Problem on Contest Flyer

John has a large collection of 8-cent stamps and 15-cent stamps (and no other denominations). He realizes that there are some postage values he can obtain using his stamps, like 38¢ = 2(15¢) + 1(8¢) or 40¢ = 5(8¢). And there are some postage values that he cannot get, like 37 cents or 41 cents, using only 8-cent stamps and 15-cent stamps. What is the **largest** postage value that he can**not** get using only 8-cent stamps and 15-cent stamps? (Assume that there is no limit to the number of stamps that he can use.)

Solution: 97 cents

Solution 1

That the postage value 97 cents cannot be obtained using 8-cent and 15-cent stamps is not hard to show. If k $(k \geq 0)$ is the number of 15-cent stamps used, then the remainder of the postage, $n - 15k$, must be greater than or equal to 0 and must be divisible by 8. Also, k must be an odd number since, if k is even, $15k$ is even and adding a multiple of 8 will still be even. But checking $k = 1, 3, 5, 7$ reveals

$$97 - 1(15) = 82, \quad 97 - 3(15) = 52, \quad 97 - 5(15) = 22 \text{ and } 97 - 7(15) = -8.$$

None of these is both greater than or equal to 0 and divisible by 8, and for larger values of k, $97 - 15k$ will be negative. So it is not possible to obtain 97 using 8-cent and 15-cent stamps.

It remains to show that every number larger than 97 can be obtained using 8-cent and 15-cent stamps. Consider that the next 8 consecutive integers can be obtained as follows:

98 = 6(15) + 1(8)
99 = 5(15) + 3(8)
100 = 4(15) + 5(8)
101 = 3(15) + 7(8)
102 = 2(15) + 9(8)
103 = 1(15) + 11(8)
104 = 13(8)
105 = 7(15)

Every number larger than 105 is a multiple of 8 added to one of these. So every number larger than 97 can be obtained using 8-cent and 15-cent stamps.

Solution 2

Let n be a postage value (in cents) that is to be obtained using 8-cent stamps and 15-cent stamps. Let k be the number of 15-cent stamps used $(k \geq 0)$. Then the remainder of the postage, $n - 15k$, is to be covered using 8-cent stamps. So $n - 15k$ must be divisible by 8 or $n - 15k \equiv 0$ **mod** 8.

Now $15 \equiv 7 \equiv -1$ **mod** 8. So if $n - 15k \equiv n + k \equiv 0$ **mod** 8, then $k \equiv -n$ **mod** 8.

Let $k = -n$ **mod** 8 where $k \in \{0, 1, 2, 3, 4, 5, 6, 7\}$.

As long as $n - 15k \geq 0$, the postage value n can be obtained using k 15-cent stamps and $\frac{n - 15k}{8}$ 8-cent stamps.

If $n \geq 105$, then $n - 15k \geq 0$, since $k \leq 7$.

Also, if $n = 98, 99, 100, 101, 102, 103, 104$, then $k = 6, 5, 4, 3, 2, 1, 0$, respectively, and $n - 15k \geq 0$.

But when $n = 97$, $k = 7$ and $n - 15k < 0$.

So 97 cents is the largest postage value that cannot be obtained using only 8-cent stamps and 15-cent stamps.