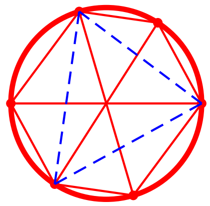
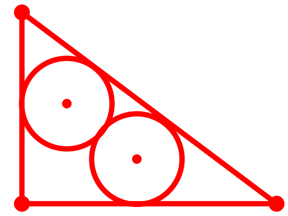
The Ninth Colonial Maths Challenge took place Thursday 23 April 2009. The winners of the individual competition were Ellen Kim (scored 13), Robert Johnson (scored 12), and Andrey Popov (scored 12). All three were from Masterman School. The winning team was also from Masterman school.

Ninth Colonial Mathematics Challenge





Mathematics Department 23 April 2009 Community College of Philadelphia

Name:	**School:**
Sex (Optional):	**Age (Optional):**
1:	**11:**
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7:	**17:**
8:	**18:**
9:	**19:**
10:	**20:**

Instructions: *This is a closed-book individual examination. Calculators, rulers, compasses, and square-grid paper are allowed. Contestants consulting with other contestants or with teachers during the examination will be disqualified.*

Time: 90 minutes.

1 If

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)=\frac{a}{b}$$

expressed as a fraction in lowest terms, determine the integer a^2+b^2.

2 A water-tank is shaped as a rectangular box of dimensions base-width-height of $50 \times 30 \times 60$ cubic inches. It is two thirds full of water. When a rock is put into the tank, the depth of the water increases to 42 inches. Find the volume of the rock, in cubic inches.

3 Solve the equation $\dfrac{x-357}{643}=\dfrac{x-643}{357}$ for x.

4 In figure 1, $OA=6$, $AB=8$, and $\angle BAO=\angle BYX=90^\circ$. If $XY=4$, determine the length of BX.

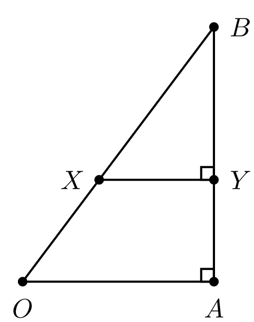


Figure 1: Problem 4.

5 Let n be the smallest strictly positive integer such that $2009n$ is a perfect cube. Find n.

6 Let n be the smallest positive integer with 100 digits whose sum of digits is 102. How many many zeros does n have when written down?

7 Given that $a - b = 2$ and $ab = 1$, determine the exact value of $a^3 - b^3$.

8 Determine the exact value of

$$9\,876\,543\,210^2 - 9\,876\,543\,212 \cdot 9\,876\,543\,208 - 9\,876\,543\,208^2 + 9\,876\,543\,209 \cdot 9\,876\,543\,207.$$

9 How many integers $n > 0$ are there for which $3n - 6$ is divisible by $n - 1$?

10 The pages of a book are numbered 1 through 64. It is known that the book is missing consecutive pages and that the sum of the numbers of those pages remanining in the book is 2006. Find the product of the numbers of the missing pages.

11 Two semicircles of radius R are contiguous, and a third semicircle of radius R is centered at the point of contact of the other two semicircles, as shown in figure 2. A circle of radius r is tangent to the three semicircles, as seen in the figure. Determine the value of the fraction $\frac{r}{R}$.

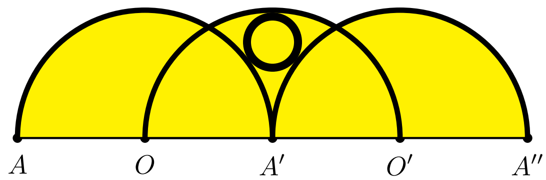


Figure 2: Problem 11.

12 The rector of a certain school is happy! Under her plan to inflate passing rates by hiring unqualified new faculty, the new hires obtained a passing rate of 85% of students in all classes. By giving a closer look to the statistics, it was noticed that, 82% of the males passed and 95% of the females passed. She is ecstatic, as she didn't have to round up or down any of the given percentages, and rounding is not one of her strengths. If the number of males that passed is known to be between 1110 and 1180, find the exact number of females that passed.

