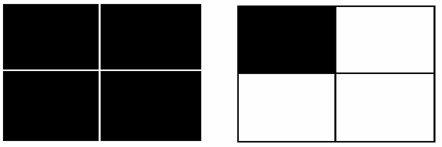
$$\begin{array}{r} 0.375 \\ 8\overline{)3.000} \\ 30 \\ \underline{24} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

So, $\frac{3}{8} = 0.375$. We now change the obtained decimal to a percent by moving the decimal point 2 places to the right, 0.375 = 37.5%.
Answer: 37.5% of the 8 biggest countries in the world are English speaking.

Example 12.6 If this is one unit



What percent represents the shaded area?



Solution:
Let us first express the shaded area as a fraction $\frac{5}{4}$. Then, change $\frac{5}{4}$ to a decimal $\frac{5}{4} = \frac{5\times 25}{4\times 25} = \frac{125}{100} = 1.25$, and finally to a percent 1.25 = 125% (recall, more than one unit always represents more than 100%).
Answer: The shaded area represents 125% of a unit.

Example 12.7 What percent of 25 is 4?

Solution:

The number 25 represents one unit. 4 out of 25 gives us $\frac{4}{25} = \frac{4 \times 4}{25 \times 4} = \frac{16}{100} = 0.16$

Answer: 4 is 16% of 25 (*).

Solving percent problems: finding the amount

*The **fraction** of all current governors in the USA who are women is* $\frac{7}{50}$. *How many women governors are serving right now?*

Recall, that in order to find what is $\frac{7}{50}$ of 50, we multiply the numbers:

$$\frac{7}{50} \times 50 = 7 .$$

Answer: Currently, there are 7 women governors in the USA.

In the same manner, to find a percent of a given number, we perform multiplication.

What is 4% of 6? It is $4\% \times 6$

To perform the operation of multiplication of percent by a number, we must change the percent to a decimal and then multiply the result by the given number.

$4\% \times 6 = 0.04 \times 6 = 0.24$

So if the question about women governors is asked in terms of percents:

*The **percent** of all current governors in the USA who are women is 14%. How many women governors are serving right now?*

To answer the question we need to perform the following operations.

Change 14% to a decimal: $14\% = 0.14$

Multiply the decimal by 50 (the number of all governors): $0.14 \times 50 = 7.0 = 7$

Answer: Currently, there are 7 women governors in the USA.

Example 12.8 22.5% of a class ~~speaks~~ Spanish fluently. If the class consists of 40 students, how many students speak fluent Spanish?

Solution:

(*) The example is not set up in any specific real life situation. If we wish, we can always assign some interpretation to this example. For instance, we can think about it as

- We invited 25 people to a party and 4 is the number that could not attend. Then we could say that 16% of those invited could not come.
- We spent 25 day vacation in Hawaii and 4 is the number of days it was raining. Then we could say 16% of all days were rainy.
- We watched 25 movies last year and 4 is the number of movies we did not like. We could say that we did not like 16% of movies we saw.

Notice that although all real life descriptions are different, from a mathematical point of view there is no difference between them. The mathematical model is the same.

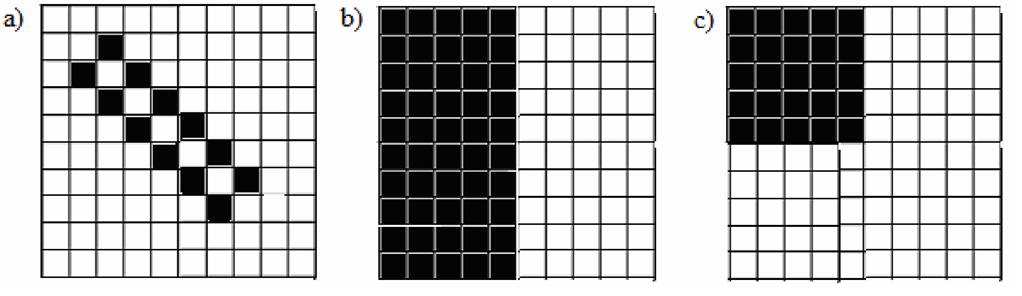
The question is really asking what is 22.5% of 40?
$22.5\% = 0.225$
$0.225 \times 40 = 9$
9 students in the class speak Spanish fluently.

Example 12.9 What is 7% of 30?

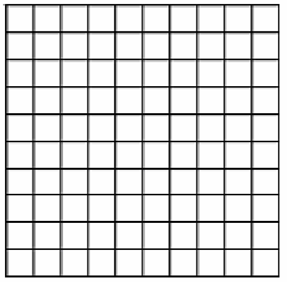
Solution:
$7\% = 0.07$
$0.07 \times 30 = 2.10 = 2.1$
7% of 30 is equal to 2.1 (*).

Exercises with Answers (For answers see Appendix A)

Ex.1 Using the percent notation, write the number representing the portion of the area that is shaded. Assume each figure represents one whole and parts are equal.



Ex.2 This is one unit. Assume that parts are equal.

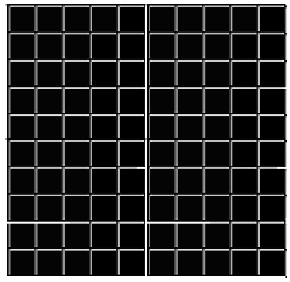


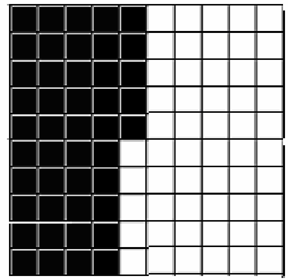
(*) Once again this example is not set up in any specific real life situation but we can always assign one. For instance, we can think about it as

- We buy an item for 30 dollars and in addition we must pay 7% tax. 7% tax on 30 is 2.1 dollars.
- During a 30 kilometer bike ride Maria needed to rest after completing 7% of the ride. This means she rode 2.1 kilometers before she got to rest.

Again, although all real life descriptions are different, from a mathematical point of view there is no difference between them. The mathematical model is the same.

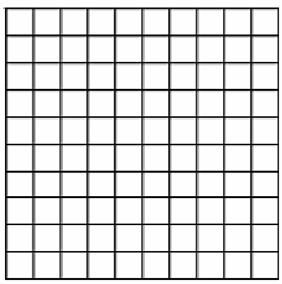
Using the percent notation, write the number representing the portion of the area that is shaded.



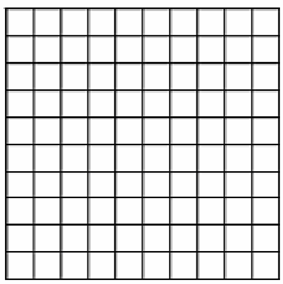


Ex.3 Shade the area corresponding to

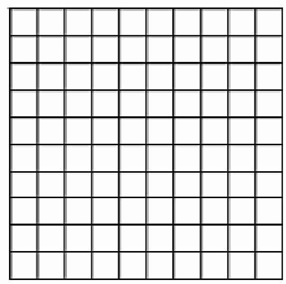
a) 3%



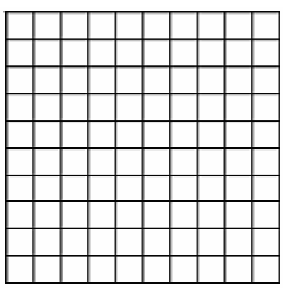
b) 20%



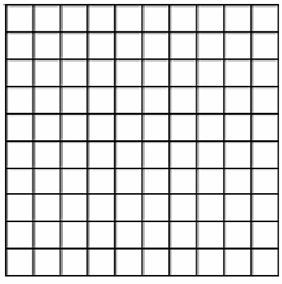
c) 80%

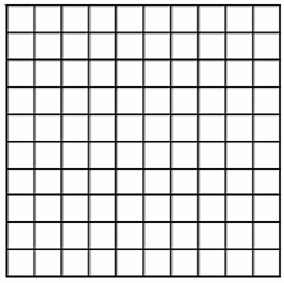


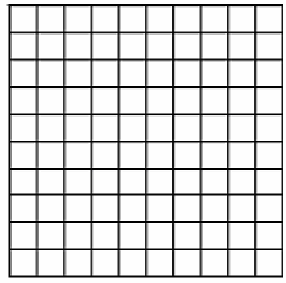
Ex.4 This is one unit.



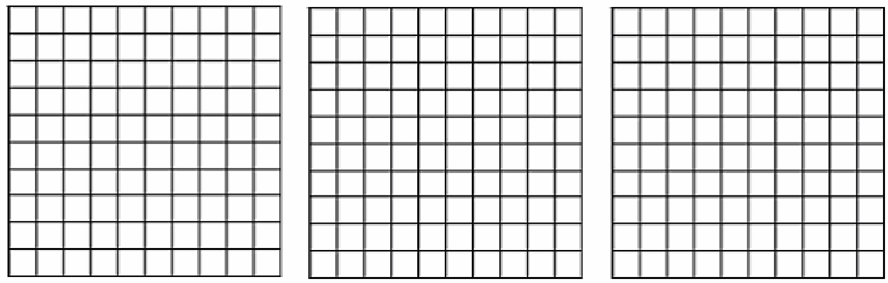
a) Shade the area corresponding to 120%.







b) Shade the area corresponding to 203%



Ex.5 Write the following in decimal notation.

a) 78% b) 0.2%
c) 234% d) 3.78%
e) 0.56% f) 2%

Ex.6 Write the following decimals as percents

a) 0.14 b) 3
c) 2.357 d) 0.02
e) 70 f) 0.5

Ex.7 In a survey of eighth graders, 4 out of 5 students preferred using pens over pencils. What percent of the students preferred using pens?

