The 2013 Colonial Mathematics Challenge was held on 6 May 2013. Eighty-five students from fifteen schools participated. Helping to run the contest were: professors Reid Huntsinger, Dan Jacobson, Elena Koublanova, Clark Loveridge, Wimayra Luy, Eric Neumann, Deivy Petrescu, Geoff Schulz, Sanda Shwe, Mohammed Teymour and Brent Webber; staff members Deb Coley and Ruth Al-Ameen; students Ryan Ivey, Khylil Robinson and Reynaldo Noel Jeune; and former student Gianni Ferrara.

The individual contest was won by Michael Leggerie from Girard Academic Music Program (GAMP), with 15 problems out of 20 correct. In second place was Alexander Palmer also from GAMP, with 11 out of 20 correct. And in third place was Marinos Rrapaj also from GAMP, with 10 out of 20 correct

First Place

Michael Leggerie

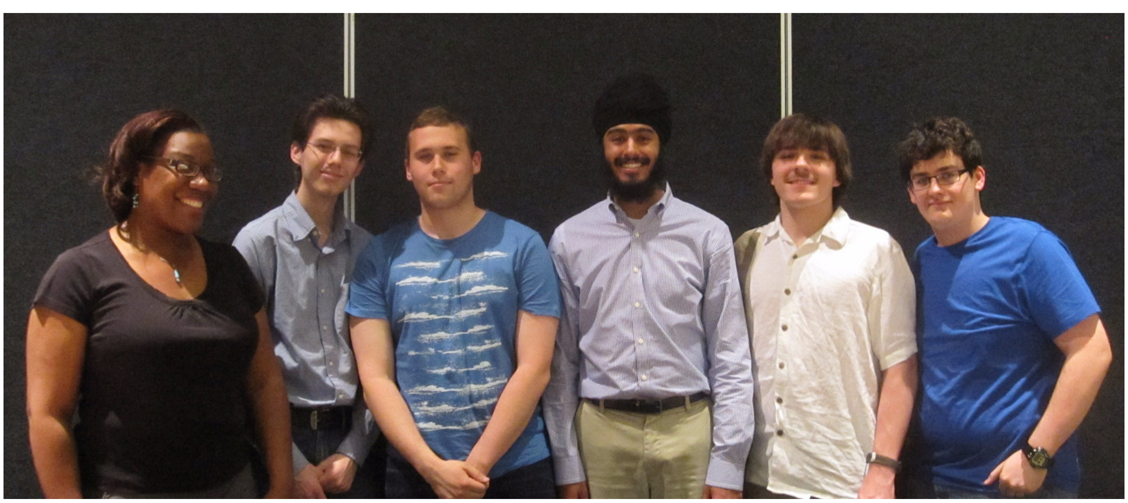
Second Place

Alexander Palmer

Third Place

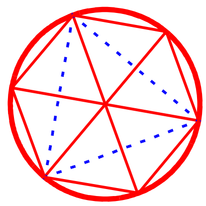
Marinos Rrapaj

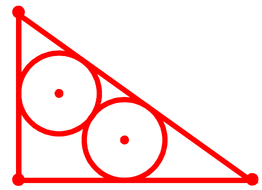
The team contest was won by the team from Girard Academic Music Program with team members Michael Leggerie, Matthew Mutter, Alexander Palmer, Marinos Rrapaj and Harnail Sembhi.



l to r: Teja Smith (teacher), Marinos Rrapaj, Alexander Palmer, Harnail Sembhi, Michael Leggerie, Matthew Mutter

2013 Colonial Mathematics Challenge





Mathematics Department 6 May 2013 Community College of Philadelphia

Name:	**School:**
Sex (Optional):	**Age (Optional):**
1:	**11:**
2:	**12:**
3:	**13:**
4:	**14:**
5:	**15:**
6:	**16:**
7:	**17:**
8:	**18:**
9:	**19:**
10:	**20:**

Instructions: *This is a closed-book individual examination. Calculators, rulers, compasses, and square-grid paper are allowed. Cell phones and other communication devices must be turned off during the examination. Contestants consulting with other contestants, teachers or any one else during the examination will be disqualified.*

Time: 90 minutes

1 Point P is outside circle C in the plane. At most how many points on C are a distance 3 units from P?

2 If $\log_7(\log_4(\log_5 x)) = 0$, then find the value of $x^{-1/2}$.

3 Judy's birthday is May 6. In 2013, this falls on a Monday. In what year will her birthday next fall on a Monday?

4 Find the number of digits in $4^{16}5^{25}$, written in the usual base 10 form.

5 Line segment AB is both a diameter of a circle of radius 2 and a side of an equilateral triangle ABC. The circle also intersects AC and BC at points D and E, respectively. Find the **exact** length of AE.

