The X Colonial Math Challenge took place May 12 2011. Professors Reid Huntsinger, Dan Jacobson, David Santos, Muhammad Teymour, Si Yoo, Yun Yoo and Brenton Webber, and student Gianni Ferrara helped with the competition.

The individual competition was won by Yun Teng Zheng from Masterman School, with 15 problems out of 20 correct. In a tie for second place, with 14 out of 20 correct were Lily Zhao and Chase Middleman, also from Masterman School.

Figure 1: Yun Teng ZHENG

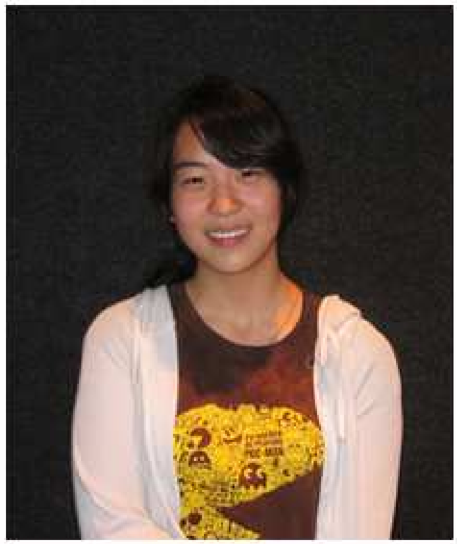


Figure 2: Lily ZHAO

Figure 3: Chase MIDDLEMAN

The team competition was also won by Masterman School, with team members Jenny Zhang, Yun Teng Zheng, Long Nguyen, Derrick Kan, Chase Middleman, Kalvin Khan, and Lily Zhao.

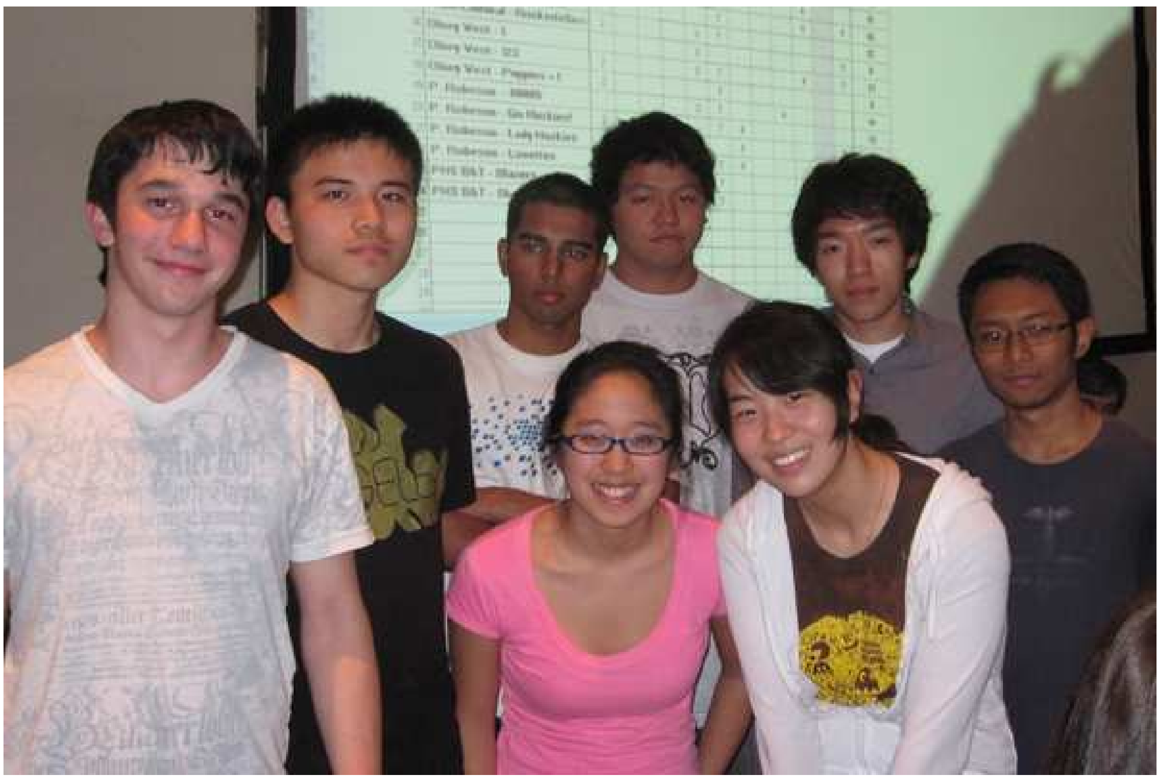
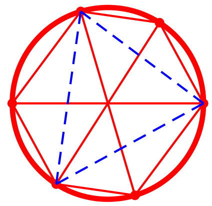
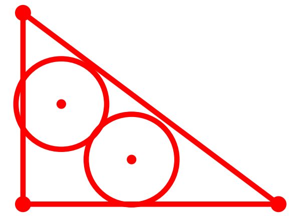


