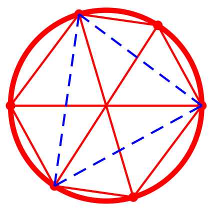
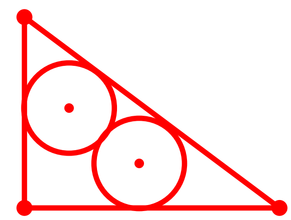
The Eighth Colonial Mathematics Challenge took place Thursday, 24 April 2008. There were 79 students representing nine schools participating. The individual contest winners, in a 3-way tie for First Place, were Van Dinh, *Masterman*, Yongfeng Gao, *South Philadelphia* and Tex Kubacki, *Masterman*. The Team Contest winners were from *Masterman Team* #2: Manjima Dhar, Patrick Henry, Robert Johnson, Spencer Katz

Eighth Colonial Mathematics Challenge





Mathematics Department | 24 April 2008 | Community College of Philadelphia

Name:	School:
Sex (Optional):	Age (Optional):
1:	11:
2:	12:
3:	13:
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6:	16:
7:	17:
8:	18:
9:	19:
10:	20:

1 Find the value of

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}.$$

2 In the magic square, the three numbers in each row, in each column, and in each diagonal add up to the same number. When the magic square below is completed, what is the sum of the missing entries?

13		
	10	
9		7

3 Given that there is a unique digit $d \in \{0,1,2,3,4,5,6,7,8,9\}$ so that the nine-digit number $19700019d$ is prime, find it.

4 Gabriel walks along a level road and then up a mountain. Upon reaching the summit, he immediately turns and walks back to his starting point. He walks 4 km/h on level ground, 3 km/h uphill, and 6 km/h downhill. If the entire walk takes 6 hours, what was the distance covered, in km, over the entire trajectory?

5 A greyhound chases a rabbit which is 60 rabbit-jumps ahead of it. Three greyhound-jumps are equivalent in length to seven rabbit-jumps. While the greyhound executes 6 jumps, the rabbit executes 9. How many greyhound-jumps must the greyhound execute in order to catch up the rabbit?

6 The positive integers a, b, c satisfy the equations

$$a^2 - b^2 - c^2 = 101, \qquad bc = 72.$$

Find the value of their sum $a + b + c$.

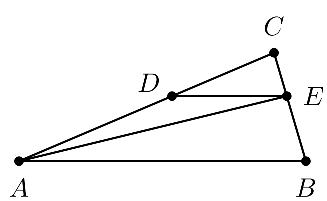
7 Determine the sum of the 23 fractions:

$$\frac{2}{1 \cdot 2 \cdot 3} + \frac{2}{2 \cdot 3 \cdot 4} + \frac{2}{3 \cdot 4 \cdot 5} + \cdots + \frac{2}{23 \cdot 24 \cdot 25}.$$

8 How many different sums can be made when two non-necessarily distinct numbers from the set $\{1, 3, 4, 5, 7\}$ are taken?

9 How many degrees does the hour hand travel when running from $2:30$ AM to $2:50$ AM?

10 In the figure below, the segment AE bisects $\angle A$ of $\triangle ABC$. Given that $DE \parallel AB$, $CD = 4$, $CE = 3$ and $DE = 6$, find the length of EB.



11 Find the least positive integer n satisfying the inequality

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{10}.$$

12 Judith was imprisoned by a band of mathematicians and sent to Guantánamo for crimes against Mathematics. Through the mercy of the Brahmin mathematician, she was given the choice of being released after 10 years or be given freedom if she climbed the 100 steps of a 100-step staircase subject to the following rules:

1. She climbs up or down only one step per day.
2. She climbs up on every day of January, March, May, July, September, and November.
3. She goes down on every day of February, April, June, August, October, and December.

Being adept at climbing, she chose this later option. If Judith started on January 1 2001, when will she gain her freedom? Give the date in the form MM-DD-YYYY.

13 Let $A = \underbrace{9\ldots9}_{30 \text{ nines}}$ and $B = \underbrace{9\ldots9}_{20 \text{ nines}}$. How many 8's are in the product AB?