6 If the sequence $\{a_n\}$ is defined by

$$a_1 = 2$$
$$a_n = a_{n-1} + 2n \qquad \text{for } n > 1$$

then find a_{100}.

7 If $\frac{b}{a} = 2$ and $\frac{c}{b} = 3$, what is the ratio $\frac{a+b}{b+c}$?

8 The odd positive integers, 1, 3, 5, 7, . . . , are arranged in five columns (*A*, *B*, *C*, *D*, *E*) continuing with the pattern shown below. In which column (*A*, *B*, *C*, *D*, or *E*) will the number 2013 occur?

A	*B*	*C*	*D*	*E*
	1	3	5	7
15	13	11	9	
	17	19	21	23
31	29	27	25	
	33	35	37	39
47	45	43	41	
	49	51	53	55
.	.	.	.	
	.	.	.	.
.	.	.	.	
	.	.	.	.

9 A cryptographer devises the following method for encoding positive integers. First the integer is expressed in base 5. Second, a one-to-one correspondence is established between the sets $\{0, 1, 2, 3, 4\}$ and $\{V, W, X, Y, Z\}$. Using this correspondence, the cryptographer finds that three consecutive positive integers in increasing order are coded as *VYZ*, *VYX*, *VVW*, respectively. What is the base-10 expression for the integer coded as *XYZ*?

10 Let $x = .123456789101112. . . 998999$, where the digits are obtained by writing the integers 1 through 999 in order. Find the 2013th digit to the right of the decimal point.

11 There are two natural ways to inscribe a square in a given isosceles right triangle. If it is done as in Figure 1 below, then the area of the square is 441 cm^2. What is the area (in cm^2) of the square inscribed in the same triangle as shown in Figure 2 below?

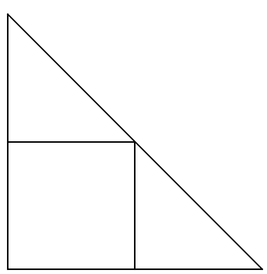


Figure 1

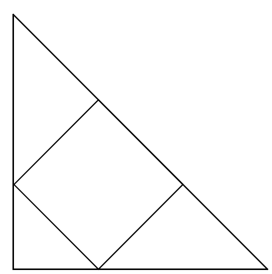
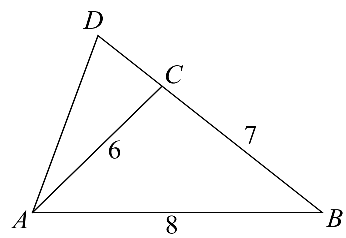


Figure 2

12 Find the product of all real roots of the equation $x^{\log x} = 10$. $(\log x = \log_{10} x)$

13 How many distinguishable rearrangements of the letters in CONTEST have both vowels first?

14 In ΔABC, $AB = 8$, $BC = 7$ and $CA = 6$. If side BC is extended to a point D so that ΔDAB is similar to ΔDCA, find the length of CD.



15 Find the **exact** sum of all real solutions of the equation $|x - |2x + 1|| = 3$.

16 The first four terms of an arithmetic sequence are $a, x, b, 2x$. Find the ratio $\frac{a}{b}$.

17 A wooden cube with edge length n units (where n is an integer > 2) is painted black on all sides. By slices parallel to its faces, the cube is cut into n^3 smaller cubes of unit edge length. If the number of smaller cubes with just one face painted black is equal to the number of smaller cubes with no paint, what is n?

18 The 120 permutations of CRAZY are listed in alphabetical order as if each were an ordinary 5-letter word. What is the last letter in the 94th word in this list?