Figure 4: Team Masterman, winner of the team competition.

Tenth Colonial Mathematics Challenge





Mathematics Department May 12 2011 Community College of Philadelphia

Name:	School:
Sex (Optional):	Age (Optional):
1:	11:
2:	12:
3:	13:
4:	14:
5:	15:
6:	16:
7:	17:
8:	18:
9:	19:
10:	20:

Instructions: *This is a closed-book individual examination. Calculators, rulers, compasses, and square-grid paper are allowed. Contestants consulting with other contestants or with teachers during the examination will be disqualified.*

Time: **90 minutes.**

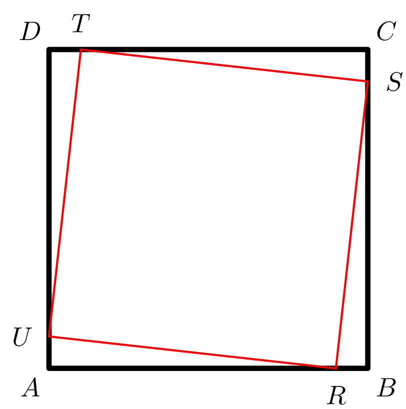
1 The numbers $49, 8, 48, 3, 43, 25, 32, 26, 19, 2$ are grouped into pairs so that the sum of every pair is the same. What is the other number in the pair containing 19?

2 The NCAA basketball tournament starts with 64 teams. Winners play winners until only one team is left. What is the total number of games that must be played to determine the winner (not counting the game between the two losers in the final four)?

3 Square $ABCD$ has sides of length 1. Points R, S, T, U are on sides AB, BC, CD, and DA, respectively, satisfy

$$RB = SC = TD = UA = \frac{AB}{10}.$$

Find the area of the square $RSTU$.



4 Arthur's birthday is May 12. In 2011, this falls on a Thursday. In what year will his birthday next fall on a Thursday?

5 Let a, b, c, d, e, f denote non-zero decimal digits. Two 3-digit natural numbers abc and def satisfy $abc + def = 1000$. Find the sum $a + b + c + d + e + f$.

6 The function ϕ is defined as

$$\phi(x, y) = x - \frac{1}{y},$$

where $y \neq 0$. Find the value of $\phi(2, \phi(2, \phi(2, 2)))$.

7 Find the sum of the squares of all the solutions to the equation

$$x^2 - 3|x - 1| = 1.$$

8 A *palindrome* is a strictly positive integer whose decimal expansion is symmetric. If the palindromes of three digits are written in increasing order

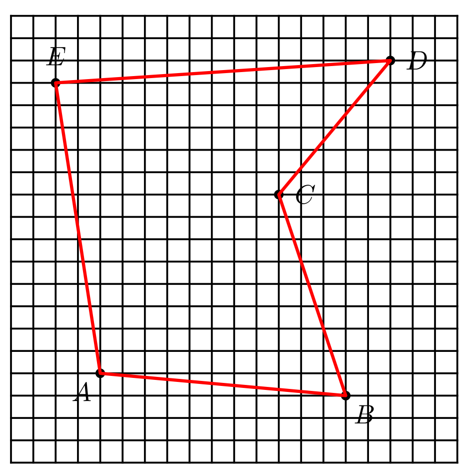
$$101, 111, 121, \ldots, 191, 202, \ldots, 989, 999$$

we see a gap of 10 between some consecutive palindromes, for example $111 - 101 = 10$ and a gap of 11 between others, for example $202 - 191$. What is the smallest possible gap between consecutive palindromes when the four-digit palindromes

$$1001, 1111, \ldots, 9889, 9999,$$

are written in increasing order?