13 The integers from 1 to 1000 inclusive are written in order around a circle. Starting at 1, every 30-th number is marked. Thus $1, 31, 61, \ldots,$ are marked. We continue going around the circle in this way until a previously marked number is marked a second time. How many unmarked numbers are left?

14 Windshield wipers each of length a, are centered at points a units apart. Each covers semicircular overlapping areas (of radius a), as shown in figure 3. What combined area of the windshield do they cover?

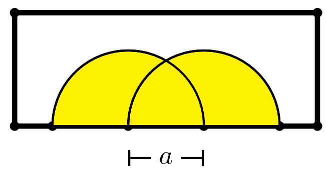


Figure 3: Problem 14.

15 The sequence

$$1, 2, 3, 2, 2, 5, 2, 2, 2, 7, 2, 2, 2, 2, 9, 2, 2, 2, 2, 2, 11, \ldots$$

consists of the consecutive positive odd integers interlaced with blocks of n 2's in the n-th block. Find the sum of the first 1000 terms. You may use the fact that for integer $n \geq 1$,

$$1 + 3 + 5 + \cdots + 2n - 1 = n^2, \qquad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

16 Determine all numbers $p > 0$ for which all of

$$p, \qquad p^2 + 2, \qquad p^2 + 4, \qquad p^2 + 20,$$

are all prime numbers.

17 A jeweler has three golden rings of identical radius a welded together as in figure 4, where the points A and B will be welded together to form the encasing of a pearl, as in figure 5. Find the radius of the largest pearl that he can use.

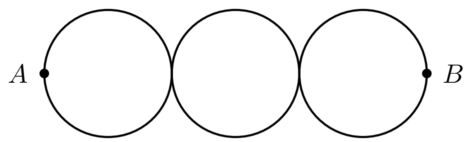


Figure 4: Problem 17.

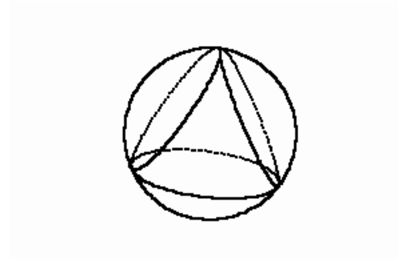


Figure 5: Problem 17.

18 In parallelogram $ABCD$ in figure 6, M is the midpoint of diagonal BD, and P is a point on AD such that $AD = 4PD$. If the area of the quadrilateral $ABMP$ is a, find the area of the shaded $\triangle PMD$.

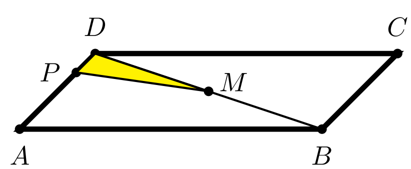


Figure 6: Problem 18.

19 I bought a certain amount of avocados at four for \$2; I kept a fifth of them, and then sold the rest at three for \$2. If I made a profit of \$2, how many avocados did I originally buy?

20 If the equation $x^4 - ax^3 - bx^2 - cx = 2007$, where a, b and c are real numbers has three distinct solutions, all of which are integers, determine the largest value of b.

21 (**3 minutes**) Four students, Adam, Beatrice, Carlos, and Dwayne, were taking a group math competition. After the competition, it was discovered that the table around which they were sitting had a piece of chewing gum which was not there before they arrived. Asking for the perpetrator, a teacher obtains the following responses:

- "It wasn't me," said Adam.
- "It was Carlos," said Beatrice.
- "It was Dwayne," said Carlos.
- "It wasn't me," said Dwayne.

As expected, only the perpetrator lied. Who stuck the gum on the table?

22 (**3 minutes**) The binary operation $\otimes$ satisfies $x \otimes y = xy + x - y$ for any pair of real numbers x and y. Find a value of a such that

$$3 \otimes a = a \otimes 5.$$

23 (**4 minutes**) You use three standard dice in order to write three digits numbers. Thus 111 is the smallest number you may write, and 666 is the largest number you may write. Some three digits numbers between 111 and 666, for example 117, cannot be written with these three dice. How many numbers between 111 and 666 cannot be written by using the three dice?