19 A drawer in a darkened room contains 100 red socks, 80 green socks, 60 blue socks and 40 yellow socks. John selects socks one at a time from the drawer but is unable to see the colors of the socks drawn. What is the smallest number of socks that must be selected to guarantee that the selection contains at least 10 pairs? (A pair of socks is two socks of the same color. No sock may be counted in more than one pair.)

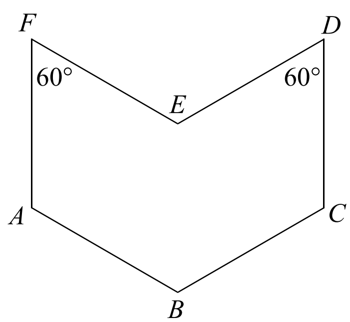
20 A ball was floating in a lake when the lake froze. The ball was removed, without breaking the ice, leaving a hole 24 cm across at the top and 8 cm deep. What is the radius of the ball in centimeters?

Team: ______________________ School: ______________________

21 (3 minutes)

Answer: ______________________

In the hexagon *ABCDEF*, sides *AF* and *CD* are parallel, as are sides *AB* and *FE*, and sides *BC* and *ED*. Each side has length 1. Also, $\angle AFE = \angle CDE = 60°$. Find the exact area of the hexagon.



Team:____________________ School:____________________

22 (4 minutes)

Answer: ________________

Find the units digit of $3^{1001}7^{1002}13^{1003}$.

Team: ______________________ School: ______________________

23 (5 minutes)

Answer: ______________________

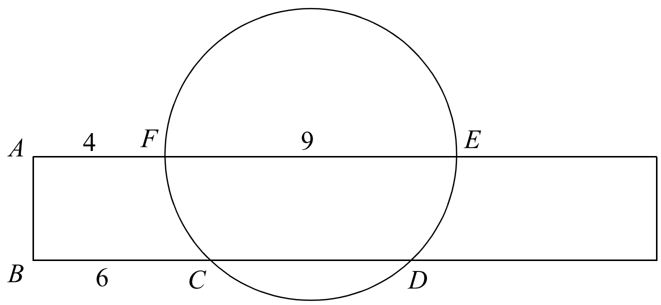
Find the value of $\dfrac{1000^2}{252^2 - 248^2}$.

Team: ______________________ School: ______________________

24 (4 minutes)

Answer: ______________________

A rectangle intersects a circle as shown. $AF = 4$, $FE = 9$ and $BC = 6$. Find CD.



Team:____________________ School:____________________

25 (4 minutes)

Answer: ________________

Using a table of a certain height, two identical blocks of wood are placed as shown in Figure 1. The distance from the top of the block on the floor to the top of the block on the table is 32 inches. After rearranging the blocks as in Figure 2, the distance from the top of the block on the floor to the top of the block on the table is 20 inches. How high is the table in inches?

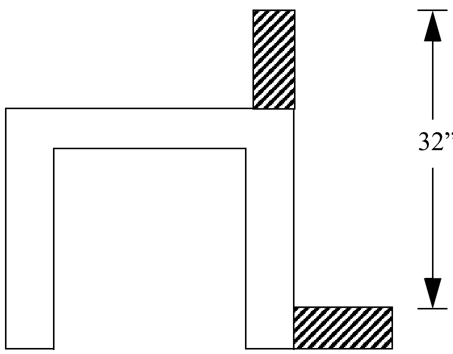


Figure 1

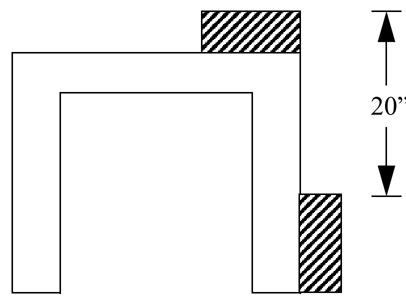


Figure 2

Team: ____________________ School: ____________________

26 (3 minutes)

Answer: ____________________

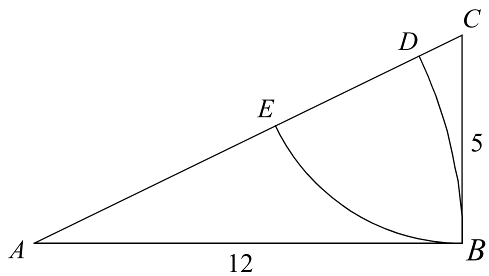
Let p, q and r be distinct prime numbers and let a, b and c be positive integers. If $p^a q^b r^c$ is the smallest positive perfect cube having pq^2r^4 as a divisor, then find $a + b + c$.

