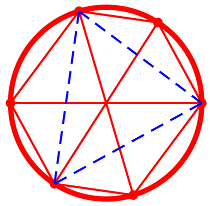
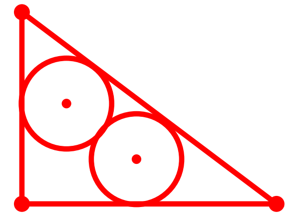
Sixth Colonial Mathematics Challenge





Mathematics Department **Community College of Philadelphia**

The Sixth Colonial Maths Challenge took place 1 March 2007. The individual competition was won by Mary Kate Daley of the Philadelphia Academy Charter High School. Samantha Fargas of the Philadelphia Academy Charter High School and Ja'saun Hoggard of the Fitzsimons High School tied for the second place. The team competition was won by the Bok Technical High School team, whose team members were: Latasha Harrison, Chhay Heng, Ly Chhay Kim, Jasper Marshall, Huong Phan, Leangheng Sou, and Vy To.

The following members of the Mathematics Department helped running the competition: Elena Koublanova, Joanne Darken, Mohammed Teymour, Robert Smith, Howard Wachtel, Dan Jacobson, David Santos. Our former student, Kyle Hofler, also contributed time.

Name:	**School:**
Sex (Optional):	**Age (Optional):**
1:	**11:**
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8:	**18:**
9:	**19:**
10:	**20:**

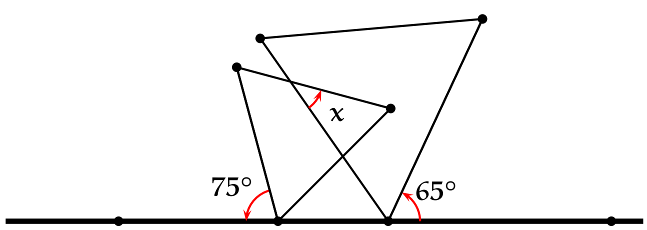
1 How many perfect cubes are there between 2 and 100?

2 Write as a fraction in lowest terms:

$$\frac{25^5 + 25^5 + 25^5}{5^{10} + 5^{10} + 5^{10} + 5^{10} + 5^{10}}.$$

3 One side of an isosceles triangle measures 1 and another measures 3. What is the length of the missing side?

4 In the figure below, both triangles are equilateral. Find the angle x, in degrees.



5 The sum of two numbers is 2 and the sum of their reciprocals is 6. What is the sum of their squares?

6 A *palindrome* is a strictly positive integer whose decimal expansion is symmetric. The sequence of palindromes is written in increasing order

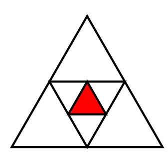
$$1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, \ldots.$$

Thus 11 occupies the tenth position, 22 the eleventh, etc. Which position is occupied by 1003001?

7 Peter works in office A and Paul works in office B, which are two different franchises of a messenger company. One day Peter has to deliver a message for office B and Paul one for office A. They start biking towards each opposite office at the same time in a straight line, at constant, but different speeds. After crossing each other, Peter has to travel an extra 16 minutes in order to reach B, and Paul has to travel an extra 9 minutes in order to reach A. If a is the total time of Peter's trip, in minutes, and b is the total time, also in minutes, of Paul's trip, find $a + b$.

8 Three pastures are covered with grass of the same density, which grows uniformly with the same speed. The pastures have areas of $3\frac{1}{3}$, 10, and 24 acres. Bulls are grazing on the pastures while the grass is growing. If the first pasture can feed 12 bulls for 4 weeks and the second pasture can feed 24 bulls for 9 weeks, how many bulls can the third pasture feed for 18 weeks?

9 In the figure below, all appearing triangles are equilateral. If the perimeter of the bounding triangle is P, what is the perimeter of the shaded triangle?



10 There exist two integers $s < t$ such that

$$\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1$$

for all x between s and t. Find $t + s$.

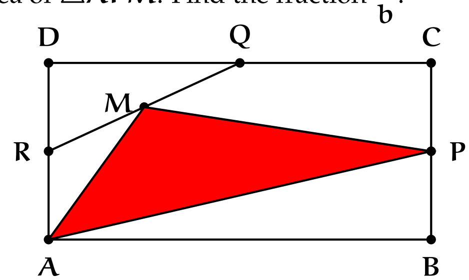
11 Yevgeny loves caviar sandwiches! He prepares them with ingredients bought at the neighborhood grocery, paying for all the ingredients a total of A. But this year the price of bread has increased 5%, butter 26%, and caviar 20%. He has noticed that if he eliminates butter—without changing anything else—the total only increases 60 cents. He also has noticed that if he reduces caviar by 15% —without altering anything else—the total again only increases by 60 cents. What was A?