Ex.8 On a test there were 40 questions. Gail had 26 correct answers. What percent of all answers did Gail have correct? What percent of all answers did Gail have incorrect?

Ex.9 What percent of 500 is 22?

Ex.10 What percent of 1 is 1.33?

Ex.11 Geoff bought a book for $20 on line. He paid $5 for shipping and handling.

a) What percent of his *total bill* was for shipping and handling?
b) What percent of his *total bill* was the cost of the book?
c) What percent of the cost of the book was for shipping and handling?
d) What percent of shipping and handling was the cost of the book?

Ex.12 What percent of 0.8 is 0.02?

Ex.13 The regular price of a text was $70. It was reduced by $10.50. By what percent was it reduced?

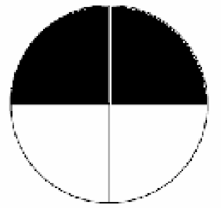
Ex.14 What percent of 25 is 0.5?

Ex.15 Out of the 8 people at the office, 3 have been working there more than 20 years.

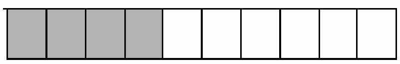
a) What percent of office workers has worked there *no more* than 20 years?
b) What percent of office workers has worked there *more* than 20 years?

Ex.16 What percent of 200 is 3?

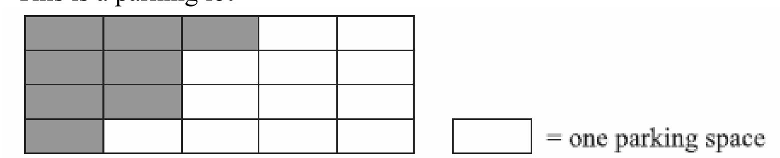
Ex.17 What percent of the area is shaded (assume that parts are equal)?



Ex.18 What percent of the area is shaded (assume that parts are equal)?

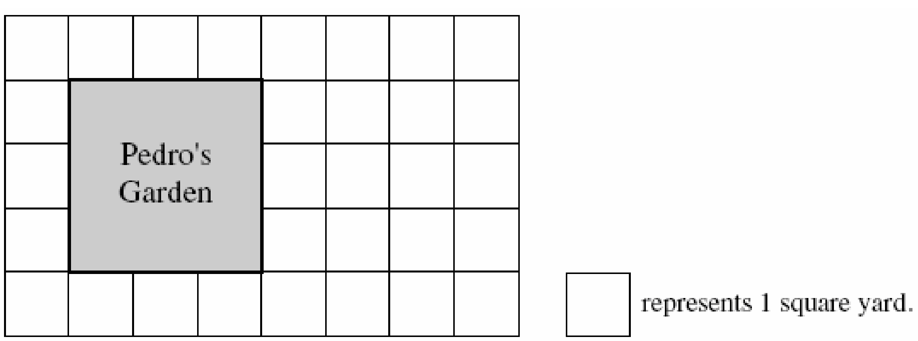


Ex.19 This is a parking lot



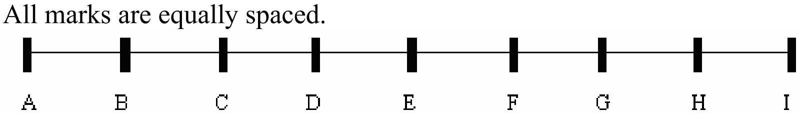
The shaded parts of the diagram represent the spaces that are reserved. What percent of the spaces is reserved?

Ex.20 This is Pedro's land and his garden on his land.



What percent of the land owned by Pedro is his garden?

Ex.21 All marks are equally spaced.



a) What percent of the segment AI is AB?
b) What percent of the segment AI is AE?
c) What percent of the segment AE is AI?

Ex.22 20% of the books on a shelf are about the history of the USA. If there are 85 books on the shelf, how many of them are about the history of the USA?

Ex.23 What is 25% of 160?

Ex.24 What is 0.02% of 124?

Ex.25 Lilla paid $15 for her lunch. She decided to leave a 15% tip. How much did she leave for the tip ?

Ex.26 What is 250% of 50?

Ex.27 David earns $3000 per month. If he spends 22.5% of his monthly earnings on his housing, what are his monthly housing expenses?

Ex.28 What is 3.3% of 20?

Ex.29 Dan invested $500 and he will get 110% of his investment back after a year. How much money will Dan get after one year?

Ex.30 What is 67.82% of 1000?

Ex.31 How does 70% of a *positive* number compare with that number?

a) It is greater than that number.
b) It is less than that number.
c) It is equal to that number.
d) It depends on the number.

Ex.32 How does 135% of a *positive* number compare with that number?

a) It is greater than that number.
b) It is less than that number.
c) It is equal to that number.
d) It depends on the number.

APPENDIX A: ANSWERS TO EXERCISES

Lesson 1

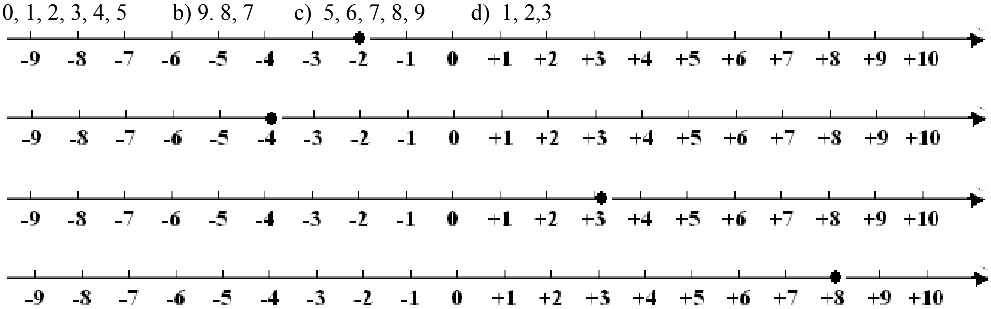
1. 3 and 7 are called factors. Numbers 0, 1, 2, 3, ... are called natural numbers.
   Operation of division by zero is not defined. Zero divided by any number except zero is equal to zero.
   The result of addition is called sum. The result of multiplication is called product.
   In the expression $15 \div 5 = 3$, 5 is called divisor, 3 is called quotient.
   In the sum 4+5, 4 and 5 are called the terms of addition.
2. b) c) d)
3. a) 4×5 b) 4×7 c) $3\times(2-5)$ d) $8\times9+4\times6$ e) $34-3\times4$ f) $(10-2)\times(4\div2)$
4. a) $\frac{456}{33}$ b) $\frac{56}{3}$
5. a) division, quotient, 11 b) subtraction, difference, 11 c) multiplication, product, 48
   d) addition, sum, 124 e) division, quotient, 5 f) multiplication, product, 10
6. a) $3\times8 = 24$ b) $73-3 = 70$ c) $100+2=102$ d) $5\times0=0$ e) $10\div2=5$ or $\frac{10}{2}=5$ f) $15\div3=5$ or $\frac{15}{5}=5$
7. a) cannot be performed b) 0 c) cannot be performed d) 1 e) 0 f) 15
8. a) 1 b) 1 c) 0 d) 0
9. a) The opposite operation to subtraction is addition. $15-9+9=15$
   b) The opposite operation to division is multiplication. $32\div8\times8=32$
   c) The opposite operation to addition is subtraction. $2+5-5=2$
   d) The opposite operation to multiplication is division. $4\times12\div12=4$
10. 40986, because of the commutative property of multiplication.
11. a) 200 b) 330 c) 5000 d) 45000 e) 100000 f) 0
12. Multiplication: a) and d) Division: d)
13. 5 is called the base, 25 is called exponent.
14. a) 5^{14} b) 12^3 c) 10^2 d) 7^{13}
15. a) $12\times12\times12$ b) $251\times251\times251\times251$ c) 7 d) $8\times8\times8\times8\times8$
16. a) 4^6 b) 58^4 c) 3^48^2 d) $7^3(8)(9)$ e) $33(3^4)$ f) $4+4+4+4^2$ g) $(2+3)^3$ h) $(12+8)^2+12+8$
17. a) $9^2=81$ b) 16 c) 100000 d) 64 e) 1000000 f) 100000000 g) 56 h) 1
18. a)
19. a) 6 b) 3 c) 1
20. No. For example, $2^3=8$ and $3^3=9$ (They are not equal)
21. a) division $14\div7\times2=2\times2=4$ b) addition (parentheses) $(8+2)6=(10)6=60$ c) subtraction $10-5+2=5-2=3$
    d) multiplication $14-2\times3=14-6=8$ e) multiplication (parentheses) $(2\times4)^2=8^2=64$
    f) exponentiation $3\times10^4=3\times10000=30000$
22. Many good answers are possible. For example: a) $5\times8+2$ b) $(63\div3)\times2$ c) $15+3^3$ d) $(5+2)(7-1)$
23. a) $(3+7)\times8=80$ $3+7\times8=59$ b) $(2\times50)^3=1000000$ $2\times50^3=250000$
    c) $20\div(5\times2)=2$ $20\div5\times2=8$ d) $14-(2+3)=9$ $14-2+3=15$
24. a) $(3+2)\times1000=5000$ b) $(44\div44)\times16=16$ c) $10-8+5=7$ d) $7+3\times6=25$ e) $18\div(7-1)=3$
    f) $(5\times2)^7=10000000$ g) $2^3\times10=80$ h) $18-(36\div9)=14$ i) $30-2\times7$ j) $1^5+2=3$
25. a) 100 b) 18 c) 12 d) 27 e) 12 f) 2 g) 32 h) 512 i) 7
    j) 0 k) 3 l) 25 m) 30 n) 1400 o) cannot be performed p) 16 q) 97900
    r) 38 s) 300 t) 21 u) 30 v) 23 w) 402 x) 36 y) 9 z) 2
26. yes
27. a) $7^5\neq5\times7$ b) $7^5=7\times7\times7\times7\times7$ c) $7^5\neq5\times5\times5\times5\times5\times5\times5$ d) $4\times3^5\neq(4\times3)^5$
    e) $65213\times678=678\times65213$ f) $9\times34^3=34^3\times9$ g) $12+690\div345=12+(690\div345)$
    h) $6534-356\neq356-6534$ i) $2345\times0=0\times45$ j) $\frac{35}{35}\neq0$
28. a) 4758 b) 50076
29. a) $10\times74008=740080$ b) $42\times42\times42=74008$ c) $74008+1=74009$ d) $42^3\times1000=74008000$
    e) $74008-74009=0$ f) $1\times74008=74009$