Team:____________________ School:____________________

27 (4 minutes)

Answer: ____________________

Triangle *ABC* is a right triangle with legs $AB = 12$ and $BC = 5$. Two arcs of circles are drawn, one with center A and radius 12, the other with center *C* and radius 5. The arcs intersect the hypotenuse, *AC*, at points *D* and *E*. Find the length *DE*.



Team: ______________________ School: ______________________

28 (6 minutes)

Answer: ______________________

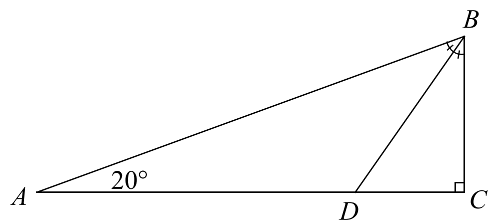
Compute the product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{9^2}\right)\left(1-\frac{1}{10^2}\right)$.

Team: ______________________ School: ______________________

29 (3 minutes)

Answer: ______________________

In the figure, ΔABC has a right angle at C and $\angle A = 20°$. If BD is the bisector of $\angle ABC$, then find the measure of $\angle BDC$ in degrees.



Team: ______________________ School: ______________________

31 (4 minutes)

Answer: ______________________

A *lattice point* is a point in the plane with integer coordinates. How many lattice points are on the line segment whose endpoints are (−3, 11) and (42, 275)? (Include both endpoints of the segment.)

Team:____________________ School:____________________

32 (5 minutes)

Answer: ________________

In ΔABC, $AB = 9$, $BC = 10$ and $CA = 8$. Point *D* is the midpoint of side *AB* and *E* is the foot of the altitude from *A* to *BC*. Find the length of *DE*.

Team:____________________ School:____________________

34 (5 minutes)

Answer: ________________

An $11 \times 11 \times 11$ wooden cube, formed by gluing together 11^3 unit cubes, is sitting on a table. You can take a photograph of the cube from any point. What is the greatest number of unit cubes that can be seen in the photograph?

1 $\boxed{2}$ The points in question are the points of intersection of the original circle C with the circle of radius 3 centered at P. Two circles with different centers intersect in at most two points.

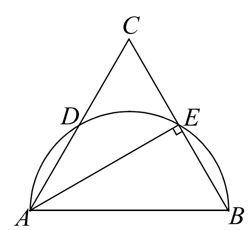
2 $\boxed{\frac{1}{25}}$ $\log_7(\log_4(\log_5 x)) = 0 \Rightarrow \log_4(\log_5 x) = 1 \Rightarrow \log_5 x = 4 \Rightarrow x = 5^4 \Rightarrow x^{-1/2} = (5^4)^{-1/2} = 5^{-2} = \frac{1}{25}$

3 $\boxed{2019}$ Note that $365 = 52 \cdot 7 + 1$, so Judy's birthday will advance one day in non-leap years and two days in leap years. Since 2016 is a leap year, her birthday will fall on the following days:

2014 Tuesday
2015 Wednesday
2016 Friday
2017 Saturday
2018 Sunday
2019 Monday

4 $\boxed{28}$ $4^{16}5^{25} = 2^{32}5^{25} = 2^7 10^{25} = 128 \cdot 10^{25}$. So the 3 digits in 128 are followed by 25 zeros for a total of 28 digits.

5 $\boxed{2\sqrt{3}}$ ΔABE is inscribed in a semicircle, so $\angle AEB = 90°$ and $AB = 4$ since AB is a diameter. Then AE is an altitude of equilateral ΔABC, so $AE = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$.