9 Find the area, in square units, of the polygon below.



10 There are various ways of inserting exactly one pair of parentheses in the expression $40 \div 4 + 6 \times 3$, for example as $40 \div (4 + 6) \times 3$ or as $(40 \div 4) + 6 \times 3$. What is the largest number obtained when exactly one pair of parentheses is inserted in the expression?

11 The sequence

$$5, 2, -3, -1, 4, 5, 2, -3, -1, 4, 5, 2, -3, -1, 4, \ldots$$

is formed by periodically repeating the block of integers $5, 2, -3, -1, 4$, in this order. Starting with the first term, consecutive terms are added together, obtaining the sum

$$5 + 2 + (-3) + (-1) + 4 + 5 + 2 + (-3) + (-1) + 4 + 5 + 2 + (-3) + (-1) + 4 + \cdots ,$$

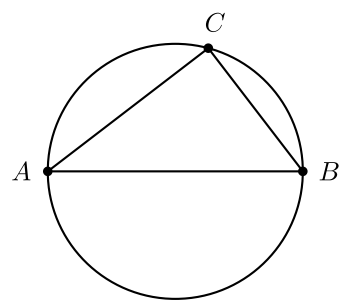
and eventually stopping. What is the smallest number of terms that must be added so that the sum be greater than 1000?

12 Find the difference between the largest and the smallest prime factors of $2^{16} - 1$.

13 If $a + b = 5$ and $a^2 + b^2 = 23$, find $|a^2 - b^2|$.

14 The fraction $\frac{2}{7}$ can be written uniquely as $\frac{2}{7} = \frac{1}{a} + \frac{1}{b}$, for positive integers $a < b$. Determine $a + b$.

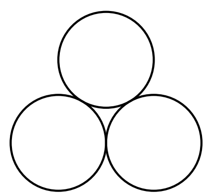
15 In the figure below, the circle shown has diameter $AB = 10$ and $\triangle ABC$ has area 11. Find the perimeter of $\triangle ABC$.



16 A set of 45 distinct strictly positive integers has mean and median 100. What is the largest value that the largest element of this set can be?

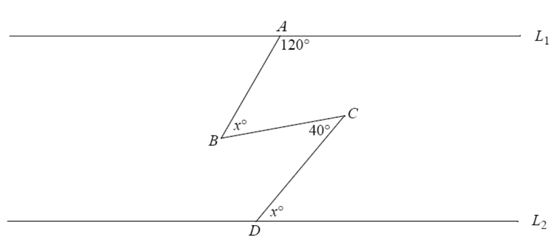
17 How many positive integer solutions are there to the equation $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{500}$?

18 Three circles of radius 1 are mutually tangential as shown in the figure below. What is the exact area of the region surrounded by the three circles?



19 There are three differently-colored bags: a red bag, a yellow bag, and a blue bag. Five distinct pieces of candy are to be distributed among the three bags. The red bag and the yellow bag must each receive at least one piece of candy. The blue bag may remain empty. How many different ways of distributing the candy among the three bags are possible?

20 In the figure below, lines L_1 and L_2 are parallel and all angle measures are in degrees. Find x, the measure of angle B.



21 (**2 minutes**)

School:____________________ Team:____________________

Place a $+, -, \times$ or $\div$ between the 4s so that the statement becomes true:

$$4 \quad \square \quad 4 \quad \boxed{.} \quad 4 \quad \square \quad 4 \quad = \quad 7$$

22 (**2 minutes**)

School:________________ Team:________________

John wrote the ten-digit integer 3906825147 on the blackboard. Mary erased five of its digits so that the remaining five digits form as large a five-digit integer as possible, when read from left to right. What is the sum of the digits of the integer left on the board?