14 Find the least value of $abc + def$, where $\{a, b, c, d, e, f\} = \{1, 2, 3, 4, 5, 6\}$.

15 A *rep-digit* is an integer all whose digits are the same. For example, 1, 222, and 9999999 are rep-digits. How many digits has the smallest positive rep-digit that is divisible by 847?

16 The set S is formed according to the following rules:

1. 2 belongs to S;
2. if n is in S then $n+5$ is also in S;
3. if n is in S then $3n$ is also in S.

Find the largest integer in the set

$$\{1, 2, 3, \ldots, 2008\}$$

that does not belong to S.

17 How many polynomials $p(x)$ of degree at least one and integer coefficients satisfy

$$16p(x^2) = (p(2x))^2,$$

for all real numbers x?

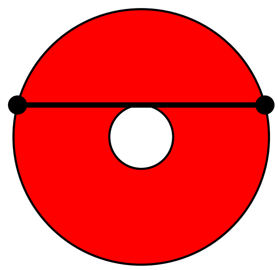
18 The gluttonous race of *Sweet-toothers* is planning to conquer planet *Caramel*. Planet Caramel is in the form of a cube, with an edge of 1 km. The planet is enveloped by a dense anti-glutton foam atmosphere in order to impede the landing of the Sweet-toothers to at least $\frac{1}{2}$ km from the surface of the planet. What is the least volume of the foam that will accomplish this?

19 Recall that in a triangle, the orthocenter is the point of concurrency of the altitudes of the triangle and the circumcenter is the point of concurrency of the perpendicular bisectors to the sides of the triangle. $\triangle ABC$ has orthocenter H, circumcenter O, M as the midpoint of segment BC and F as the foot of the altitude from A. If $HOMF$ is a rectangle with $HO = 11$ and $OM = 5$, compute the length of BC.

20 Five burglars stole a purse with gold coins. The five burglars took each different amounts according to their meanness, with the meanest among the five taking the largest amount of coins and the meekest of the five taking the least amount of coins. Unfair sharing caused a fight that was brought to an end by an arbitrator. He ordered that the meanest burglar should double the shares of the other four burglars below him. Once this was accomplished, the second meanest burglar should double the shares of the other three burglars below him. Once this was accomplished, the third meanest burglar should double the shares of the two burglars below him. Once this was accomplished, the fourth meanest burglar should double the shares of the meekest burglar. After this procedure was terminated, each burglar received the same amount of money. How many coins were in a purse if the meanest of the burglars took 240 coins initially?

21 (**3 minutes**) Find the smallest positive integer n for which for which n^2+n+11 is a composite integer.

22 (**3 minutes**) The two circles below are concentric. The chord shown is tangent to the inner circle and has length $2a$. What is the area of the annular region shaded?



23 (**4 minutes**) Compute

$$\frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{3}-\frac{1}{4}}\cdot\frac{\frac{1}{4}-\frac{1}{5}}{\frac{1}{5}-\frac{1}{6}}\cdot\frac{\frac{1}{6}-\frac{1}{7}}{\frac{1}{7}-\frac{1}{8}}\cdots\frac{\frac{1}{48}-\frac{1}{49}}{\frac{1}{49}-\frac{1}{50}}.$$

24 (**6 minutes**) Given that $2^{32}-1$ has exactly two divisors a and b satisfying the inequalities

$$50<a<b<100,$$

find the product ab.

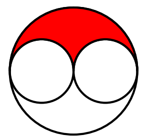
25 (**4 minutes**) If $49^x+49^{-x}=7$, find 7^x+7^{-x}.

26 (**7 minutes**) Suppose there are constants A, B, C, D such that the equality

$$x^3=A+B(x-3)+C(x-3)(x-2)+D(x-3)(x-2)(x-1)$$

holds for all real values of x. Find the value of $A+B+C+D$.