12 Consider non-zero real numbers a, b, c, and d such that c and d are solutions of $x^2 + ax + b = 0$ and a and b are solutions of $x^2 + cx + d = 0$. Find $a + b + c + d$.

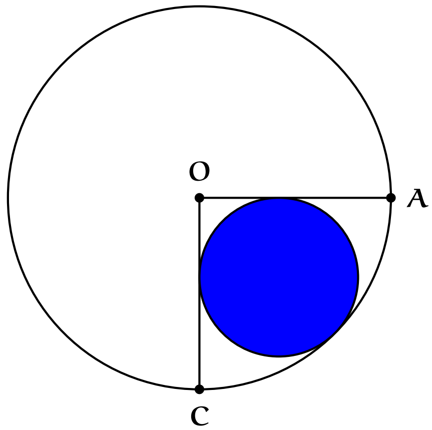
13 Each of the six faces of a cube is painted in a different color. The cube has now a fixed color scheme. A die can be formed by painting the numbers $\{1, 2, 3, 4, 5, 6\}$ in such a way that the opposing faces add up to 7. How many different dice can be formed?

14 If $a^3 + b^3 + c^3 = -4$, $a + b + c = 4$, and $a^2 + b^2 + c^2 = 2$, find abc.

15 In rectangle $ABCD$ of area a, P, Q and R are the midpoints of sides BC, CD and AD, respectively, and M is the midpoint of segment QR. Let b be the area of $\triangle APM$. Find the fraction $\frac{a}{b}$.



16 A circle is inscribed in a quarter circle, as shown in the figure below. If the radius of the larger circle is R, find the radius of the smaller circle.



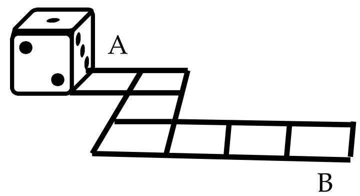
17 The Hydra of Lerna is made of heads and necks, with a neck connecting exactly two heads. With one swing of his sword Hercules can cut off all necks connected to one head, but the head immediately grows necks to all other heads it was previously not connected to (one neck to each head). Hercules can kill the Hydra if he manages to split it into two unconnected parts. What is the smallest number of swings required to kill the Hydra if it has 100 necks?

18 When writing all the integers from 1 to 2007, inclusive, how many 0's are used?

19 For each integer $n > 0$, let n^* be the number obtained by moving the rightmost digit to be the first digit on the left. For example, $9845^* = 5984$. Solve the equation $7x^* = 2x$.

20 A *lattice point* is a coordinate point (x, y) with both x and y integers. How many lattice points (x, y) satisfy $|x| + |y| < 100$?

21 (3 minutes) A die is initially with **1** on its upper face and **2** on its frontal face. It rolls seven times on its sides, starting from point **A** to point **B** as indicated in the figure. How many points will the top show upon reaching point **B**? You may wish to know that the sum of the points of opposite faces in a normal die is 7.



22 (3 minutes) Fill in the blanks so that the sum of the numbers in any three adjacent spots is equal to 10:

1								2

23 (3 minutes) Write

$$\left(\sqrt{5+\sqrt{12}}-\sqrt{5-\sqrt{12}}\right)^2$$

in the form $a + b\sqrt{c}$, where a, b, c are integers, and c is not divisible by a square greater than 1.

24 (4 minutes) Wanda went jogging for a long 5 hour run. First, she ran along a horizontal path, then climbed up the hill and finally returned back along the same path. If Wanda's speed on the horizontal path is 4 mph, the speed during the climb is 3 mph and the speed during the descent is 6 mph, find the total distance of her run.

25 (5 minutes) The hexagon $ABCA'B'C'$ in figure 1 is inscribed in a circumference, such that the diagonals AA', BB' and CC' are diameters of the circumference, and the area of $\triangle ACB'$ is 1. Find the area of the hexagon $[ABCA'B'C']$.