23 (**4 minutes**)

School:________________ Team:________________

Square $ABCD$ has side 1 unit, and $\triangle ABR$ is equilateral, where R is inside the square. Find the measure of obtuse $\angle DRC$.

24 (**3 minutes**)

School: ________________ Team: ________________

The positive non-squares are written in ascending order:

$$2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, \ldots.$$

What is the 100th term of this sequence?

25 (**3 minutes**)

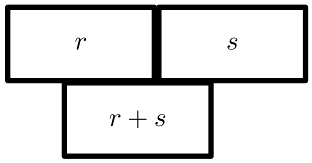
School:____________________ Team:____________________

A month with 31 days has the same number of Mondays and Fridays. What day is the first day of this month?

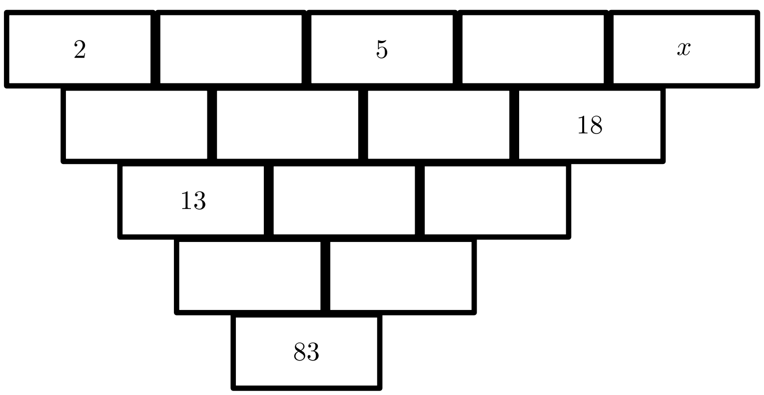
26 (**7 minutes**)

School:____________________ Team:____________________

Rule: given two consecutive cells on a row, add them together and put the sum in the cell below, as follows.



The following number pyramid is formed by following the stated rule.



Determine the value of x.

27 (**4 minutes**)

School:________________ Team:________________

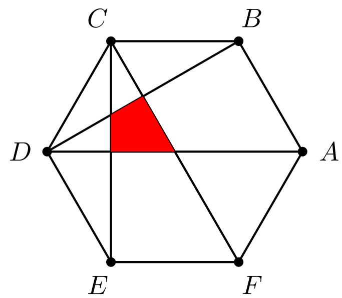
What is the exact value of the integer

$$17761776 \times 17761775 - 2 \times 17761774 \times 17761775 + 17761776 \times 17761781 \ ?$$

28 (**4 minutes**)

School:____________________ Team:____________________

The figure below shows regular hexagon $ABCDEF$ of area 666. Find the area of the shaded region inside the hexagon.



29 (**5 minutes**)

School:________________ Team:________________

Find the sum of the digits of the number obtained after calculating

$$777\,777\,777\,777\,777^2 - 222\,222\,222\,222\,223^2.$$

30 (**4 minutes**)

School:____________________ Team:____________________

A room has at least one person. In the room, $\frac{2}{5}$ of all the people are wearing gloves and $\frac{3}{4}$ of the people are wearing hats. What is the minimum number of people in the room wearing both a hat and gloves?

31 (**4 minutes**)

School:____________________ Team:____________________

What is the largest value of $x + y + z$ when

$$\frac{xy}{z} = -10, \qquad \frac{xz}{y} = -\frac{2}{5}, \qquad \frac{yz}{x} = -\frac{5}{2}?$$

1 32. Each pair adds up to $\dfrac{49+8+48+3+43+25+32+26+19+2}{10} = 51$. Hence, the number that goes with 19 is $51 - 19 = 32$.

2 63. There must be 63 losers and only one winner.