27 (**3 minutes**) Consider three circles as shown in the figure below. The two smaller circles are congruent and mutually tangential, their point of tangency being the center of the large circle. They are also internally tangent to the larger circle. If the area of the shaded region is 2π, find the area of the larger circle.



28 (**5 minutes**) The positive real numbers r, s, t, u satisfy

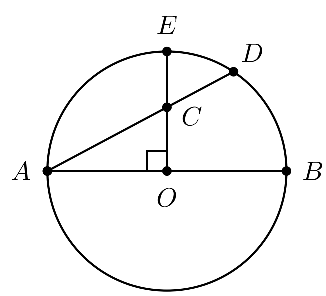
$$\frac{5r+s}{5t+u} = \frac{6r+s}{6t+u}, \qquad \frac{7r+s}{7t+u} = 9.$$

Find the value of

29 (**4 minutes**) Find all real numbers x satisfying the equation

$$\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} = 3.$$

30 (**7 minutes**) The circle with center O below has radius 15. The radius EO is perpendicular to the diameter AB. AD meets OE at C. If $OC = 8$, find the length of CD.



31 (**5 minutes**) Let a and b be, respectively, the largest and the smallest integers in the set $\{1, 2, 3, \ldots, 100\}$ for which $x^{2009} + x^{2008}$ is a square. Find $a + b$.

32 **3 minutes** A textbook is opened at random. If the product of the numbers of the two facing pages is 930, what is the sum of the page numbers?

33 **3 minutes** The students in Ms. Smith's gym class stand equally spaced around a circle. John is standing directly across from Linda. They count off around the circle starting with Mary. That is, Mary says **"one"** and the next person says **"two"** and continue counting off all the way around the circle. If John says **"seven"** and Linda says

"thirty-three", how many students are in the circle?

34 3 minutes Multiply and simplify the product

$$(x-a)(x-b)(x-c)\cdots(x-y)(x-z).$$

1 Plainly,

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}}=\frac{1}{1+\frac{1}{1+\frac{1}{2}}}=\frac{1}{1+\frac{1}{\frac{3}{2}}}=\frac{1}{1+\frac{2}{3}}=\frac{1}{\frac{5}{3}}=\frac{3}{5}.$$

2 The completed square follows.

13	6	11
8	10	12
9	14	7

The sum of the missing entries is thus $6+8+12+14+11=51$.

3 d cannot be even, as then the number would be divisible by 2. Now, $1+9+7+0+0+0+1+9+d=27+d$. If $d\in\{3,6,9\}$, then the sum of the digital sum would be divisible by 3, and so the number would be divisible by 3. If $d=5$, the number would be divisible by 5. If $d=7$, the number would be divisible by 197. This leaves $d=1$ as the only possible digit, and so it must be this one.

4 Let d be the distance covered from the starting point along the level road to the foot of the mountain, and d' the distance from the foot of the mountain to the summit. We want $2(d+d')$. Now,

$$6=\frac{2d}{4}+\frac{d'}{3}+\frac{d'}{6}\implies 6=\frac{d+d'}{2}\implies 2(d+d')=24.$$

He covered 24 km.

5 Let g be the number of greyhound-jumps that it takes the greyhound to catch up the rabbit, and let r be the number of rabbit-jumps that the rabbit executes until it is caught. The given data yields

$$\frac{g}{6}=\frac{r}{9},\quad \frac{g}{60+r}=\frac{3}{7}\implies g=72, r=108.$$

So, 72 greyhound-jumps.

6 We have

$$a^2 - b^2 - 2bc - c^2 = 101 - 2 \cdot 72 = -43 \implies a^2 - (b+c)^2 = -43 \implies (a-b-c)(a+b+c) = -43.$$

Since 43 is prime, we must have $a + b + c = 43$.