24 (**4 minutes**) Determine all the values of the positive integer N such that $N^3 - 1000$ is a positive prime number.

25 (**4 minutes**) A cube is inscribed in a sphere of radius R. If the volume of the cube is V and the volume of the sphere is V', determine the ratio $\frac{V}{V'}$.

26 (**5 minutes**) The numbers a, b, c satisfy

$$ab = 1, \quad bc = 2, \quad ca = 3.$$

Find the value of $a^2 + b^2 + c^2$.

27 (**6 minutes**) In the adjoining figure, the points M' and N' are on the circle with center O and the lines XM' and XN' intersect the circle at M and N, respectively. Furthermore, $XN = NN' = 6$ and $MM' = 10$. Determine the value of XM.

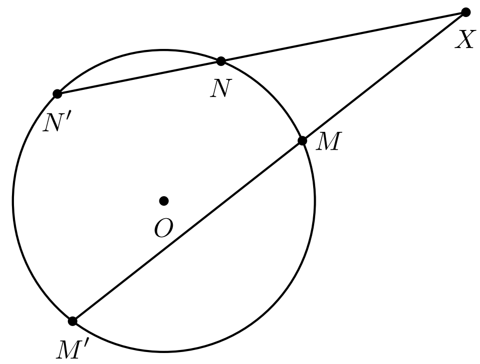


Figure 7: Problem 27

28 (**5 minutes**) The regular heptagon with vertices $I - II - III - IV - V - VI - VII$ is inscribed in the circle of center O, as in the adjoining figure 8. A bug jumps in counter-clockwise direction from one point to another. If the bug is on an odd-numbered vertex, it will make one jump to the next vertex. If the bug is on an even-numbered vertex, it will make two jumps. If the bug started at vertex III, where will it be after 2009 steps?

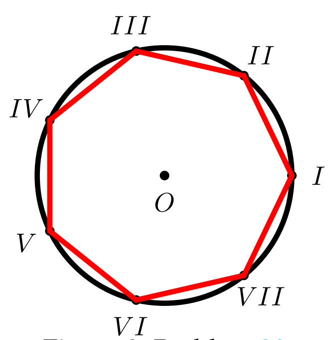


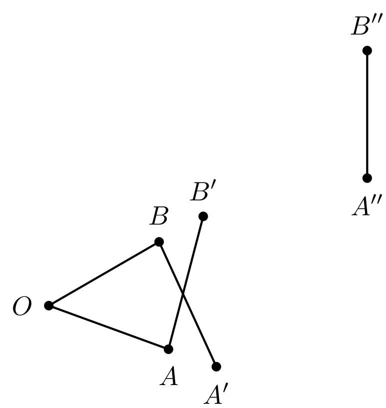
Figure 8: Problem 28.

29 (7 minutes) Consider the 2008×2008 matrix

$$\begin{matrix} 1 & 2 & 3 & \cdots & 2008 \\ 2009 & 2010 & 2011 & \cdots & 4016 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 4030057 & 4030058 & 4030059 & \cdots & 2008^2 \end{matrix}$$

2008 numbers are chosen at random, with the condition that no two numbers chosen belong to either the same row or the same column. What is the sum of the numbers chosen?

30 (**7 minutes**) Karl is playing with five matches, each of the same length. He arranges four of them in a design that resembles the adjoining figure. He wants to put the fifth so that the figure forms a (self-intersecting) pentagon of sides of identical length and so that the the points O, A, A' be collinear, as well as the points O, B, B' be collinear. What should the value of $\angle AOB$ be, in degrees?



1 We have

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)=\left(\frac{3}{4}\right)\left(\frac{8}{9}\right)\left(\frac{15}{16}\right)\left(\frac{24}{25}\right)=\frac{3}{5}.$$

So $3^2+5^2=34$.

2 Observe that $\frac{2}{3}\cdot 60=40$, so the original volume is $50\cdot 30\cdot 40=60000$ cubic inches. After the rock is added the new volume is $50\cdot 30\cdot 42=63000$ cubic inches, so the volume of the rock is $63000-60000=3000$ cubic inches.