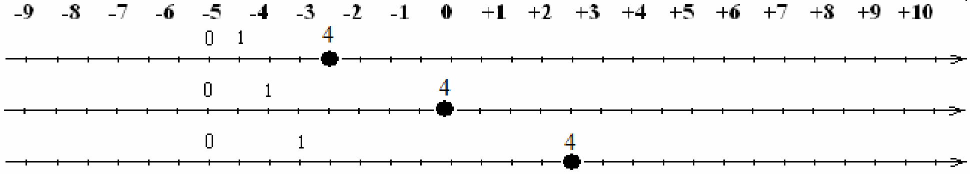
30. a) 34008 b) 340080 c) 34009 d) 0 e) 34008 f)34000

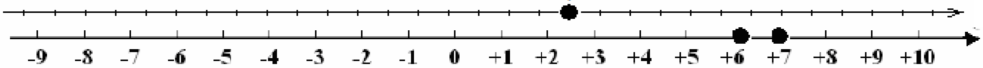
31. a) $348 \times 10 = 3480$ b) $3^3 = 27$ c) $459+781=781+459$ d) $15 \div 3 = 5$ e) $7-2=4+1$ f) $57 \times 100 = 5700$
g) $\frac{135}{135} = 1$ h) $6^2 = 36$ i) $\frac{823}{1} = 823$ j) $24 \div 6 = 2+2$ k) $67 \times 835 = 835 \times 67$ l) $25+5=30$
m) $100 \div 2 = 50$ n) $5 \times 7 = 30+5$ o) $746-6=740$

Lesson 2

1. All positive numbers are to the right of zero on a number line.
All negative numbers are to the left of zero on a number line.
2. a) $3<5$ b) $4 > -2$
3. a) C b) B c) 16 d) 5 e) A, C, D, E, F f) B, D g) −13, −8, −2, 0, 3, 8
4. a) F, b) A c) C d) E e) D f) −14, −12, −5, −1, 0, 6, 8
5. a) $5 < 8$ b) $3 > -4$ c) $0<10$ d) $0 > -1$ e) $-4 > -5$ f) $-2 > -3$ g) $-2 = -2$ h) $-99 > -100$
6. a) −10, −7, −3, 0, 5, 8 b) −7, −5, −2, −1, 2, 3 c) −63, −62, −51, −43, 48, 49
7 Many correct answers are possible. * can be replaced by one of the following:
a) 0, 1, 2, 3, 4, 5 b) 9. 8, 7 c) 5, 6, 7, 8, 9 d) 1, 2,3

8. a) b) c) d)


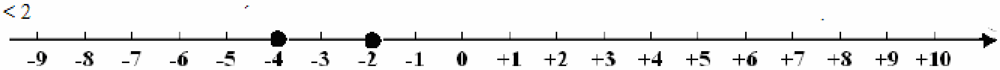
9. a) b) c)


10 a)


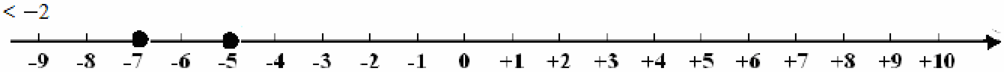
$6 < 7$

b)


$-1 < 2$

c)


$-4 < -2$

d)


$-7 < -5$

11. 4, 3, 2, 1, 0, −1, −2
12. More than one correct answers are possible.
a) 6, 7, 8, ... b) -8,-9,-10, ... c) 2, 3, 4 d) -6,-5,-4, -3, -2,-1, 0
13. a) −4 b) 16 c) 102 d) 0
14. a) b)


15. a) 5−7= −2 b) −2+6=4 c) −4−3=−7
16. a) \$40 − \$60= −\$20 b) −\$25 − \$25 = −\$50
17. a) $-6^0F-8^0F = -14^0F$ b) $-6^0F + 10^0F = 4^0F$

18. 5 inches; `-7 + 12 = 5`

19. a) 2 b) −20 c) −2 d) 20 e) 0 f) 0

20. a) positive; 3 b) negative; −19 c) negative; −3

21. a) −1 b) −5 c) 0 d) 3 e) −3 f) 4 g) −5 h) 1 i) −4
j) −7 k) 1 l) −18 m) −4 n) 0 o) 2 p) 5 q) −3 r) 0

22. a) 5 b) −13 c) 1 d) −12 e) −4 f) 11 g) 3 h) −6 i) 42
j) −8 k) −8 l) −21 m) 24 n) 0 o) −12 p) 15 q) 4 r) −30

23. a) −4 b) −4 c) 14 d) −34 e) 5 f) 10 g) −20 h) −7 i) 0 j) −19 k) −2 l) 0

24. After the series of steps a) – d) you get back the original number, you started with. You get the same number after repeating a) – d) again 3 or 5 times.

25. a) −5+3=−2 b) 1−10=−9 c) −2−2=−4 d) 31−31=0 e) −2+5=3 f) −5−2=−7
g) −5−3=−8 h) 7−2=2+3 i) −1−1=−3+1

Lesson 3

1. Several correct answers are possible. For example:
a) 7+(−3)+4=7+(−3)+4=7+4−3=−3+7+4 b) −2+(−5)+12 = −5+12−2 =12−5−2

2. 1−3−8+2 −8−3+2+1 2+3−8+1 2+1−3−8 −3+1−2+8

3. a) 0 b) −27 c) −7 d) 0 e) −2 f) −15 g) −10 h) −20 i) −4 j) −5 k) 7 l) 0

4. a) 0 b) −15 c) 10 d) −10 e) 12 f) 5 g) −12 h) −15 i) −5 j) 6 k) −70 l) 13

5. a) 5 b) 22 c) 20 d) −5 e) −13 f) −6 g) 4 h) − i) 11 j) 6 k) −8 l) −2

6. a) $100\times(-2)=-200$ b) (−3(−9)=27 c) (−20)6=−120 d) (−100)(−10)=1000

7. a) −6 b) 4200 c) −8 d) −63 e) 18 f) 48 g) −77000 h) 0 i) 40 j) −36 k) 56 l) −63 m) 36 n) −345

8. a) $14\div(-2)$ or $\frac{14}{-2}=-7$ b) $-25\div(-5)$ or $\frac{-25}{-5}=5$ c) $-36\div 6$ or $\frac{-36}{6}=-6$ d) $-40\div 8$ or $\frac{-40}{8}=-5$

9. a) −7 b) 2 c) −9 d) 8 e) −6 f) 10 g) 0 h) not possible i) −3 j) 7 k) −4 l) −9

10. a) 8(5)(−10)= −400 b) (−1) (−100)(−3)= −300 c) (−7)(3)(−2)=42

11. a) 54 b) 27 c) 27 d) −3400 e) −48 f) −54000 g) −60 h) 0 i) 1 j) −48000 k) −32 l) −4

12. a) negative b) negative c) positive d) negative

13. a) 48 b) 40 c) 1 d) 40000 e) −7 f) −42 g) 6 h) not possible i) −6 j) 14

14. Several correct answers are possible. a) 8(−7)(−9)=(−9)(8)(−7) b) (14)(−8)(−3)(−2)

15. a) (−1)(−1)(4)=4 b) (−1)(−1)(−1)(13)= −13 c) (−1)(−1)(7)=7 d) (−1)(−1)(−1)(−1)(5)=5

16. a) −26 b) 45

17. a) −(−9) and −(−(−9)) , 0 and 0 , −(−12) and −12

18. a) $-5-7\neq 5+7$ b) 12−7=−7+12 c) $5\times2\times4=2\times4\times5$ d) $-(-5)\neq-(-(-5))$
e) (−3)(−7)(8) = −3(7)(8) f) −7+3−2=3−2−7 g) 2−(−4)+(−5)=2+4+5

19. a) −479−832=−479−832 b) 439−512=−512+439 c) 5−6−1=−6−1+5 d) 2−8+3=3+2−8 e) $-10\times56=-560$
f) $543\times(-784)=-784\times543$ g) $-20\div5=-4$ h) $-4\times7=-28$ i) −7(−2)=10+4 j) $-1-1-1=-1\times3$

20. a) 15570 b) −15570 c) 155700 d) − 15570

Lesson 4

1. a) $(-5)^6$ b) 8^7 c) $(-4)^2$ d) $(-7)^8$

2. a) 32 b) −71

3. a) -8^4 b) $(-8)^3$ c) $(-8)^3-8-8$ d) -8^3-8^4 e) 5^27^4 f) $-(-8)^38^4$ g) 2^3-8^3 h) 3^22^3

4. a) $-11\times11\times11$ b) (−11) (−11) (−11) (−11)