6 $\boxed{10100}$ From the definition of the sequence $a_n - a_{n-1} = 2n \quad (n > 1)$. Then

$$\begin{aligned} a_1 &= 2 \cdot 1 \\ a_2 - a_1 &= 2 \cdot 2 \\ a_3 - a_2 &= 2 \cdot 3 \\ &\vdots \\ a_{100} - a_{99} &= 2 \cdot 100 \end{aligned}$$

Adding these equations, $a_{100} = 2(1 + 2 + 3 + \ldots + 100) = 2\left(\frac{100 \cdot 101}{2}\right) = 10100$.

7 $\boxed{\frac{3}{8}}$ $\frac{b}{a} = 2 \Rightarrow b = 2a$. $\frac{c}{b} = 3 \Rightarrow c = 3b = 6a$. Then $\frac{a+b}{b+c} = \frac{a+2a}{2a+6a} = \frac{3}{8}$.

8 $\boxed{B}$ The rows in pairs of two have the form

A	B	C	D	E
	$16k+1$	$16k+3$	$16k+5$	$16k+7$
$16k+15$	$16k+13$	$16k+11$	$16k+9$	

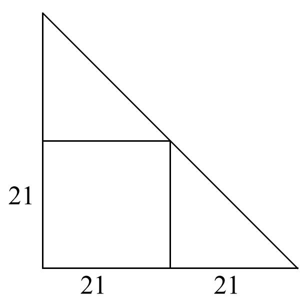
$(k = 0, 1, 2, \ldots)$. Since $2013 = 16(125) + 13$, 2013 is in column *B*.

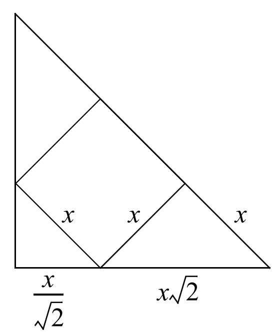
9 $\boxed{108}$ In base 5, $VYZ + 1 = VYX$, so $X = Z + 1$. Also, $VYX + 1 = VVW$. This implies that $X = 4$, $W = 0$, $Z = 3$ and $V = Y + 1$. So $Y = 1$ and $V = 2$. Then $XYZ = 413_5 = 108_{10}$.

10 $\boxed{7}$ $.\underbrace{123456789}_{A}\underbrace{1011\ldots99}_{B}100101\ldots$

Consider the groupings above. Group *A* (the single digit numbers) has 9 digits. Group *B* (the 2-digit numbers) has $90 \cdot 2 = 180$ digits. Together they total 189 digits. To get to digit 2013 we need $2013 - 189 = 1824$ more digits or $1824/3 = 608$ more three digit numbers. The 3-digit numbers start with 100 so the 608th 3-digit number is 707 and the 2013th digit is 7.

11 $\boxed{392}$ In the first triangle, the side of the square is 21 cm, so the side of the large triangle is 42 cm. In the second figure, let the side of the square be *x*. The small triangle with hypotenuse *x* has side $x/\sqrt{2}$ and the triangle with legs *x* has hypotenuse $x\sqrt{2}$. Then one leg of the large triangle is $42 = x/\sqrt{2} + \sqrt{2}x = 3x/\sqrt{2}$. Then $x = 14\sqrt{2}$ and $x^2 = 196(2) = 392$.





12 $\boxed{1}$ Given $x^{\log x} = 10$, take the common log (base 10) of both sides:

$\log(x^{\log x}) = \log 10 \Rightarrow (\log x)^2 = 1 \Rightarrow \log x = \pm 1 \Rightarrow x = 10$ or $x = \frac{1}{10}$. Then the product of the roots is $10 \cdot \frac{1}{10} = 1$.

13 $\boxed{120}$ There are 2 ways to order the vowels. If the consonants were all distinct there would be $5! = 120$ orderings of the consonants. But since there are two T's which are indistinguishable, there are $120/2 = 60$ orderings of the consonants. Thus there are $2(60) = 120$ rearrangements of CONTEST with the vowels first.