3 Let $a=357$, $b=643$. Then $a+b=1000$, and $\frac{x-a}{b}=\frac{x-b}{a} \implies ax-a^2=bx-b^2 \implies x(a-b)=a^2-b^2 \implies x=\frac{a^2-b^2}{a-b}=\frac{(a-b)(a+b)}{a-b}=a+b=1000.$

4 By similar triangles, $\frac{BY}{XY}=\frac{BA}{OA} \implies BY=\frac{8}{6}\cdot 4=\frac{16}{3}$. By the Pythagorean Theorem,

$$BX=\sqrt{XY^2+YB^2}=\sqrt{16+\frac{256}{9}}=\frac{20}{3}.$$

5 Since $2009=7^2\cdot 41$, the smallest such n will be $n=7\cdot 41^2=11767$.

6 To minimize the number, its first digit from right to left must be a 1 and its last digits from right to left must be 9's. Now, $\lfloor\frac{100}{9}\rfloor=11$, so we must use eleven 9's and one 1 and one 2, since $9\cdot 11+1+2=102$. The number looks like

$$1\underbrace{0\ldots0}_{87\text{ zeros}}2999999999$$

7 We have

$$a^2+b^2=(a-b)^2+2ab=2^2+2(1)=6.$$

We conclude that

$$a^3-b^3=(a-b)(a^2+ab+b^2)=(2)(6+1)=14.$$

8 Put $a=9\,876\,543\,210$. Then we want

$$a^2-(a+2)(a-2)-(a-2)^2+(a-1)(a-3)=a^2-(a^2-4)-(a^2-4a+4)+(a^2-4a+3)=3.$$

9 We have

$$\frac{3n-6}{n-1} = \frac{3n-3-3}{n-1} = 3 - \frac{3}{n-1}.$$

If this is an integers then we must have $n-1$ dividing 3, and so $n-1=1$, $n-1=-1$, $n-1=3$, or $n-1=-3$. This gives $n \in \{2, 0, 4, -2\}$, of which only two are strictly positive integers.

10 The sum of the numbers on the 64 pages is

$$1 + 2 + \cdots + 64 = \frac{64 \cdot 65}{2} = 2080.$$

If the missing page numbers are $l+1, l+2, \cdots, l+n$ then

$$(l+1) + (l+2) + \cdots + (l+n) = \frac{n(2l+n+1)}{2} = 2080 - 2006 = 74 \implies n(2l+n+1) = 148.$$

Now, for each missing page there are two numbers, and so n must be even. Observe that n and $2l+n+1$ are integers of different parity and that $n < 2l+n+1$. Moreover, $n \leq 64$ and $l \leq 64$. Let us factor 148 as the product of two integers of different parity with a factor less than 64: $148 = 4 \cdot 37$. If $n = 4$ then $2l+n+1 = 37 \implies l = 16$. This means that pages $l+1 = 17, l+2 = 18, l+3 = 19, l+4 = 20$ are missing. So $17 \cdot 18 \cdot 19 \cdot 20 = 116280$.

11 Let the center of the small circle be B. Then $BO = r + R$ and $BA' = R - r$. Also, $OA' = R$ and $\triangle OA'B$ is right-angled at A'. Thus by the Pythagorean Theorem,

$$(R-r)^2 + R^2 = (R+r)^2 \implies R^2 - 2rR + r^2 + R^2 = R^2 + 2rR + r^2 \implies R^2 = 4rR \implies \frac{r}{R} = \frac{1}{4}.$$

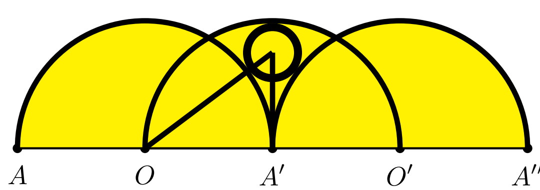


Figure 9: Problem 11.