5. a) 1000000 b) −16 c) −123 d) 1 e) −64 f) −1 g) −49 h)27

6. a) positive b) negative c) negative d) negative e) positive f) negative

7. a) multiplication; −19 b) subtraction (parentheses); 5 c) multiplication (parentheses); 36 d) exponentiation;−18
e) addition (parentheses); −3 f) subtraction; 7 g) division; −6 h) subtraction; −12

8. a) 15 b) 1 c) −32 d) cannot be evaluated e) 64 f) 4 g) 49 h) −2 i)2 j) 3 k) 6 l)12 m) 2 n) −10
o) −1 p) 0 q) 0 r) −72 s) 8 t) cannot be evaluated u) −42 v) −1 w) −1000 x) 3 y) −1 z) 1

9. a) (−3)(−2)(4)=24 b) 2−7+5=0 c) $(-4)\div2-9=-11$ d) 5+7(−3)= −16 e) $(4-6)^3=-8$ f) $(-5+8)\div(-3)=-1$
g) $(-3\times2)^2=36$ h) $30-(-3)^3=57$ i) −11−(5+7)= −23 j) $1-(-1)^{22}=0$ k) $(-5-2)\times1000=-7000$

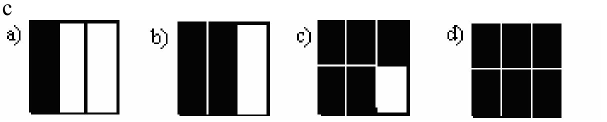
10. a) $(-2)^3 = -8$ b) $1^{27} = 1$ c) $-4 \times 2+567 = -8+567$ d) $67(-8+3) = 67(-5)$

11. a) $-7+(-2)^4 = -7+16$ b) $(-5+10-2)^7 = 3^7$ c) $x^{14} = -1$ d) $-341+(-25) \div 5 = -341+(-5)$

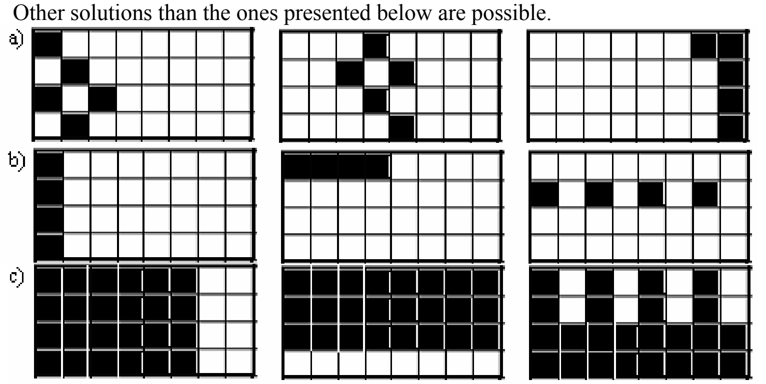
12. a) -2240 b) 22400 c) 0

Lesson 5

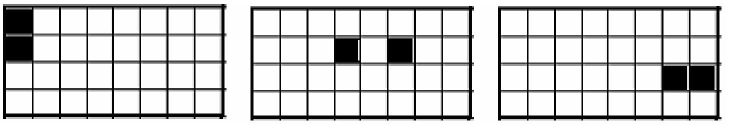
1. Numerator: 2 ; denominator: 3
2. $\frac{29}{354}$
3. The top number of a fraction is called numerator, the bottom number is called denominator. The denominator names the number of equal parts the unit has been divided and the numerator names how many of those parts we take.
4. a) $\frac{2}{4}$ b) $\frac{1}{4}$ c) $\frac{3}{8}$ d) $\frac{5}{8}$
5. a) $\frac{2}{4}$ b) $\frac{2}{4}$ c) $\frac{1}{2}$ d) $\frac{2}{3}$ e) $\frac{1}{2}$
6. C
7. B
8. a) $\frac{64}{100}$ b) $\frac{80}{100}$ c) $\frac{95}{100}$
9. To find $\frac{1}{8}$ one has to divide it into 8 equal parts and take 1 of them.

   To find $\frac{2}{8}$ one has to divide it into 8 equal parts and take 2 of them.
10. $\frac{6}{8}$ numerator:6; denominator: 8
11. $\frac{5}{9}$
12. a) $\frac{4}{14}$ b) $\frac{2}{14}$ c) $\frac{10}{14}$
13. c
14. 

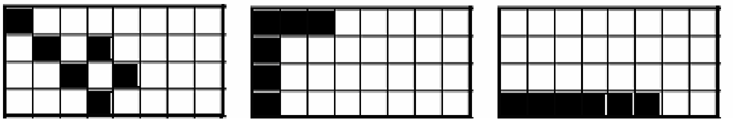
15. a) 4 b) 1 c) 3 d) 2
16. Other solutions than the ones presented below are possible.



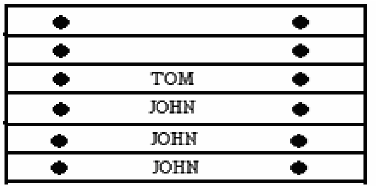
d)



e)



17. c
18. 5 M&M's
19.



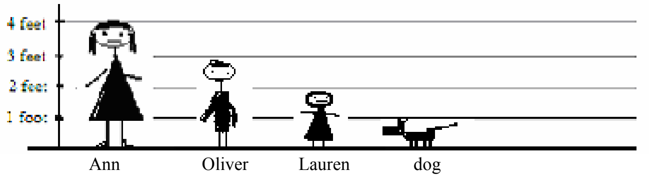
$\frac{2}{6}$ (or $\frac{1}{3}$) of all drawers is not used.

20.

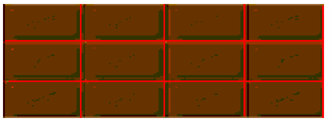
Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
		1 math	math	math	math	math
math	math	math	math	biology	biology	biology
biology	biology	biology	biology	French	French	French
French	French	French	chemistry	chemistry	chemistry	chemistry
			30			

There are 4 days left; $\frac{4}{30}$ (or $\frac{2}{15}$) of all days in October Austin does not have to study.

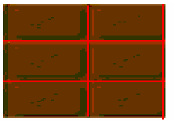
21. a) $\frac{1}{3}$ b) $\frac{2}{3}$ c) $\frac{1}{2}$ d) taller



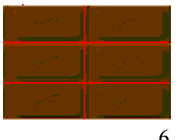
22. a)



part for the teacher

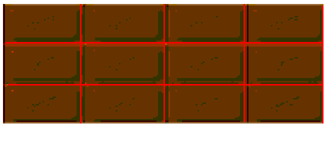


part for the friend

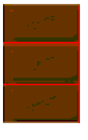


part that is left: You get $\frac{6}{24}$ (or $\frac{1}{4}$) of a chocolate bar.

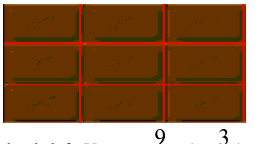
b)



part for the teacher



part for the friend



part that is left: You get $\frac{9}{24}$ (or $\frac{3}{8}$) of the chocolate bar.

23. $\frac{1}{8}$

24. $\frac{1}{30}$

25. $\frac{80}{100}$

26. $\frac{4}{9}$

27. $\frac{3}{15}$

28. 3

29. a) 8 b) 4

30. 30 minutes

31. 5 miles

32. 75 pages

33. 20 miles

34. 2 halves 3 thirds

35. $67 \div 89$

36. $\frac{4}{5}$

37. a) $\frac{1}{2}$ b) $\frac{1}{3}$ c) $\frac{19}{782}$

38. a) $\frac{3}{14}$ b) $\frac{4}{17}$ c) $-\frac{7}{26}$ d) $-\frac{5}{7}$

39. $\frac{3}{1}$ $\frac{1}{1}$ $-\frac{23}{1}$ $\frac{7}{1}$ $\frac{0}{1}$

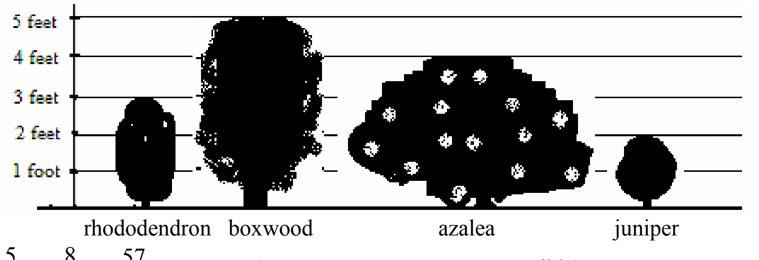
40. $\frac{103}{103}$, $\frac{4}{8}$, $\frac{1235}{1235}$, $-\frac{30}{15}$, $-\frac{60}{12}$, $\frac{3}{5}$, $\frac{15}{12}$

41. $\frac{9}{6}$

42. $\frac{7}{2}$

43. a) $\frac{6}{5}$ b) $\frac{7}{5}$ c) $\frac{8}{5}$

44. a) $\frac{2}{3}$ b) $\frac{3}{5}$ c) $\frac{3}{4}$ d) taller



45. $\frac{5}{2}$ $\frac{8}{11}$ $\frac{57}{32}$ (many correct answers are possible)

46. a) $\frac{7}{5} > 1$ b) $\frac{86}{85} > 1$ c) $\frac{4}{5} < 1$ d) $\frac{13}{5} > 1$ e) $\frac{14}{14} = 1$ f) $\frac{4}{89} < 1$

47. a) 9, 10 … b) 1, 2 … 6 c) 1, 2 … 122 d) 123 a) and b) and c) have many correct answers

48. proper fractions: $\frac{7}{17}$, $\frac{19}{51}$ improper fractions: $\frac{8}{3}, \frac{3}{3}, \frac{8}{8}, \frac{34}{34}$