7 Observe that

$$\frac{2}{(k-1)k(k+1)} = \frac{1}{(k-1)k} - \frac{1}{k(k+1)},$$

from where

$$\begin{aligned} \frac{2}{1\cdot 2\cdot 3} + \frac{2}{2\cdot 3\cdot 4} + \frac{2}{3\cdot 4\cdot 5} + \cdots + \frac{2}{23\cdot 24\cdot 25} &= \left(\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}\right) + \left(\frac{1}{2\cdot 3} - \frac{1}{3\cdot 4}\right) + \cdots \left(\frac{1}{23\cdot 24} - \frac{1}{24\cdot 25}\right) \\ &= \frac{1}{1\cdot 2} - \frac{1}{24\cdot 25} \\ &= \frac{299}{600}. \end{aligned}$$

8 From the addition table

$+$	1	3	4	5	7
1	2	4	5	6	8
3	4	6	7	8	10
4	5	7	8	9	11
5	6	8	9	10	12
7	8	10	11	12	14

the different sums belong to the set $\{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14\}$, and so there are eleven different sums.

9 10°. When running from $2:30$ AM to $2:50$ AM, the minute hand travels $\dfrac{20}{60} = \dfrac{1}{3}$ of a circumference. The hour hand, proportionally, travels $\dfrac{1}{3}$ of the way of the 30° that there are between 2 and 3, that is, 10°.

10 We have $\triangle ABC \sim \triangle DEC$ and so corresponding sides are proportional. Thus $\frac{AB}{CA} = \frac{DC}{DE} = \frac{6}{4} = \frac{3}{2}$. By the angle bisector theorem,

$$\frac{AB}{CA} = \frac{BE}{EC} \implies BE = \frac{AB}{CA} \cdot EC = \frac{3}{2} \cdot 3 = \frac{9}{2}.$$

11 From the identity $x^2 - y^2 = (x - y)(x + y)$ and using the fact that $\sqrt{n} < \sqrt{n+1}$, we obtain

$$n + 1 - n = 1 \implies (\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) = 1 \implies \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} > \frac{1}{2\sqrt{n+1}}.$$

Hence,

$$\frac{1}{2\sqrt{n+1}} < \frac{1}{10} \implies 5 < \sqrt{n+1} \implies 25 < n + 1 \implies n > 24.$$

Since $5.1^2 = 26.01 > 26$, we have $\sqrt{26} < 5.1$. Hence,

$$\sqrt{26} - \sqrt{25} < 5.1 - 5 = \frac{1}{10},$$

and so $n = 25$ fulfills the inequality.

12 Let us see what happens in a typical non-leap year, and in a typical leap year.

In a non-leap year

by the end of	she has climbed
31 January	31 steps
28 February	$31 - 28 = 3$ steps
31 March	$3 + 31 = 34$ steps
30 April	$34 - 30 = 4$ steps
31 May	$31 + 4 = 35$ steps
30 June	$35 - 30 = 5$ steps
31 July	$31 + 5 = 36$ steps
31 August	$36 - 31 = 5$ steps
30 September	$30 + 5 = 35$ steps
31 October	$35 - 31 = 4$ steps
30 November	$30 + 4 = 34$ steps
31 December	$34 - 31 = 3$ steps

In a leap year

by the end of	she has climbed
31 January	31 steps
29 February	$31 - 29 = 2$ steps
31 March	$2 + 31 = 33$ steps
30 April	$33 - 30 = 3$ steps
31 May	$31 + 3 = 34$ steps
30 June	$34 - 30 = 4$ steps
31 July	$31 + 4 = 35$ steps
31 August	$35 - 31 = 4$ steps
30 September	$30 + 4 = 34$ steps
31 October	$34 - 31 = 3$ steps
30 November	$30 + 3 = 33$ steps
31 December	$33 - 31 = 2$ steps

Now, $100 - 36 = 64$. Let us see how many years it takes her to climb 64 steps. By the end of the four-year range $2001 - 2004$, she climbs $3 + 3 + 3 + 2 = 11$ steps. By the end of the four-year range $2005 - 2008$, she has climbed 22 steps. By the end of the four-year range $2009 - 20012$, she has climbed 33 steps. By the end of the four-year range $2013 - 2016$, she has climbed 44 steps. By the end of the four-year range $2017 - 2020$, she has climbed 55 steps. By the end of the four-year range $2021 - 2024$, she has climbed 66 steps. In fact, by 31 December 2023 she has climbed 64 steps, and by 31 July 2024 she has climbed $64 + 35 = 99$ steps. This means that she needs to go into 2025. By 31 March 2025 she has climbed $66 + 34 = 100$ steps and she is now free. Answer: 03-31-2025.