14 $\boxed{9}$ By similar triangles, $\frac{CD}{AD} = \frac{AC}{AB} = \frac{AD}{BD}$. Let $CD = x$. Then $\frac{x}{AD} = \frac{6}{8} = \frac{AD}{x+7}$. From the first part of this equation, $AD = \frac{4x}{3}$, and from the last part, $AD = \frac{3(x+7)}{4}$. Equating the two expressions for *AD*, and solving yields $x = CD = 9$.

15 $\boxed{\frac{2}{3}}$ $x - |2x+1| = 3$ or $x - |2x+1| = -3$

Case 1 $2x + 1 \geq 0$ $\left(x \geq -\frac{1}{2}\right)$. Then $x - (2x+1) = 3$ or $x - (2x+1) = -3$.

$x - 1 = 3$ or $x - 1 = -3$

$x = -4$ or $x = 2$ But $-4 < -\frac{1}{2}$, so -4 is not a solution.

Case 2 $2x + 1 < 0$ $\left(x < -\frac{1}{2}\right)$. Then $x + (2x+1) = 3$ or $x + (2x+1) = -3$.

$x = \frac{2}{3}$ or $x = -\frac{4}{3}$ But $\frac{2}{3} > -\frac{1}{2}$, so $\frac{2}{3}$ is not a solution.

So there are two real solutions and their sum is $2 + \left(-\frac{4}{3}\right) = \frac{2}{3}$.

16 $\boxed{\frac{1}{3}}$ Let k be the difference between successive terms. Then the difference between the 4th and 2nd terms is $2k$.

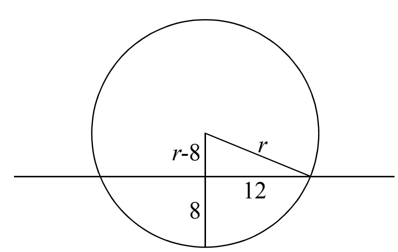
$2k = 2x - x$ and $k = \frac{x}{2}$. Then $a = x - \frac{x}{2} = \frac{x}{2}$ and $b = x + \frac{x}{2} = \frac{3x}{2}$. So $\frac{a}{b} = \frac{\frac{x}{2}}{\frac{3x}{2}} = \frac{1}{3}$.

17 $\boxed{8}$ The large cube has 6 faces containing at least one painted face on each smaller cube. The unpainted smaller cubes come from removing the 6 faces of the large cube, so there are $(n-2)^3$ unpainted smaller cubes. The smaller cubes with one painted face come from the 6 faces of the larger cube without the edges, so there are $6(n-2)^2$ smaller cubes with one painted face. If $(n-2)^3 = 6(n-2)^2$, then $n - 2 = 6$ and $n = 8$.

18 $\boxed{\text{A}}$ There are $4! = 24$ permutations starting with each of the 5 letters - A, B, C, D, E. Listed in alphabetical order, the 96th permutation is the last one starting with Y, namely YZRCA. Then the 95th is YZRAC and the 94th is YZCRA. So the last letter in the 94th word is A.

19 $\boxed{23}$ For any selection, at most 4 socks (one sock of each color) will be left unpaired. This occurs when an odd number of socks of each color is selected. So if 24 socks are selected, at least 20 socks are paired guaranteeing at least 10 pairs. But since 23 is not the sum of four odd numbers, if 23 socks are selected there are at most 3 colors with an unpaired sock, so again, there are at least 10 pairs. Since $22 = 5 + 5 + 5 + 7$, there could be 4 unpaired socks and only 9 pairs. So 23 is the smallest number of socks that must be selected to guarantee at least 10 pairs.

20 $\boxed{13}$ The figure shows a cross-section of the ball still in the ice. In the right triangle, $r^2 = (r-8)^2 + 12^2$. Solving for r, $r = 13$.

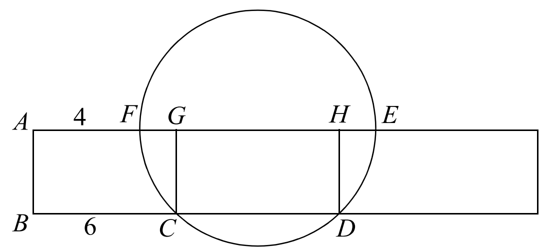


21 $\boxed{\sqrt{3}}$ Draw segments AE, BE and CE. There are now 4 equilateral triangles with side length 1. So the total area is $4\left(\frac{1}{2}\right)(1)\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$.