3 $\dfrac{41}{50}$. The area of the square is

$$TU^2 = UD^2 + DT^2 = \frac{1}{10^2} + \frac{9^2}{10^2} = \frac{82}{100} = \frac{41}{50}.$$

4 2016. Observe that $365 = 52 \cdot 7 + 1$, so Arthur's birthday will advance one day in non-leap years and two days in leap years. Since 2012 and 2016 are leap years, Arthur's birthday will fall in the following days:

2012	Saturday
2013	Sunday
2014	Monday
2015	Tuesday
2016	Thursday

5 28. We must have

$$c + f = 10, \quad 1 + b + e = 10, \quad 1 + a + d = 10,$$

whence $a + b + c + d + e + f = 10 + 10 + 10 - 2 = 28$.

6 $\dfrac{5}{4}$. We have

$$\phi(2, \phi(2, \phi(2, 2))) = \phi(2, \phi(2, \frac{3}{2})) = \phi(2, \frac{4}{3}) = \frac{5}{4}.$$

7 21. If $x \geq 1$ then

$$x^2 - 3|x-1| = 1 \implies x^2 - 3(x-1) = 1 \implies x^2 - 3x + 2 = 0 \implies (x-1)(x-2) = 0 \implies x \in \{1, 2\}.$$

If $x < 1$ then

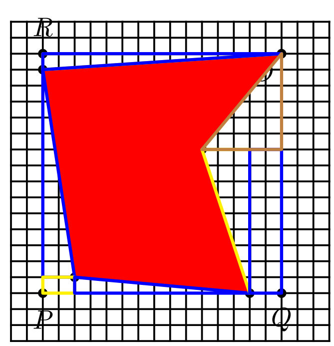
$$x^2 - 3|x-1| = 1 \implies x^2 - 3(1-x) = 1 \implies x^2 + 3x - 4 = 0 \implies (x+4)(x-1) = 0 \implies x \in \{-4\}.$$

The sum of the squares of the solutions is therefore $1^2 + (-4)^2 + 2^2 = 21$.

8 11. The gap between 1001 and 1111 is of $1111 - 1001 = 110$. The gap between 1991 and 2002 is $2002 - 1991 = 11$, and it is easy to check that this will be the smallest gap.

9 $\dfrac{303}{2}$. The area of the polygon $ABCDE$ is equal to the area of the square $PQDR$ minus the areas of the shaded regions (5 triangles and 2 rectangles).

$$\text{area } ABCDE = 15^2 - \frac{15 \cdot 1}{2} - \frac{13 \cdot 2}{2} - 2 \cdot 1 - \frac{11 \cdot 1}{2} - \frac{8 \cdot 3}{2} - \frac{15 \cdot 1}{2} - 2 \cdot 8 - \frac{7 \cdot 5}{2} = \frac{303}{2}.$$



10 48. To maximize the expression, minimize the divisor of 40 and increase the multiplicand of 3. Thus the maximum is $(40 \div 4 + 6) \times 3 = 48$.

11 712. Each block sums to 7. Now, $\llbracket \frac{1000}{7} \rrbracket = 142$, So adding the $142 \cdot 5 = 710$ integers in the first 142 blocks gives a total of $142 \cdot 7 = 994$. The sum of the first 709 integers is 990, the sum of the first 708 is 991 and the sum of the first 707 is 994. The sum of the first 711 is 999 and the first of the first 712 is 1001.

12 254. We have

$$2^{16} - 1 = (2^8 - 1)(2^8 + 1) = (2^4 - 1)(2^4 + 1)(2^8 + 1) = 3 \cdot 5 \cdot 17 \cdot 257,$$

and so the difference is $257 - 3 = 254$.

13 $5\sqrt{21}$. $2ab = (a+b)^2 - (a^2+b^2) = 2$, so $(a-b)^2 = a^2 + b^2 - 2ab = 21 \implies |a-b| = \sqrt{21}$. So $|a^2 - b^2| = 5\sqrt{21}$.

14 32. Since $a < b$, $\frac{2}{a} > \frac{1}{a} + \frac{1}{b} = \frac{2}{7} \implies 7 > a$. Since $\frac{1}{3} > \frac{2}{7}$, we must have $4 \leq a < 7$. So we let $a = 4, 5, 6$ and see what happens. But already for $a = 4$ we get

$$\frac{2}{7} - \frac{1}{4} = \frac{1}{28}.$$

Since we are told that the solution is unique, we must have $a = 4$ and $b = 28$, so $a + b = 32$.