13 We have

$$A = 10^{30} - 1, \quad B = 10^{20} - 1 \implies AB = (10^{30} - 1)(10^{20} - 1) = 10^{50} - 10^{30} - 10^{20} + 1.$$

Now,

$$T := (10^{50} + 1) - 10^{30} = \underbrace{9\ldots9}_{\text{20 nines}} \; \underbrace{0\ldots0}_{\text{29 zeroes}} \; 1$$

and

$$T - 10^{20} = \underbrace{9\ldots9}_{\text{19 nines}} \; 89999999999 \; \underbrace{0\ldots0}_{\text{19 zeroes}} \; 1,$$

so there is only one 8.

14 By the Arithmetic-Mean-Geometric Inequality,

$$M + N \geq 2\sqrt{MN} = 2\sqrt{abcdef} = 2\sqrt{6!} \approx 53.67,$$

and thus $M \approx N \approx 26.83$. There will never be equality, since $M + N$ is an integer and $2\sqrt{6!}$ irrational. One achieves the minimum by letting $\{a, b, c\} = \{1, 5, 6\}$ and $\{d, e, f\} = \{2, 3, 4\}$, giving

$$1 \cdot 5 \cdot 6 + 2 \cdot 3 \cdot 4 = 54 = \lceil 2\sqrt{6!} \, \rceil.$$

We also obtain 54 on taking $\{a, b, c\} = \{1, 4, 6\}$ and $\{d, e, f\} = \{2, 3, 5\}$.

15 Let $a > 0$ be a digit. The integer sought has the form

$$N = \underbrace{a\ldots a}_{n \text{ times}} = \frac{a}{9} \cdot (10^n - 1).$$

Since $847 = 7 \cdot 11^2$, the integer sought must be divisible by 7 and by 11. Since it is divisible by 11, n must be even, say $n = 2k$. But then $\dfrac{N}{11}$ is also divisible by 11 and

$$\frac{N}{11} = \underbrace{0a0a\ldots0a}_{k \text{ times}}$$

indicates that k must be a multiple of 11. Thus the smallest possible n is $n = 2 \cdot 11 = 22$.

16 Observe that applying k times the second rule, $n+5k$ is in S. Similarly, $3^k \cdot 2$ is in S by applying k times the third rule. Since 2 is in S, $2+5k$ is in S, that is, numbers that leave remainder 2 upon division by 5 are in S. This means that

$$\{2, 7, 12, \cdots 2002, 2007\} \subseteq S.$$

Since $3 \cdot 2 = 6$ is in S, then the numbers $6 + 5k = 1 + 5(k+1)$ are in S, that is, numbers 6 or higher that leave remainder 1 upon division by 5. Thus the numbers

$$\{6, 11, 16, \cdots 2001, 2006\} \subseteq S.$$

Since $3 \cdot 6 = 18$ is in S, then the numbers $18 + 5k = 3 + 5(k+3)$ are in S, that is, numbers 18 or higher that leave remainder 3 upon division by 5. Thus the numbers

$$\{18, 23, 28, \cdots 2003, 2008\} \subseteq S.$$

Since $3 \cdot 18 = 54$ is in S, then the numbers $54 + 5k = 4 + 5(k+10)$ are in S, that is, numbers 54 or higher that leave remainder 4 upon division by 5. Thus the numbers

$$\{54, 59, 64, \cdots 2004\} \subseteq S.$$

Now, we claim that there are no multiples of 5 in S. For by combining the rules every number in S has the form $3^a \cdot 2 + 5b$, with $a \geq 0$, $b \geq 0$ integers. Since $3^a \cdot 2$ is never a multiple of 5, this establishes the claim. Hence the largest element of

$$\{1, 2, 3, \ldots, 2008\}$$

not in S is 2005.