22 $\boxed{9}$ Any power of either $7 \cdot 13 = 91$ or $3^4 = 81$ has a units digit of 1. So the units digit of $3^{1001}7^{1002}13^{1003} = (3^4)^{250}(7 \cdot 13)^{1002} \cdot 3 \cdot 13$ is the same as the units digit of $3 \cdot 13 = 39$ which is 9.

23 $\boxed{500}$ $\dfrac{1000^2}{252^2 - 248^2} = \dfrac{1000^2}{(252+248)(252-248)} = \dfrac{1000}{500} \cdot \dfrac{1000}{4} = 2 \cdot 250 = 500$

24 $\boxed{5}$ Let G and H be points on segment FE so that $CDHG$ is a rectangle. $AG = BC = 6$ so $FG = 2$. Similarly $EH = 2$. Then $CD = GH = 9 - 2 - 2 = 5$.



25 $\boxed{26}$ Let the blocks be $(a \text{ inches}) \times (b \text{ inches})$ where $a < b$, and let the height of the table be h inches. From the first arrangement $h - a + b = 32$ and from the second $h + a - b = 20$. Adding these two equations, $2h = 52$ and $h = 26$.

26 $\boxed{12}$ If $p^a q^b r^c$ is the smallest positive perfect cube having pq^2r^4 as a divisor, then a, b and c are all multiples of 3, and $a \geq 1$, $b \geq 2$ and $c \geq 4$. The smallest such are $a = 3$, $b = 3$ and $c = 6$, so $a + b + c = 3 + 3 + 6 = 12$.

27 $\boxed{4}$ By the Pythagorean theorem, $AC = \sqrt{12^2 + 5^2} = 13$. $AD = 12$ so $DC = 1$. $CE = 5$ so $AE = 8$. Then $DE = AC - DC - AE = 13 - 1 - 8 = 4$.

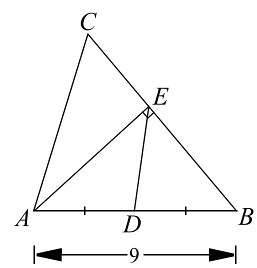
28 $\boxed{\frac{11}{20}}$ $1 - \frac{1}{n^2} = \left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{n}\right) = \left(\frac{n-1}{n}\right)\left(\frac{n+1}{n}\right)$. So $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{9^2}\right)\left(1 - \frac{1}{10^2}\right)$

$$= \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{7}{8} \cdot \frac{9}{8} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{9}{10} \cdot \frac{11}{10} = \frac{1}{2} \cdot \frac{11}{10} = \frac{11}{20}$$

29 $\boxed{55}$ $\angle ABC = 90° - 20° = 70°$. $\angle DBC = \frac{1}{2}\angle ABC = 35°$. $\angle BDC = 90° - 35° = 55°$.

31 $\boxed{4}$ The slope of the line segment is $\dfrac{275 - 11}{42 + 3} = \dfrac{88}{15}$. Since 88 and 15 are relatively prime, the next lattice point after $(-3, 11)$ is $(-3 + 15, 11 + 88) = (12, 99)$. Continuing in this way we get lattice points $(27, 187)$ and $(42, 275)$. So there are 4 lattice points on the segment.

32 $\boxed{\frac{9}{2}}$ DE is the median to the hypotenuse of right ΔABE, so it is half as long as the hypotenuse. $DE = \frac{1}{2}AB = \frac{9}{2}$.



34 $\boxed{331}$ At most three of the cubes six faces can be seen at one time. There is one unit cube that is on all three faces. The three edges each contain 10 unit cubes not counting the one that is on all three faces. And the three faces each contain 10^2 unit cubes not counting the ones on the three edges. So the greatest number of unit cubes that can be seen from a single point is $1 + 3(10) + 3(10^2) = 331$.

or

The cubes that can**not** be seen form a cube that is 10×10×10 unit cubes. So the number of cubes that can be seen is $11^3 - 10^3 = 331$.

Note: Problems numbered 30 and 33 were not used in the team contest.