49. a) $\frac{-7}{-4} = \frac{7}{4}$ b) $\frac{-1}{2} = -\frac{1}{2}$ c) $\frac{5}{-6} \neq \frac{-5}{-6}$ d) $\frac{-3}{8} = \frac{3}{-8}$ e) $-\frac{4}{9} \neq \frac{-4}{-9}$ f) $\frac{3}{5} = -\left(\frac{-3}{5}\right)$

50. $-\frac{35}{478} = \frac{-35}{478} = \frac{35}{-478}$

51. $\frac{9}{-11} = \frac{-9}{11} = -\frac{9}{11}$

52. D

53. $\frac{4}{6}, \frac{2}{3}$

54. a) $\frac{3 \times 5}{7 \times 5} = \frac{15}{35}$ b) $\frac{50 \div 10}{60 \div 10} = \frac{5}{6}$ c) $\frac{28 \div 4}{36 \div 4} = \frac{7}{9}$ d) $\frac{3 \times 6}{2 \times 6} = \frac{18}{12}$

55. $\frac{8}{14}, \frac{12}{21}, \frac{20}{35}$ (answers vary)

56. Yes. $\frac{3 \times 2}{4 \times 2} = \frac{6}{8}$ $\frac{9}{12}, \frac{15}{20}, \frac{30}{40}$

57. $\frac{30}{50}, \frac{24}{40}, \frac{-6}{-10}$

58. $\frac{20}{32}$

59. $-\frac{12}{16}$

60. $\frac{24}{64}$

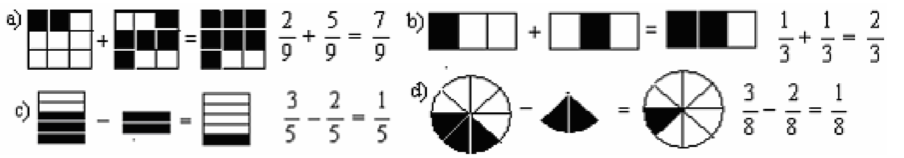
61. a) $\frac{3}{16} = \frac{6}{32}$ b) $\frac{5}{9} = \frac{500}{900}$ c) $5 = \frac{30}{6}$ d) $\frac{5}{9} = \frac{15}{27}$ e) $\frac{3}{9} = \frac{30}{90}$ f) $-7 = -\frac{56}{8}$

62. a) $\frac{1}{4}$ b) $\frac{1}{20}$ c) $\frac{3}{4}$ d) $\frac{3}{10}$ e) $-\frac{2}{7}$ f) 3 g) $\frac{2}{3}$ h) $-\frac{10}{3}$

63. a) $\frac{1}{2}$ f) -8

Lesson 6

1. $\frac{3}{7} + \frac{4}{9} = \frac{4}{9} + \frac{3}{7}$ Subtraction is not commutative.

2.


3. a) $\frac{8}{5}$ b) $\frac{1}{3}$ c) $\frac{4}{5}$ d) $\frac{23}{6}$ e) 1 f) $\frac{11}{2}$ g) $\frac{3}{5}$ h) 1 i) $\frac{1}{2}$

4. a) $\frac{29}{30}$ b) $\frac{43}{72}$ c) $\frac{13}{30}$ d) $\frac{23}{20}$ e) $\frac{17}{12}$ f) $\frac{55}{36}$ g) $\frac{41}{24}$ h) $\frac{15}{14}$

5. a) $\frac{-1+5}{6}=\frac{4}{6}=\frac{2}{3}$ b) $\frac{2+(-3)}{8}=\frac{-1}{8}=-\frac{1}{8}$ c) $-\frac{5}{25}=-\frac{1}{5}$ d) 0 e) $-\frac{4}{7}$

f) -1 g) -2 h) $-\frac{1}{14}$ i) $\frac{7}{9}$ j) $\frac{1}{2}$

6. a) $-\frac{1}{2}$ b) $-\frac{31}{33}$ c) $-\frac{2}{15}$ d) $-\frac{41}{63}$ e) $-\frac{5}{3}$ f) $-\frac{9}{20}$ g) $\frac{1}{3}$ h) $-\frac{25}{8}$

7. a) $-\frac{11}{12}$ b) $\frac{45}{56}$ c) $-\frac{11}{36}$ d) $\frac{1}{24}$ e) $-\frac{53}{63}$ f) $\frac{31}{49}$ g) $\frac{161}{50}$ h) $-\frac{9}{20}$ i) $-\frac{5}{18}$ j) $\frac{1}{6}$

8. a) $-\frac{37}{20}$ b) $\frac{5}{21}$ c) $-\frac{31}{36}$ d) $\frac{11}{24}$ e) $-\frac{33}{7}$ f) -1 g) $\frac{22}{63}$ h) $-\frac{67}{36}$ i) $\frac{5}{6}$ j) $\frac{1}{28}$

k) $-\frac{10}{7}$ l) $\frac{1}{22}$ m) $-\frac{29}{45}$ n) $\frac{5}{18}$ o) 0 p) $\frac{29}{24}$ q) $-\frac{3}{10}$ r) $-\frac{1}{8}$ s) $-\frac{12}{5}$ t) 0

9. a) $\frac{2+1}{8}=\frac{3}{8}$ b) $\frac{3}{13}+\frac{6}{13}=\frac{9}{13}$ c) $\frac{3}{13}+\frac{6}{13}=\frac{9}{13}$ d) $\frac{4}{5}-\frac{1}{5}=\frac{3}{5}$ e) $-\frac{2}{9}-\frac{2}{9}=-\frac{4}{9}$

f) $-\frac{7}{11}+\frac{7}{11}=0$ g) $-\frac{2}{7}-\frac{3}{7}=-\frac{5}{7}$ h) $\frac{1}{5}+\frac{3}{10}=\frac{2}{10}+\frac{3}{10}$ i) $\frac{2}{3}-\frac{1}{5}=\frac{10}{15}-\frac{3}{15}$

j) $\frac{1}{3}+\frac{2}{3}=1$ k) $\frac{3}{5}+\frac{2}{5}=1$ l) $\frac{11}{8}-\frac{3}{8}=1$

Lesson 7

1. $\frac{3}{7}\times\frac{4}{5}=\frac{4}{5}\times\frac{3}{7}$ Division of rational numbers is NOT commutative.

2. a) $4\times\frac{1}{5}=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{4}{5}$ b) $\frac{3}{13}\times 5=\frac{3}{13}+\frac{3}{13}+\frac{3}{13}+\frac{3}{13}+\frac{3}{13}=\frac{15}{13}$

c) $2\times\frac{23}{107}=\frac{23}{107}+\frac{23}{107}=\frac{46}{107}$

3. a) $\frac{1}{5}x\frac{2}{3}=\frac{1\times 2}{5\times 3}=\frac{2}{15}$ b) $-\frac{9}{7}\times\frac{8}{5}=-\frac{9\times 8}{7\times 5}=-\frac{72}{35}$ c) $\frac{7}{3}x\left(-\frac{2}{3}\right)=-\frac{7\times 2}{3\times 3}=-\frac{14}{9}$

d) $\left(-\frac{5}{4}\right)\left(\frac{9}{2}\right)=-\frac{5\times 9}{4\times 2}=-\frac{45}{8}$ e) $\left(-\frac{19}{3}\right)\left(-\frac{1}{8}\right)=\frac{19\times 1}{3\times 8}=\frac{19}{24}$ f) $\frac{8}{7}x\frac{10}{9}=\frac{8\times 10}{7\times 9}=\frac{80}{63}$

4. a) $\frac{12}{7}$ b) $-\frac{4}{3}$ c) 4 d) $-\frac{3}{4}$ e) $-\frac{2}{3}$ f) 19

5. a) $\frac{2}{3}$ b) $\frac{21}{5}$ c) $\frac{2300}{27}$ d) -4 e) 16 f) $-\frac{8}{5}$

6. a) $\frac{4}{21}$ b) $-\frac{4}{21}$ c) $-\frac{1}{16}$ d) $\frac{1}{14}$ e) $-\frac{3}{4}$ f) -8

7. a) $\frac{8}{13}$ b) $\frac{12}{55}$ c) $\frac{50}{21}$ d) -8 e) $-\frac{4}{5}$ f) $-\frac{8}{45}$ g) -1 h) $-\frac{3}{16}$ i) $-\frac{2}{3}$ j) $-\frac{1}{62}$

8. Austin ate $\frac{1}{16}$ of the pizza.

9. Noah ate $\frac{1}{20}$ of the chocolate bar.

10. There are 15 blue marbles in the jar.

11. Jon studied biology $\frac{1}{6}$ of an hour.

12. $\frac{35}{6}$

13. a) $\frac{1}{9}\times\frac{1}{9}\times\frac{1}{9}\times\frac{1}{9}$ b) $\left(-\frac{9}{8}\right)\left(-\frac{9}{8}\right)\left(-\frac{9}{8}\right)$ c) $-\left(\frac{9}{8}\right)\left(\frac{9}{8}\right)\left(\frac{9}{8}\right)$ d) $\frac{9\times9\times9}{8}$

14. a) $\left(\frac{2}{9}\right)^7$ b) $-\left(\frac{5}{12}\right)^3-\left(\frac{3}{4}\right)^2$ c) $\left(-\frac{7}{8}\right)^2\left(\frac{3}{5}\right)^3$ d) $-\frac{5^2}{6^3}$

15. a) $\frac{81}{64}$ b) $-\frac{1}{16}$ c) $\frac{64}{9}$ d) $-\frac{25}{6}$ e) $\frac{1}{16}$ f) $\frac{8}{27}$