17 Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_n \neq 0$, $n \geq 1$. Then

$$16p(x^2) = (p(2x))^2 \implies 16(a_n x^{2n} + a_{n-1}x^{2n-2} + \cdots + a_1 x^2 + a_0) = \left(2^n a_n x^n + 2^{n-1}a_{n-1}x^{n-1} + \cdots + 2a_1 x + a_0\right)^2$$

Since the coefficients on both sides of the equality must agree, we must have

$$16a_n = 2^{2n}a_n^2 \implies 2^4 = 2^{2n}a_n$$

since $a_n \neq 0$. As a_n is an integer, we must have the following cases: $n = 1, a_n = 4$, $n = 2, a_n = 1$. Clearly we may not have $n \geq 3$. Thus such polynomials are either linear or quadratic. Also, for $x = 0$, $16p(0) = (p(0))^2$ and therefore either $p(0) = 0$ or $p(0) = 16$.

For $n = 1$ we seek $p(x) = 4x + a$. Solving

$$16(4x^2 + a) = (8x + a)^2 \implies a = 0,$$

whence $p(x) = 4x$.

For $n = 2$, let $p(x) = x^2 + ax + b$. Solving

$$16(x^4 + ax^2 + b) = (4x^2 + 2ax + b)^2 \implies a = 0.$$

Since $p(0) = 0$ or $p(0) = 16$, we must test $p(x) = x^2$ and $p(x) = x^2 + 16$. It is easy to see that only $p(x) = x^2$ satisfies the desired properties.

In conclusion, $4x$ and x^2 are the only two such polynomials..

18 The cloud must be made of at least six parallelepipeds of dimension $\dfrac{1}{2} \cdot 1 \cdot 1$ km^3 , contiguous to each face of the cube; twelve quarter-cylinders of height 1 km and radius $\dfrac{1}{2}$ km, each aligned to the edges of the cube; and eight eights-of-a-sphere, each of radius $\dfrac{1}{2}$ km, with center at the eight vertices of the cube. Thus the minimum volume is

$$6 \cdot \frac{1}{2} \cdot 1 \cdot 1 + 12 \cdot \frac{1}{4} \cdot \pi \cdot 1^2 \cdot \frac{1}{2^2} + 8 \cdot \frac{1}{8} \cdot \frac{4}{3} \cdot \pi \cdot \frac{1}{2^3} = 3 + \pi\frac{11}{12}\,\text{km}^3.$$

19 The centroid G lies on the line $\overleftrightarrow{HO}$ (this is the *Euler line*) and since $GM = \frac{1}{3}AM$, we must have $HF = \frac{1}{3}AF$, whence $AF = 15$. Triangles $\triangle BFH$ and $\triangle AFC$ are similar because they are both right triangles with $\angle HBC =$

$90^\circ - \angle BCA = \angle CAF$. This yields

$$BF \cdot FC = FH \cdot AF = 75.$$

Hence,

$$BC^2 = (BF - FC)^2 + 4BF \cdot FC = (2MF)^2 + 4 \cdot 75 = 22^2 + 300 = 784$$

and so $BC = \sqrt{784} = 28$.

20 Let there be a coins in the purse. There are five stages. The fifth stage is when all the burglars have the same amount of money. Let

$$a_k, b_k, c_k, d_k, e_k$$

be the amount of money that each burglar has, decreasing lexicographically, with the a's denoting the amount of the meanest burglar and e_k denoting the amount of the meekest burglar. Observe that for all k we have

$$a_k + b_k + c_k + d_k + e_k = a.$$

On stage five we are given that

$$a_5 = b_5 = c_5 = d_5 = e_5 = \frac{a}{5}.$$

On stage four

$$a_4 = b_4 = c_4 = \frac{a}{5}, \quad d_4 = \frac{a}{5} + \frac{a}{10} = \frac{3a}{10} \quad e_4 = \frac{a}{10}.$$