12 If f is the number of females and m is the number of males, then

$$.85(f+m) = .95f + .82m \implies 10f = 3m.$$

Now,

$$1110 \leq .82m \leq 1180 \implies 1353 \leq m \leq 1439.$$

Since $\frac{82m}{100} = \frac{41m}{50}$ is an integer, and since the only multiple of 50 between 1353 and 1439 is 1400, we must have $m = 1400$. This means that

$$f = \frac{3 \cdot 1400}{10} = 420.$$

Now, 95% of 420 is 399.

13 The first time around the numbers of the form $1 + 30n$ are deleted. The last one of this satisfies $1 + 30n \leq 1000 \implies n = \frac{1000-1}{30} = 33$, so $1 + 30 \cdot 33 = 991$ is the last number deleted. In this first round we have deleted $1 + 33 = 34$ numbers. The next number to be deleted is $991 + 30 - 1000 = 21$. Thus in the next round numbers of the form $21 + 30n$ are deleted. The last one of this satisfies

$$21 + 30n \leq 1000 \implies n = \frac{1000-21}{30} = 32,$$

so $21 + 30 \cdot 32 = 981$ is the last number deleted. In the second round we have deleted $1 + 32 = 33$ numbers. The next number to be deleted is $981 + 30 - 1000 = 11$. In the next round, $1 + 32 = 33$ numbers of the form $11 + 30n$ are deleted, with $11 + 30 \cdot 32 = 971$ being the last one deleted. The next number to be deleted is $971 + 30 - 1000 = 1$, which already has been deleted. So we have deleted $34 + 33 + 33 = 100$ numbers and $1000 - 100 = 900$.

14 The area sought is the area of the two semicircles minus the area of the "Gothic window" which they intersect, as this area is counted twice. The area of the two semicircles is $\frac{\pi a^2}{2} + \frac{\pi a^2}{2} = \pi a^2$. The area of the Gothic window is the area of an equilateral triangle of side a (which is $\frac{\sqrt{3}a^2}{4}$) plus the area of two circular segments of a circle of radius a subtending an angle of $\frac{\pi}{3}$. The area of the sector to which this segment belongs is $\frac{1}{2} \cdot \frac{\pi}{3} \cdot a^2 = \frac{\pi a^2}{6}$. Thus the area of a segment is $\dfrac{\pi a^2}{6} - \dfrac{\sqrt{3}a^2}{4}$ and the area of the Gothic window is $2\dfrac{\pi a^2}{6} - \dfrac{\sqrt{3}a^2}{4}$. Hence the area sought is finally,

$$\pi a^2 - \left(2\frac{\pi a^2}{6} - \frac{\sqrt{3}a^2}{4}\right) = \frac{2\pi a^2}{3} + \frac{\sqrt{3}a^2}{4}.$$

15 Recall that

$$1 + 3 + 5 + \cdots 2n - 1 = n^2, \qquad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Consider the k-th block of 2's, which is written after $2k - 1$. After the last 2 in this block has been written, we have listed

$$k + 1 + 2 + 3 + \cdots + k = k + \frac{k(k+1)}{2}$$

numbers. If $k + \dfrac{k(k+1)}{2} = 1000$ then $k \approx 43$. After writing the last 2 in the block of 43 2's, we have listed

$$43 + \frac{43(43+1)}{2} = 989$$

integers. The 990th is $2 \cdot 44 - 1 = 87$, and then we need the first ten 2's of the block containing 44 2's. The desired sum is thus

$$1 + 3 + 5 + \cdots + 87 + 2 + (2+2) + (2+2+2) + \cdots + \underbrace{(2 + \cdots + 2)}_{43\ 2's} + 10 \cdot 2.$$

Now

$$1 + 3 + 5 + \cdots + 87 = 44^2 = 1936,$$

$$2 + (2+2) + (2+2+2) + \cdots + \underbrace{(2 + \cdots + 2)}_{43\ 2's} = 2(1 + 2 + 3 + \cdots + 43) = 1892,$$

and so the sum of the first thousand terms is

$$1936 + 1892 + 20 = 3848.$$

16 Observe that for $p = 3$, all of $3, 11, 13, 29$ are prime. The question is, are there any other p for which this is true? Let us prove that this is not the case. Any prime $p > 3$ either $p = 3k + 1$ or $p = 3k + 2$. In the former case,

$$p^2 + 2 = (3k+1)^2 + 2 = 9k^2 + 6k + 1 + 2 = 3(3k^2 + 2k + 1),$$

a multiple of 3. In the latter case,

$$p^2 + 20 = (3k+2)^2 + 20 = 9k^2 + 6k + 4 + 20 = 3(3k^2 + 2k + 8),$$

another multiple of 3.