15 22. Let $AC = b$, $BC = a$. Then we are given that

$$\frac{ab}{2} = 11 \implies ab = 22, \qquad a^2 + b^2 = 10^2.$$

where we have invoked Pythagoras' Theorem on the last equality. This gives

$$(a+b)^2 = a^2 + b^2 + 2ab = 100 + 44 = 144 \implies a + b = 12,$$

since $a + b > 0$. The perimeter is thus

$$a + b + 10 = 12 + 10 = 22.$$

16 1816. Let x_k, $k = 1, \ldots, 45$ be the integers listed in increasing order. We must have $x_{23} = 100$ and

$$x_1 + \cdots + x_{45} = 45 \cdot 100 = 4500.$$

We minimize $x_1, \ldots, x_{22}$, by taking $x_k = k$ for these. In order to force the mean to be 100 we would need $x_j = 100 + (j - 23)$, for $j = 24, \ldots, 44$. Hence

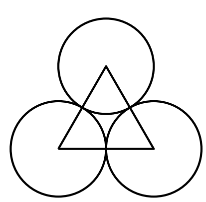
$$\underbrace{1 + 2 \cdots + 22}_{\text{22 terms}} + 100 + \underbrace{101 + 102 \cdots + 121}_{\text{21 terms}} + x_{45} = 4500 \implies 253 + 100 + 2331 + x_{45} = 4500 \implies x_{45} = 1816.$$

17 35. The relation $\frac{1}{a} + \frac{1}{b} = \frac{1}{500}$ implies that both $a$and b must be greater than 501, and that

$$(a - 500)(b - 500) = 500^2 = 5^6 \cdot 2^4.$$

So $a - 500$ and $b - 500$ are positive divisors of 500^2 and there are $(6+1)(4+1) = 35$ of those.

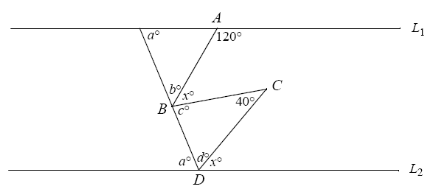
18 $\sqrt{3} - \frac{\pi}{2}$. The area sought is the area of the equilateral triangle formed joining the centers of the three circles minus the three angular sectors. Since the equilateral triangle has side 2, it has area $\sqrt{3}$. Since 60° is $1/6$ of the circumference, each of the angular sectors has area $\frac{\pi}{6}$. Thus the area sought is $\sqrt{3} - \frac{\pi}{2}$.



19 180. If there were no restrictions on the number of candies per bag, then each piece of candy could be distributed in 3 ways. In this case there would be 3^5 ways to distribute the candy. But this counts the cases where the red bag or the yellow bag is empty. If the red bag remained empty, then the candy could be distributed in 2^5 ways. Similarly if the yellow bag is empty there are 2^5 ways to distribute the candy. But these both include the one case where the red bag and yellow bag are both empty. So there $2^5 + 2^5 - 1$ ways to distribute the candy such that either the red bag or the yellow bag is empty. Then the number of ways to distribute the candy so that the red and yellow bags each receive at least one piece of candy is $3^5 - (2^5 + 2^5 - 1) = 180$.

Alternatively, let $A = \{1, 2, 3, 4, 5\}$ denote the set of the candies, and let $B = \{\text{red}, \text{yellow}, \text{blue}\}$ be the set of the bags. We want the number of onto functions from A to B and the number of onto functions from A to $\{\text{red}, \text{yellow}\}$. There are $3^5 - \binom{3}{2}2^5 + \binom{3}{1}1^5 = 150$ functions of the first type, and $2^5 - \binom{2}{1}1^5 = 30$ functions of the second type. The total is therefore $150 + 30 = 180$.

20 50. Draw the segment through D and B intersecting line L_1. Let a, b, c and d be the measures in degrees of the angles as shown. We have the equalities $a + b = 120$, $c + d = 140$, and so $a + b + c + d = 260$. Furthermore, $b + c + x = 180$ and $a + d + x = 180$. This yields $2x + a + b + c + d = 360$ from where $2x = 100 \implies x = 50$.