On stage three we have

$$a_3 = b_3 = \frac{a}{5}, \qquad c_3 = \frac{a}{5} + \frac{3a}{20} + \frac{a}{20} = \frac{2a}{5} \quad d_3 = \frac{3a}{20} \quad e_3 = \frac{a}{20}.$$

On stage two we have

$$a_2 = \frac{a}{5}, \quad b_2 = \frac{a}{5} + \frac{a}{5} + \frac{3a}{40} + \frac{a}{40} = \frac{a}{2}, \quad c_2 = \frac{a}{5} \quad d_2 = \frac{3a}{40} \quad e_2 = \frac{a}{40}.$$

On stage one we have

$$a_1 = \frac{a}{5} + \frac{a}{4} + \frac{a}{10} + \frac{3a}{80} + \frac{a}{80} = \frac{3a}{5}, \quad b_1 = \frac{a}{4}, \quad c_1 = \frac{a}{10} \quad d_1 = \frac{3a}{80} \quad e_1 = \frac{a}{80}.$$

Since $a_1 = 240$, we deduce $\dfrac{3a}{5} = 240 \implies a = 400$.

Check: On the first stage the distribution is (from meanest to meekest):

$$240, \quad 100, \quad 40, \quad 15, \quad 5.$$

On the second stage we have

$$80, \qquad 200, \quad 80 \quad 30, \quad 10.$$

On the third stage we have

$$80, \quad 80, \quad 160, \quad 60, \quad 20.$$

On the fourth stage we have

$$80, \quad 80, \quad 80, \quad 120, \quad 40.$$

On the fifth stage we have

$$80, \quad 80, \quad 80, \quad 80, \quad 80.$$

21 For $n = 0$ through $n = 10$ the sequence of outputs is

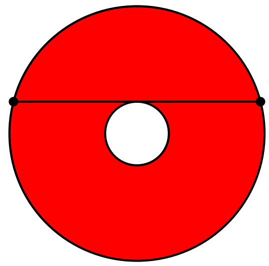
$$11, 13, 17, 23, 31, 41, 53, 67, 83, 101, 121,$$

of which all are prime, except 121. Thus the smallest such n is $n = 10$.

22 Let R be the radius of the larger circle, and r the radius of the smaller circle. The area of the annulus is $\pi(R^2 - r^2)$.

Using the Pythagorean Theorem, $R^2 = r^2 + a^2$, whence the area is

$$\pi(R^2 - r^2) = \pi a^2.$$



23 We have,

$$\begin{aligned}
\frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{3}-\frac{1}{4}}\cdot\frac{\frac{1}{4}-\frac{1}{5}}{\frac{1}{5}-\frac{1}{6}}\cdot\frac{\frac{1}{6}-\frac{1}{7}}{\frac{1}{7}-\frac{1}{8}}\cdots\frac{\frac{1}{48}-\frac{1}{49}}{\frac{1}{49}-\frac{1}{50}} &= \frac{\frac{1}{2\cdot3}}{\frac{1}{3\cdot4}}\cdot\frac{\frac{1}{4\cdot5}}{\frac{1}{5\cdot6}}\cdot\frac{\frac{1}{6\cdot7}}{\frac{1}{7\cdot8}}\cdots\frac{\frac{1}{48\cdot49}}{\frac{1}{49\cdot50}} \\
&= \frac{\frac{1}{2}}{\frac{1}{50}} \\
&= 25.
\end{aligned}$$

24 We have

$$\begin{aligned}
2^{32}-1 &= (2^{16}-1)(2^{16}+1) \\
&= (2^{8}-1)(2^{8}+1)(2^{16}+1) \\
&= (2^{4}-1)(2^{4}+1)(2^{8}+1)(2^{16}+1) \\
&= (2^{2}-1)(2^{2}+1)(2^{4}+1)(2^{8}+1)(2^{16}+1) \\
&= (2-1)(2+1)(2^{2}+1)(2^{4}+1)(2^{8}+1)(2^{16}+1).
\end{aligned}$$

Since $2^8+1=257$, a and b must be part of the product

$$(2-1)(2+1)(2^2+1)(2^4+1)=255=3\cdot5\cdot17.$$

The only divisors of 255 in the desired range are $3\cdot17=51$ and $5\cdot17=85$, whence the desired product is $51\cdot85=4335$.