17 The points of contact of the three rings form an equilateral triangle of side $2a$, whose centroid will be the center of the largest encased pearl. From figure 10 and by the Pythagorean Theorem,

$$CH^2 = 4a^2 - a^2 = 3a^2 \implies CH = \sqrt{3}a.$$

The radius of the largest encased pearl is CO, which is $\frac{2}{3}$ of the distance CH, thus the radius sought is

$$CO = \frac{2\sqrt{3}}{3}a.$$

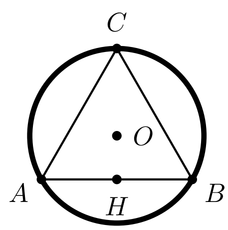


Figure 10: Problem 17.

18 By joining AM we see that $[\triangle ADM] = 4[\triangle PDM]$, as the triangles have the same height, so their areas are proportional to their bases. Also, subtracting common areas, $[\triangle APM] = 3[\triangle PDM]$. Using the same argument, $[\triangle AMB] = [\triangle AMD]$, as the bases MB and MD are equal. Therefore, $[\triangle AMB] = 4[\triangle PDM]$. This yields, upon adding the areas of $\triangle APM$ and $\triangle AMB$ that the area of the quadrilateral $[ABMP] = 7[\triangle PDM]$, so $[\triangle PDM] = \frac{a}{7}$.

19 If x is the amount originally bought, then I spent $2\left(\frac{x}{4}\right) = \frac{x}{2}$ dollars. Since I kept $\frac{x}{5}$, I must have sold $\frac{4x}{5}$ of them, making $2\left(\dfrac{\frac{4x}{5}}{3}\right) = \dfrac{8x}{15}$ dollars on this sale. My net gain is thus

$$\frac{8x}{15} - \frac{x}{2} = 2.$$

Multiplying both sides by 30, we have

$$30\left(\frac{8x}{15} - \frac{x}{2}\right) = 2(30) \implies 16x - 15x = 60.$$

I originally bought sixty avocados.

20 Let $f(x) = x^4 - ax^3 - bx^2 - cx - 2007 = (x-\alpha)^2(x-\beta)(x-\gamma)$, with α, β, γ distinct integers. Comparing coefficients,

$$\alpha^2\beta\gamma = -2007 = -3^2 \cdot 223.$$

and

$$\alpha^2 + 2\alpha(\beta+\gamma) + \beta\gamma = -b.$$

If $\alpha^2 = 1$, then $\beta\gamma = -2007$ and $-b = -2006 \pm 2(\beta+\gamma)$. In this case, the smallest value of $-b$ will be obtained when $\alpha = -1$ and $\beta+\gamma$ is maximal, giving $\beta+\gamma = 669 - 3 = 666$, or if $\alpha = 1$ and $\beta+\gamma$ is minimal, giving $\beta+\gamma = -669+3 = -666$. In either case, $b = 3338$. If $\alpha^2 = 9$, then $\beta\gamma = -223$ and $-b = -214 \pm 2\cdot 3(\beta+\gamma)$. In this case, the smallest value of $-b$ is obtained if either $\alpha = -3$ and $\beta+\gamma$ is maximal, giving $\beta+\gamma = 223-1 = 222$, or if $\alpha = 3$ and $\beta+\gamma$ is minimal, giving $\beta+\gamma = -223+1 = -222$. In either case, $b = 1546$. So $b = 3338$ is the maximal.

21 Since three students said the truth and one lied, the perpetrators could only be Carlos or Dwayne. But Dwayne is ruled out by the truth of the last statement.