16. a) $\frac{7}{2}$ b) $-\frac{5}{9}$ c) $\frac{1}{4}$ d) $-\frac{15}{14}$ e) $-\frac{1}{6}$ f) -3

17. a) $\frac{6}{25}$ b) $-\frac{9}{4}$ c) $-\frac{5}{2}$ d) 2 e) $-\frac{9}{4}$ f) $\frac{45}{26}$ g) $-\frac{32}{3}$ h) $\frac{28}{5}$

18. a) $\frac{6}{5}$ b) -49 c) $-\frac{1}{3}$ d) $-\frac{1}{16}$ e) $\frac{8}{19}$ f) $-\frac{1}{7}$ g) $-\frac{2}{9}$ h) 50

19. a) 10 b) $-\frac{3}{2}$ c) $\frac{4}{3}$ d) -6 e) $-\frac{1}{45}$ f) $-\frac{2}{55}$ g) $\frac{39}{4}$ h) $-\frac{16}{15}$

20. a) 3 b) $\frac{16}{3}$ c) $\frac{1}{27}$ d) -16 e) $\frac{35}{3}$ f) $\frac{49}{81}$ g) $-\frac{8}{11}$ h) $\frac{40}{3}$ i) $\frac{16}{7}$

j) $\frac{18}{35}$ k) -4 l) $\frac{27}{140}$ m) $-\frac{2}{21}$ n) $-\frac{1}{1000000}$ o) $\frac{10}{3}$ p) 0 q) $-\frac{9}{8}$ r) -6

s) $\frac{1}{12}$ t) $\frac{27}{64}$ u) not possible v) $-\frac{5}{2}$ w) $\frac{1}{2}$ x) -1 y) -20 z) $-\frac{5}{36}$

21. a) $\frac{2}{7}\times(-14)=-4$ b) $\left(\frac{9}{10}\right)^2=\frac{81}{100}$ c) $\left(-\frac{3}{5}\right)\left(-\frac{10}{7}\right)=\frac{6}{7}$ d) $-3\div\left(\frac{7}{8}\right)=-\frac{24}{7}$

e) $\dfrac{-\frac{2}{9}}{-\frac{1}{27}}=6$ f) $\dfrac{-\frac{3}{11}}{4}=-\frac{3}{44}$ g) $\frac{1}{7}\times(-2)\times\left(-\frac{21}{8}\right)=\frac{3}{4}$ h) $\dfrac{-\frac{3}{16}}{\frac{9}{2}}=-\frac{1}{24}$

i) $\left(-\frac{2}{5}\right)^3=-\frac{8}{125}$ j) $\left(-\frac{33}{8}\right)\left(-\frac{12}{21}\right)\left(-\frac{28}{33}\right)=-2$

22. a) $\frac{1}{2}\cdot3=\frac{3}{2}$ b) $\frac{3}{5}\div\frac{7}{8}=\frac{3}{5}\cdot\frac{8}{7}$ c) $\frac{8}{3}\times\frac{2}{7}=\frac{16}{21}$ d) $\left(\frac{2}{3}\right)^2=\frac{4}{9}$ e) $-\frac{2}{5}\cdot\frac{4}{5}=-\frac{8}{25}$

f) $\frac{7}{123}\cdot\frac{7}{123}=\left(\frac{7}{123}\right)^2$ g) $\frac{5}{6}\div1=\frac{5}{6}$ h) $\frac{5}{6}\div\frac{5}{6}=1$ i) $4\div\frac{1}{2}=8$

j) $3\div\frac{1}{2}=6$ k) $\dfrac{3}{\frac{5}{7}}=3\div\frac{5}{7}$ l) $\dfrac{3}{\frac{5}{7}}=3\cdot\frac{7}{5}$

Lesson 8

1 a) $\frac{1}{3}$ b) $-\frac{1}{9}$ c) $-\frac{1}{2}$ d) $-\frac{7}{30}$ e) $\frac{25}{64}$ f) $\frac{237}{50}$

2. a) subtraction (parentheses); -6 b) multiplication; $-\frac{58}{15}$ c) division; $\frac{10}{21}$ d) division; $-\frac{5}{9}$

e) multiplication(parentheses); $\frac{16}{25}$ f) multiplication; $-\frac{13}{20}$ g) not possible h) multiplication; $-\frac{24}{25}$

i) addition(parentheses); $-\frac{1}{5}$ j) subtraction; $-\frac{49}{6}$ k) exponentiation ; $\frac{1}{2}$ l) subtraction; $-\frac{1}{5}$

m) addition; $-\frac{3}{20}$ n) exponentiation; $-\frac{1}{36}$ o) subtraction; $\frac{16}{7}$ p) exponentiation; $-\frac{5}{8}$

q) addition; -1 r) exponentiation; $\frac{23}{48}$ s) multiplication; $\frac{37}{40}$ t) division; $-\frac{4}{21}$

u) exponentiation; $-\frac{1}{44}$ v) addition (parentheses); $\frac{1}{4}$

3. a) $\left(\frac{4}{9}+\frac{5}{9}\right)\left(-\frac{4}{13}\right)=-\frac{4}{13}$ b) $-8\div\frac{3}{4}\times\left(-\frac{1}{16}\right)=\frac{2}{3}$ c) $\frac{4}{21}+\frac{5}{7}\times\left(-\frac{2}{5}\right)=-\frac{2}{21}$ d) $2\div\left(\frac{4}{9}-\frac{3}{5}\right)=-45$

e) $\left[\left(-\frac{12}{13}\right)\left(\frac{13}{12}\right)\right]^7=-1$ f) $\left(-\frac{1}{3}\right)^3\times 10=-\frac{10}{27}$ g) $-\frac{3}{20}-\frac{5}{6}\div\left(-\frac{25}{12}\right)=\frac{1}{4}$

4. a) $-\frac{2}{7}\times\frac{14}{5}+\frac{2}{3}=-\frac{4}{5}+\frac{2}{3}$ b) $\left(\frac{2}{7}-\frac{1}{14}\right)^5=\left(\frac{3}{14}\right)^5$ c) $-3\div\frac{1}{5}\cdot\frac{2}{5}=-15\cdot\frac{2}{5}$

d) $\frac{3}{7}-\left(\frac{1}{9}\right)^2=\frac{3}{7}-\frac{1}{81}$ e) $\left(\frac{1}{4}-\frac{7}{8}\right)\div 2=-\frac{5}{8}\div 2$ f) $-\frac{1}{2}+\frac{5}{16}-\frac{11}{2}=-\frac{1}{2}-\frac{11}{2}+\frac{5}{16}$

5. a) $\frac{3}{11}<\frac{8}{11}$ b) $\frac{3}{2}>\frac{1}{5}$ c) $\frac{8}{9}<\frac{8}{3}$ d) $\frac{7}{5}>\frac{2}{5}$ e) $\frac{3}{4}>\frac{1}{4}$ f) $\frac{5}{2}>\frac{5}{11}$

6. $\frac{9}{5},\frac{5}{6}$

7. a) $\frac{2}{5}>\frac{1}{3}$ b) $\frac{11}{8}>\frac{9}{7}$ c) $\frac{3}{7}<\frac{10}{21}$ d) $\frac{2}{9}>\frac{1}{6}$ e) $\frac{3}{10}>\frac{1}{5}$ f) $\frac{5}{4}>\frac{8}{7}$

8. a) $-\frac{2}{9}>-\frac{1}{3}$ b) $-\frac{7}{9}<-\frac{3}{4}$ c) $-\frac{3}{8}>-\frac{2}{5}$ d) $\frac{-1}{4}>-\frac{2}{5}$ e) $\frac{3}{-8}<-\frac{1}{6}$ f) $-\frac{3}{10}<-\frac{29}{100}$

9. a) $-\frac{5}{4}<-\frac{9}{8}$ b) $\frac{2}{11}>\frac{3}{22}$ c) $-\frac{2}{8}>\frac{-2}{5}$ d) $\frac{-9}{7}<-\frac{2}{5}$ e) $-\frac{5}{123}>-\frac{123}{5}$

f) $\frac{8}{6}>\frac{10}{9}$ g) $-\frac{13}{4}<-\frac{13}{7}$ h) $-\frac{1}{2}<0$

10. a) Suzanne

11. ginger

12. $-\frac{5}{6},\quad -\frac{4}{5},\quad \frac{4}{5},\quad \frac{14}{15},\quad 1,\quad \frac{8}{7},\quad \frac{9}{7}$

13. a) $\frac{3}{4}<\frac{5}{4}$ b) $\frac{7}{6}<\frac{7}{4}$ c) $\frac{7}{3}>\frac{7}{8}$ d) $-\frac{5}{12}>-\frac{7}{12}$

$x = 4,3,2...$ $\quad$ $x = 5,4,3...$ $\quad$ $x = 6,5,4...$ $\quad$ $x = 6,7,8...$

Lesson 9

1. a) $1,\ 1\frac{1}{2},\ 2,\ 2\frac{1}{2},\ 3,\ 3\frac{1}{2},\ 4,\ 4\frac{1}{2},\ 5,\ 5\frac{1}{2},\ 6$

   b) $1,\ 1\frac{1}{5},\ 1\frac{2}{5},\ 1\frac{3}{5},\ 1\frac{4}{5},\ 2,\ 2\frac{1}{5},\ 2\frac{2}{5},\ 2\frac{3}{5},\ 2\frac{4}{5},\ 3,\ 3\frac{1}{5},\ 3\frac{2}{5},\ 3\frac{3}{5},\ 3\frac{4}{5},\ 4$

2. $4\frac{1}{4}$

3. $2\frac{3}{7}$

4. a) $2\frac{3}{4}$ $\quad$ b) $1\frac{3}{4}$ $\quad$ c) $1\frac{1}{4}$

5. a) $5\frac{1}{2}$ $\quad$ b) $1\frac{1}{2}$ $\quad$ c) $4\frac{1}{2}$

6. $7\frac{1}{5}$ represents 7 units and $\frac{1}{5}$ of the next unit or $\frac{36}{5}$;

   $7\left(\frac{1}{5}\right)$ represents 7 times $\frac{1}{5}$, or $\frac{7}{5}$, which is equivalent to $1\frac{2}{5}$.