25 We have

$$\left(7^x+7^{-x}\right)^2=49^x+49^{-x}+2=7+2=9 \implies 7^x+7^{-x}=3,$$

since $7^x + 7^{-x}$ is positive.

26 If $x = 3$ then

$$27 = A.$$

If $x = 2$ then

$$8 = A - B \implies B = A - 8 = 19.$$

If $x = 1$ then

$$1 = A - 2B + 2C \implies C = \frac{1 - A + 2B}{2} = \frac{1 - 27 + 38}{2} = 6.$$

Now, D must be equal to 1, since it is the coefficient of x^3 in both dextral and sinistral side. Hence,

$$A + B + C + D = 27 + 19 + 6 + 1 = 53.$$

27 Let the smaller circles have radius r. Then the large circle has radius $2r$ and area $\pi(2r)^2 = 4\pi r^2$. Double the shaded area is 4π, which is the area of the larger circle minus the area of the smaller circles. Thus

$$4\pi = 4\pi r^2 - 2(\pi r^2) = 2\pi r^2 \implies 4\pi r^2 = 8\pi,$$

whence the area of the larger circle is 8π.

28 If $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$ and $(a+c)d = (b+d)c$ and so $\frac{a+c}{b+c} = \frac{c}{d}$, that is adding the numerators and denominators of fractions in proportion, keeps the proportion. Hence, if $\frac{5r+s}{5t+u} = k = \frac{6r+s}{6t+u}$, then

$$k = \frac{(6r+s)-(5r+s)}{(6t+u)-(5t+u)} = \frac{r}{t}.$$

Since then $\frac{5r}{5t} = k$, this in turn gives

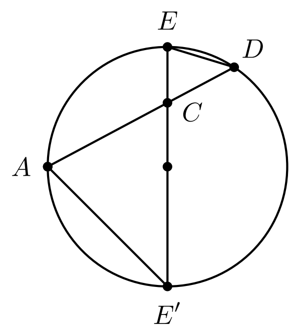
$$k = \frac{5r + s - 5r}{5t + u - 5t} = \frac{s}{t}.$$

From $9 = \frac{7r+s}{7t+u}$ we deduce $k = 9$ and so $\frac{9t+u}{9r+s} = \frac{1}{k} = \frac{1}{9}$.

29 We have

$$\begin{aligned}\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}=3 \quad &\implies \quad \sqrt{x+1}+\sqrt{x-1}=3\sqrt{x+1}-3\sqrt{x-1}\\ &\implies \quad \sqrt{x+1}=2\sqrt{x-1}\\ &\implies \quad x+1=4x-4\\ &\implies \quad x=\frac{5}{3}.\end{aligned}$$

30 By considering $\triangle AOC$, by the Pythagorean Theorem, $AC = 17$. $CE = OE - OC = 15 - 8 = 7$. Let E' be diametrically opposite to E. Then $E'C = 23$.



From the power of a point formulæ or by arguing that $\triangle DCE \sim \triangle E'CA$,

$$AC \cdot CD = EC \cdot CE' \implies CD = \frac{7\cdot 23}{17} = \frac{161}{17}.$$

31 Since $x^{2009} + x^{2008} = x^{2008}(1 + x)$ and x and $1 + x$ are relatively prime, $1 + x$ must be a square. Thus $a = 99$ and $b = 3$.

32 If x and $x+1$ are the pages, then $x(x+1) = 930$. Roughly, $x^2 \approx 930 \implies x \approx 30$. One can easily check that $30 \cdot 31 = 930$ and the desired sum is $30 + 31 = 61$.

33 Draw a line passing through John and Linda. Persons 8 through 32 constitute $32 - 8 + 1 = 25$ people on one side of the John and Linda line. There must be another 25 on the other side of the line, so there is a total of $25+25+2 = 52$ students.

34 The 24th term of the expression is $x - x$, hence the product is 0.