22 We have

$$3 \otimes a = a \otimes 5 \implies 3a + 3 - a = 5a + a - 5 \implies a = 2.$$

23 With the three dice, you can write $6^3 = 216$ numbers. Between 111 and 666 inclusive there are $666-111+1 = 556$ integers, and thus $556 - 216 = 340$ cannot be written with the dice.

24 $N^3 - 1000 = (N-10)(N^2+10N+100)$. Since $N - 10 < N^2 + 10N + 100$ and the product is a prime, we must have $N - 10 = 1$ and so $N = 11$.

25 The diagonal of the cube is $2R$. If a is the side of the cube, then $a^2 + a^2 + a^2 = 4R^2$ and so $a^2 = \dfrac{4R^2}{3} \implies a^3 = \dfrac{8R^3}{3\sqrt{3}}$. The ratio is therefore

$$\frac{V}{V'} = \frac{a^3}{\frac{4}{3}\pi R^3} = \frac{\frac{8R^3}{3\sqrt{3}}}{\frac{4}{3}\pi R^3} = \frac{2}{\sqrt{3}\pi}.$$

26 We have

$$a^2b^2c^2 = (ab)(bc)(ca) = 6.$$

Hence,

$$a^2 = \frac{a^2b^2c^2}{(bc)^2} = \frac{6}{4} = \frac{3}{2},$$

$$b^2 = \frac{a^2b^2c^2}{(ca)^2} = \frac{6}{9} = \frac{2}{3},$$

$$c^2 = \frac{a^2b^2c^2}{(ab)^2} = \frac{6}{1} = 6,$$

whence

$$a^2 + b^2 + c^2 = \frac{3}{2} + \frac{2}{3} + 6 = \frac{49}{6}.$$

27 By the power-of-a-point formulæ or by looking at similar triangles,

$$XN \cdot XN' = XM \cdot XM' \implies XM = \frac{6 \cdot 12}{XM + 10} \implies XM = \sqrt{97} - 5.$$

28 After the first jump, the bug will be at IV. After the second, it will be at VI. After the third, it will be at I. After the fourth, it will be at II. After the fifth, it will be at IV, and now the pattern continues in cycles of four: for numbers leaving remainder 0 upon division by 4, the bug is at II; for numbers leaving remainder 1 upon division by 4, the bug is at IV; for numbers leaving remainder 2 upon division by 4, the bug is at VI; for numbers leaving remainder 3 upon division by 4, the bug is at I. Since 2009 leaves remainder 1 when divided by four, the bug will be at IV.

29 Observe that the entry situated on the first column and the k-th row is $2008(k-1)+1$, $1 \le k \le 2008$. Let a (depending on k) be the difference between the chosen entry on this row and $2008(k-1)+1$. Then a attains every integral value between 0 and 2007 exactly once. Letting $n = 2008$ the desired sum is a sum of entries of the form $n(k-1)+1+a$ where a and $k-1$ attain every integral value between O and 2007. The desired sum is thus

$$(1+(n+1)+(2n+1)+\cdots+(n-1)n+1)+(0+1+\cdots+(n-1)) = n(n^2+1)/2.$$

Putting $n = 2008$ we obtain $\frac{2008\cdot(2008^2+1)}{2} = 4048193260.$

30 $\triangle BOA'$ is isosceles at B and so $\widehat{BOA'} = \widehat{BA'O} := \alpha$. Now, $\widehat{A'BO} = 180° - 2\alpha$, which means that $\widehat{B'BA'} = 2\alpha$. Since $\triangle B'A'B$ is isosceles at A', we must have $\widehat{BB'A'} = \widehat{B'BA'} = 2\alpha$. Using a completely symmetrical argument, $\widehat{AA'B'} = 2\alpha$. Hence $\triangle A'OB'$ is isosceles at O with angle sum

$$\alpha + 2\alpha + 2\alpha = 180° \implies \alpha = 36°.$$

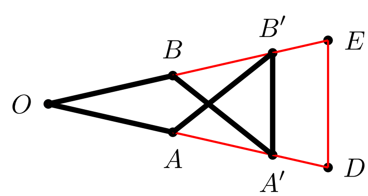


Figure 11: Problem **??**.