7. a) $\frac{23}{6}$ $\quad$ b) $-\frac{302}{3}$ $\quad$ c) $\frac{22}{5}$ $\quad$ d) $-\frac{207}{100}$ $\quad$ e) $-\frac{23}{3}$ $\quad$ f) $\frac{354}{345}$

8. a) $2\frac{3}{8} = \frac{19}{8}$ $\quad$ b) $8\frac{5}{9} = \frac{77}{9}$ $\quad$ c) $-2\frac{7}{10} = -\frac{27}{10}$

9. a) $6\frac{4}{7}$ $\quad$ b) $-10\frac{6}{10} = -10\frac{3}{5}$ $\quad$ c) $-5\frac{2}{5}$ $\quad$ d) $3\frac{6}{9} = 3\frac{2}{3}$ $\quad$ e) $3\frac{1}{6}$ $\quad$ f) $-5\frac{3}{4}$

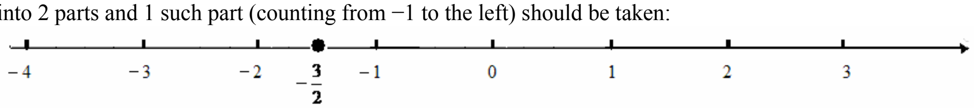
10. $\frac{5}{9} \times 6 = \frac{30}{9} = \frac{10}{3} = 3\frac{1}{3}$

11. $\frac{26}{6} = 4\frac{2}{6} = 4\frac{1}{3}$

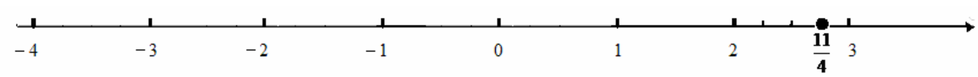
12. $\frac{9}{2} = 4\frac{1}{2}$

13. a) 0 and 1 $\quad$ b) -3 and -2 $\quad$ c) -8 and -7 $\quad$ d) -8 and -7 $\quad$ e) 2 and 3 $\quad$ f) -1 and 0.

14. a) The point representing $-\frac{3}{2}$ is between -2 and -1. The interval between -2 and -1 must be divided equally into 2 parts and 1 such part (counting from -1 to the left) should be taken:

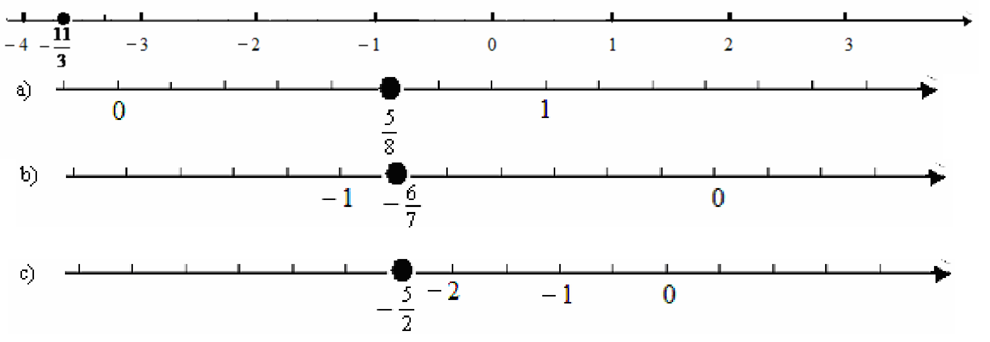


b) The point representing $\frac{11}{4}$ is between 2 and 3. The interval between 2 and 3 must be divided equally into 4 parts and 3 such parts (counting from 2 to the right) should be taken:



c) The point representing $-\frac{11}{3}$ is between -4 and -3. The interval between -4 and -3 must be divided equally into 3 parts and 2 such parts (counting from -3 to the left) should be taken:

15.



16. a) $\frac{6}{5}=1\frac{1}{5}$ b) $-\frac{6}{7}$ c) $-\frac{5}{3}=-1\frac{2}{3}$ d) $-\frac{12}{2}=-6$ e) $\frac{6}{9}=\frac{2}{3}$

17. a) $44\frac{3}{7}$ b) $108\frac{1}{5}$ c) $124\frac{2}{3}$ d) $3245\frac{4}{5}$ e) $272\frac{1}{5}$ f) $121\frac{16}{17}$ g) $50\frac{2}{3}$ h) $12\frac{14}{27}$

18. a) $64\frac{17}{24}$ b) $2004\frac{5}{7}$ c) $767\frac{9}{10}$ d) $20\frac{13}{24}$ e) $45\frac{11}{18}$ f) $50\frac{9}{10}$

19. a) $40\frac{23}{35}$ b) $350\frac{17}{24}$ c) $200\frac{1}{4}$ d) 46 e) $21\frac{11}{20}$ f) $58\frac{1}{2}$

20. a) $80\frac{1}{3}$ b) $2000\frac{17}{20}$ c) $199\frac{1}{2}$ d) 20 e) $8\frac{1}{12}$ f) $\frac{2}{11}$

21. a) $3+\frac{2}{7}=3\frac{2}{7}$ b) $2+5\frac{1}{3}=7\frac{1}{3}$ c) $5\frac{2}{7}+\frac{2}{7}=5\frac{4}{7}$ d) $2\frac{5}{9}-\frac{5}{9}=2$

e) $3\frac{8}{9}-3=\frac{8}{9}$ f) $4\frac{2}{7}-3\frac{2}{7}=1$ g) $4\frac{1}{2}+\frac{1}{2}=5$ h) $1-\frac{2}{3}=\frac{1}{3}$

22. a) $40\frac{11}{26}$ b) $562\frac{6}{35}$ c) $62\frac{7}{24}$ d) $200\frac{1}{8}$ e) $300\frac{13}{63}$ f) $66\frac{13}{33}$

23. a) $10\frac{13}{42}$ b) $78\frac{37}{56}$ c) $57\frac{5}{9}$ d) $30\frac{7}{9}$ e) $65\frac{25}{29}$ f) $9\frac{5}{6}$ g) $42\frac{7}{8}$ h) $5\frac{20}{27}$

24. a) $-30\frac{14}{25}$ b) $-22\frac{1}{21}$ c) $-28\frac{1}{6}$ d) $221\frac{7}{15}$ e) $-200\frac{4}{7}$ f) $-\frac{111}{112}$ g) $\frac{7}{10}$ h) $-101\frac{1}{100}$ i) $40\frac{4}{7}$

25. a) $20\frac{1}{2}$ b) $33\frac{11}{15}$ c) $-7\frac{3}{11}$ d) $-40\frac{5}{28}$ e) $-23\frac{16}{21}$ f) $10\frac{14}{19}$

26. a) $-\frac{7}{10}$ b) -5 c) 6 d) $3\frac{5}{31}=\frac{98}{31}$ e) $-2\frac{10}{17}=-\frac{44}{17}$ f) $4\frac{4}{5}=\frac{24}{5}$ g) $-\frac{13}{28}$ h) -5 i) $-\frac{8}{21}$ j) $-\frac{11}{18}$

27. a) $-\frac{125}{64}$ b) $\frac{81}{16}$ c) $-\frac{81}{16}$ d) $-\frac{121}{16}$

28. a) $6\frac{1}{14}=\frac{85}{14}$ b) $\frac{33}{5}=6\frac{3}{5}$ c) $-\frac{21}{2}=-10\frac{1}{2}$ d) -32 e) $\frac{44}{81}$ f) 13 g) $\frac{1}{2}$ h) $\frac{1}{36}$

i) $\frac{19}{14}$ j) $-\frac{2}{21}$ k) $-\frac{45}{44}$ l) $-\frac{11}{36}$ m) $\frac{29}{10}=2\frac{9}{10}$ n) $\frac{18}{7}=2\frac{4}{7}$ o) $-\frac{23}{20}=-1\frac{3}{20}$

p) $-\frac{9}{43}$ q) $-51\frac{1}{3}=-\frac{154}{3}$ r) $-5\frac{3}{20}$ s) $-\frac{156}{25}=-6\frac{6}{25}$ t) $-\frac{37}{3}=-12\frac{1}{3}$