21 $+\ -\ \div$. Clearly $4 + 4 - 4 \div 4 = 7$.

22 33. The largest possible number that can be formed is 98547. $9 + 8 + 5 + 4 + 7 = 33$.

23 Observe that $\triangle ARD$ and $\triangle BRC$ are isosceles and congruent. Let

$$\alpha = \angle DAR = \angle ARD = \angle CRB = \angle BCR.$$

Then $\angle CDR = \angle CRD = 90° - \alpha$ and $\angle DRC = 2\alpha$. Thus

$$360° = \angle ARD + 60° + \angle BRC + \angle DRC = 4\alpha + 60° \implies \alpha = 75°.$$

Finally, $\angle DRC = 2\alpha = 150°$.

24 In the integers from 1 to 100, inclusive, there are 10 squares. Eliminating these squares, there are 90 integers left, and so 99 is the 90th non-square. We must add ten more integers, starting with 101, whence 110 is the 100th non-square.

25 Tuesday. Monday and Friday are 4 days apart. Now, $31 = 28 + 3$, so the first of the three leftover days over 4 weeks in the month cannot be Wednesday, Thursday, or Friday, or the month would have an extra Friday. Similarly, the first of the three leftover days over 4 weeks in the month cannot be Saturday, Sunday, or Monday, or the month would have an extra Monday. The only possibility left is Tuesday.

26 11. Let the first row be

2	a	5	b	x

Then the second row is

$2+a$	$a+5$	$5+b$	18

This gives $2+a+a+5=13 \implies a=3$. The next row is

13	$13+b$	$23+b$

The next row is

$26+b$	$36+2b$

This gives $26+b+36+2b=83 \implies b=7$. Finally, $b+x=18 \implies 7+x=18 \implies x=11$.

27 177617756. Put $x=17761776$. Then we must find

$$x(x-1)-2(x-2)(x-1)+x(x+5)=10x-4=177617760-4=177617756.$$

28 37. The medians of a triangle divide the triangle into six triangles of equal area. The area shaded is $\frac{2}{6}$ of one of the 6 equilateral triangles that compose the hexagon, hence $\frac{1}{3}\cdot\frac{1}{6}=\frac{1}{18}$ of the area of the hexagon is shaded. The area sought is thus $\frac{666}{18}=37$.

29 74. Observe that

$$\begin{aligned}777\,777\,777\,777\,777^2-222\,222\,222\,222\,223^2 &= (777\,777\,777\,777\,777+222\,222\,222\,222\,223)\\ &\quad\times(777\,777\,777\,777\,777-222\,222\,222\,222\,223)\\ &= (555\,555\,555\,555\,554)(1\,000\,000\,000\,000\,000)\\ &= 555\,555\,555\,555\,554\,000\,000\,000\,000\,000.\end{aligned}$$

The sum of the digits is $5\cdot 14+4=74$.

30 3. Let H denote the set of people in the room who wear a hat, and G the set of people in the room who wear gloves. If $|X|$ denotes the number of elements of X then we want

$$|H\cap G|=|H|+|G|-|H\cup G|.$$

If there are x people in the room, $|G|=\frac{2x}{5}$ and $|H|=\frac{3x}{4}$ must be integers, and so x must be a strictly positive multiple of 20, since $x\geq 1$. Thus $\frac{x}{20}\geq 1$ and

$$|H\cap G|=|H|+|G|-|H\cup G|=\frac{3x}{4}+\frac{2x}{5}-|H\cup G|=\frac{23x}{20}-|H\cup G|\geq\frac{23x}{20}-x=\frac{3x}{20}=\frac{x}{20}\cdot 3\geq 3,$$

so at least 3 people meet both conditions.

31 Multiplying the first and second equations,

$$x^2=4.$$

Multiplying the second and third equations,

$$z^2=1.$$

Multiplying the third and first equations,

$$y^2=25.$$

From the equations, either exactly one or exactly three of the variables are negative, and thus the set of solutions (x,y,z) is

$$\{(-2,-5,-1),(-2,5,1),(2,-5,1),(2,5,-1)\}.$$

So, the largest value is $5+2-1=6$.