Lesson 10

1. Every integer has an unwritten decimal point to the *right*. To multiply a decimal by a power of 10, move the decimal point to the *right* the same number of decimal places as the number of zeros in the power of ten. Fractional part of a decimal is to the right of the decimal point. To divide a decimal by a power of 10, move the decimal point to the *left* the same number of decimal places as the number of zeros in the power of ten. The place value of hundredths is to the *right* of the place value of tenths. Place value of tens is to the left of the decimal point.
2. a) 0.1 b) 0.04
3. a) 3.2 b) 2.4
4. a) integer part: 4 fractional part:.06 b) integer part: −2 fractional part: .901 c) integer part: 0 fractional part: .901
5. a) 1 b) 4 c) 9
6. a) 3 b) 8 c) 6
7. 80.357
8. – 2607.49
9. a) 4 tenths b) 2 thousandths c) 8 hundredths d) 1 ten-thousandths
10. a) tens b) thousandths c) tenths d) ones
11. a) minus five and seven hundredths b) zero and two-hundred thirty-four thousandths
    c) one-hundred twenty-four and one ten-thousandths d) negative zero and four tenths
12. a) 3.57 b) 0.35 c) – 207.0095 d) – 44.002
13. a) 0.8=0.800 b) 95=95.0 c) $0.05 \neq 0.0$ d) 0.789=.789 e) −0.230= −.23 f) $-45 \neq -450$
    g) 0000.36=0.360 h) $-7.002 \neq -7.0002$ i) −.4= −0.4 j) 00071=71
14. 23.7000, 00023.7, and 23.70 are equal to 23.7.
15. a) 0.70 b) 0.70000
16. a) 8.0 b) −248.0
17. a) $\frac{34}{10}$ b) $-\frac{7}{10}$ c) $\frac{2005}{1000}$ d) $\frac{137}{10}$ e) $-\frac{3}{1000}$ f) $\frac{2060}{10000} = \frac{103}{500}$
18. a) 170 b) 4.6 c) 347000 d) 700 e) 7.629 f) 340
19. a) 2.31 b) 6.782 c) 0.0000067 d) 4.21 e) 0.0004 f) 0.0004
20. a) multiplication by 1000 b) division by 100 c) division by 10
21. a) 20 b) 0.007 c) 3.4 d)1200 e)2 f) 0.054907 g) 0.09004 h) 340000 i) 234.5 j) 0.072 k) 3500 l) 5.678
22. a) 100 b) 10 c) 1000 d) 10000 e) 1000 f) 10
23. a) $3.7 \times 100=370$ b) $461 \div 1000=0.461$ c) $25 \div 10=2.5$ d) $0.64 \times 10=6.4$ e) $2 \div 100=0.02$
    f) $3.7152 \times 1000=3715.2$ g) $12 \times 1000=12000$ h) $893.2 \div 100=8.932$ i) $9 \div 10=0.9$
24. a) 0.28 b) 45.5 c) – 0.005 d) 0.775 e) 4.4 f) – 2.4375 g) – 0.875 h) – 0.06
    i) 5.375 j) – 239.17 k) – 4.0102 l) 3.005
25. a) $4.3>4.003$ b) $-23.1>-25.6$ c) $-4.0<0.01$ d) $0.2345<0.2635$
    e) $-2.87>-42.874$ f) $-0.1234>-0.1824$ g) $4.56<4.650$
26. a) 0 and 1 b) – 3 and –2 c) 4 and 5 d) – 1 and 0
27. 0.3 −0.1 0.247 0.34
28. −8.7 −0.5 −0.4 −0.09 2.34 2.36 3.1 5
29. a) X can be 8 or 9 b) X can be 1 or 0 c) X can be 4 or 3 or 2 or 1 or 0. d) X must be 0.

Lesson 11

1. a) 16.85 b) 6.414 c) 8.004 d) 2.47 e) 3.88 f) 43.997
2. a) 1.069 b) – 3.1 c) – 0.77 d) – 0.13 e) – 1.29 f) –1.8
3. a) 4.6 b) −0.76 c) – 1.5 d) 7.218 e) 2.41 f) 0.6 g) – 1.7 h) – 2.6 i) 0.4 j) 0
4. a) 0.012 b) – 0.004 c) 0.16 d) – 0.603 e) 7 f) – 0.00018 g) – 0.02 h) – 0.0793
5. a) – 0.024 b) – 0.009 c) – 0.00055 d) 0.02 e) – 0.251 f) 0.00016 g) 12 h) – 0.0009 i) 0.00084 j) – 100
6. 14
7. 0.8
8. 1900
9. 62
10. a) 1.89^7 ; positive b) $(–3.4504)^{11}$; negative c) $(–0.6402)^{16}$; positive

11. a) 0.81 b) – 0.064 c) – 1.21 d) 0.0036 e) – 0.00000001 f) – 0.0000001

12. a) $-\frac{80}{33}$ b) $-\frac{200}{3}$ c) $-\frac{700}{3}$ d) 5000 e) $\frac{13}{200}$ f) $\frac{123}{340}$ g) $-\frac{7}{8}$ h) $-\frac{1}{15}$

13. a) – 0.5 b) 0.0375 c) – 0.4 d) 0.073 e) –0.0075 f) 0.04

14. a) – 12.2 b) – 0.067 c) –3.1 d) – 0.81 e) 0.15 f) –500 g) – 3.75 h) – 1.4
i) 2.62 j) 223.4 k) 0.014 l) – 2 m) – 2500 n) 0.0028 o) 0.00000001 p) 61.5

15. a) – 0.2 b) – 1.93 c) 0 d) 0.209 e) 0.008 f) 10 g) 0.00001 h) 0.007 i) – 3.5
j) – 0.98 k) undefined l) – 0.1 m) 5 n) 0.94 o) 0.008 p) – 0.706 q) –1.895 r) –100

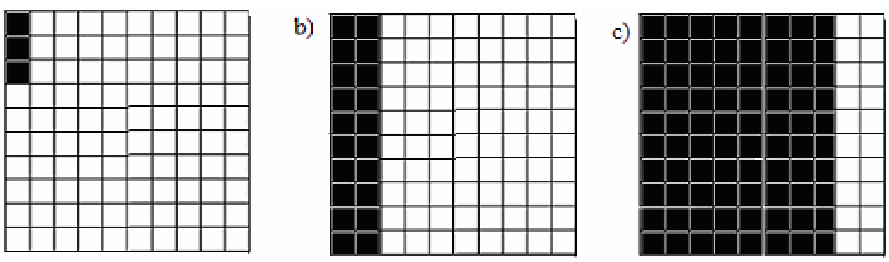
16. a) $(-0.5+0.2)\times 10=-3$ b) $(-0.3)^3 \div 0.09=-0.3$ c) $7.2+(-4.1-3)=0.1$ d) $2-(-0.03)\times(-0.05)=1.9985$
e) $0.4\div 0.02\times(-0.01)=-0.2$ f) $[(-0.4)+(-0.6)]^{17}=-1$ g) $(-0.2)\times(-0.06)\times 1000=12$

17. a) −0.173 b) 0 c) 0.0173 d) 10.173 e) 1.73 f) 0.001

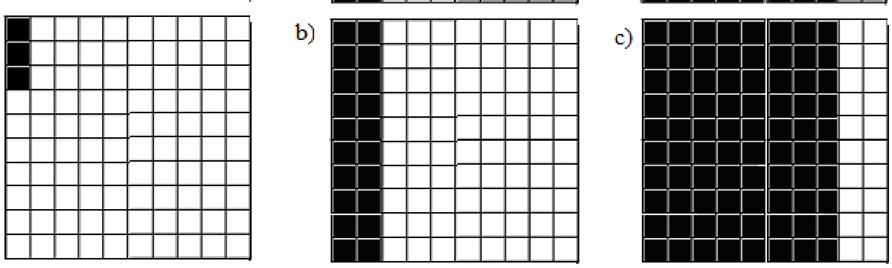
18. a) $0.2^2=0.04$ b) $0.2-0.1=0.1$ c) $0.3+0.7=1$ d) $-0.2\div(-0.2)=1$ e) $0.1^3=0.001$
f) $2.3-0.3=2$ g) $2.35-0.35=2$ h) $2\times 0.3=0.6$

Lesson 12

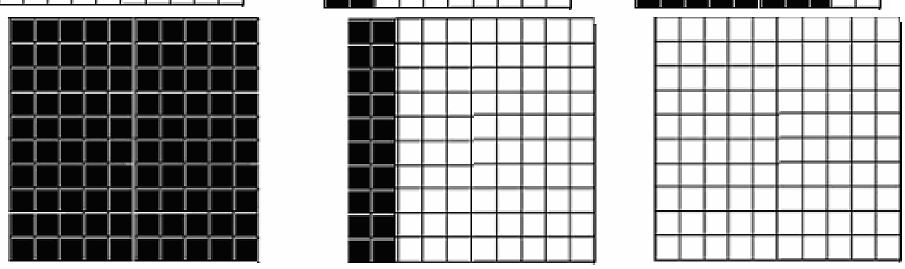
1. a) 12% b) 50% c) 25%
2. 145%
3. a) b) c)



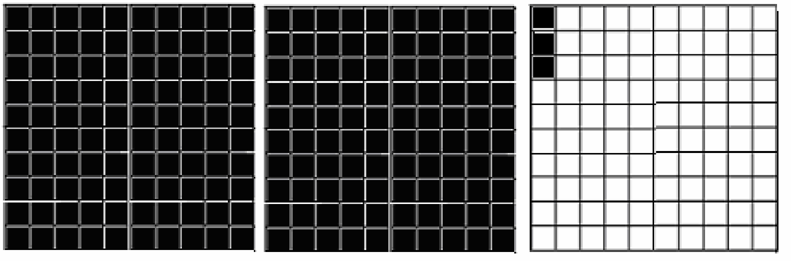
a) b) c)



4. a)



b)



5. a) 0.78 b) 0.002 c) 2.34 d) 0.0378 e) 0.0056 f) 0.02
6. a) 14% b) 300% c) 235.7% d) 2% e) 7000% f) 50%
7. 80%
8. 65% 35%
9. 4.4%
10. 133%
11. a) 20% b) 80% c)25% d)400%
12. 2.5%
13. 15%
14. 2%
15. a) 62.5% b) 37.5%
16. 1.5%
17. 50%
18. 40%
19. 40%
20. 22.5%
21. a) 12.5% b) 50% c) 200%
22. 17
23. 40
24. 0.0248
25. $2.25
26. 125
27. $675
28. 0.66
29. $550
30. 678.2
31. b)
32. a)