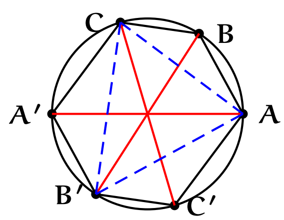


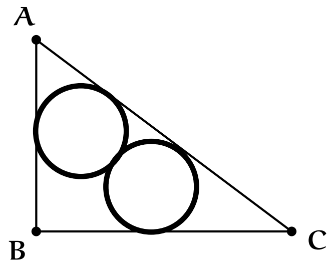
Figure 1: Problem 25.

26 (7 minutes) Write the product of fractions

$$\left(\frac{2^3-1}{2^3+1}\right)\left(\frac{3^3-1}{3^3+1}\right)\left(\frac{4^3-1}{4^3+1}\right)\cdots\left(\frac{100^3-1}{100^3+1}\right)$$

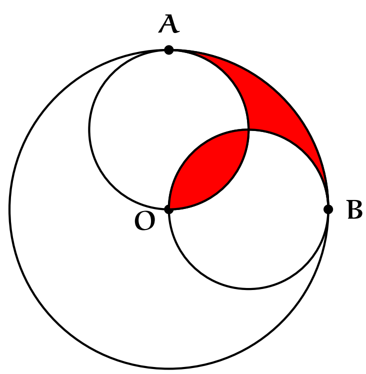
in lowest terms.

27 (9 minutes) Two circles of radius R are externally tangent. They are also internally tangent to a right triangle $\triangle ABC$ where $AB = 3$, $BC = 4$, and $AC = 5$. Moreover, both circles are tangent to the hypotenuse of the triangle, as shown in the figure. Determine the value of R.



28 (5 minutes) How many pairs of integers (x, y) are there for which $x^2 - y^2 = 81$?

29 (6 minutes) $\widehat{AB}$ is a quarter of the circumference of the circle with center O, which has radius R. Arcs $\widehat{OA}$ and $\widehat{OB}$ are half circumferences of the smaller circles, which are both congruent. Find the area of the shaded region.



30 (4 minutes) In the Microsoft software *Excel*, columns are labelled by letters. The first twenty six columns are labelled A through Z, the twenty seventh is labelled AA, the twenty eighth AB, etc. What label appears on the 2007th column?

31 (5 minutes) Let a and b be the solutions to the equation

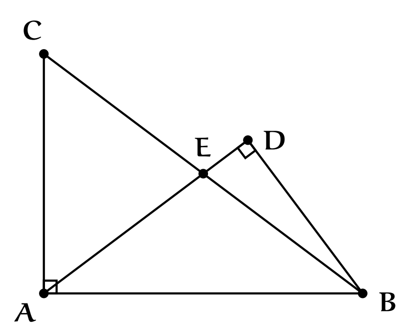
$$x^2 - 2x - 2 = 0.$$

Find

$$a^3 + b^3.$$

32 (4 minutes) Find the greatest multiple of 8 using all ten digits and with all the digits different.

33 (5 minutes) In the following figure, $\triangle ABC$ is rectangle at A, and $\triangle ADB$ is rectangle at D. Point E is the intersection of the segments $[AD]$ and $[BC]$. It is known that $AC = 15$, $AD = 16$, and $BD = 12$. Find the area of $\triangle ABE$.



(4 minutes) Find the product of all the positive solutions x of

$$(x)\left(\sqrt[x]{x^3}\right) = \frac{x^x}{x}.$$

35 (4 minutes) For how many positive integers, n, is $n^2 + n + 2$ a square?

1 There are three: $8, 27, 64$.

2 We have

$$\frac{25^5 + 25^5 + 25^5}{5^{10} + 5^{10} + 5^{10} + 5^{10} + 5^{10}} = \frac{3 \cdot 25^5}{5 \cdot 5^{10}} = \frac{3 \cdot 5^{10}}{5^{11}} = \frac{3}{5}.$$

3 The sum of the lengths of two sides of a triangle should exceed the length of the third side. Thus the missing side cannot measure 1 because $1 + 1 < 3$. Hence it must measure 3.

4 The two triangles form a triangle with angles $180° - 60° - 65° = 55°$, $180° - 60° - 75° = 45°$, and $180° - 55° - 45° = 80°$, with the $80°$ being vertical. Thus $x = 180° - 80° - 60° = 40°$.

5 Let x and y be the numbers. Then $x + y = 2$ and $\dfrac{1}{x} + \dfrac{1}{y} = 6$. Thus

$$6 = \frac{1}{x} + \frac{1}{y} = \frac{y + x}{xy} = \frac{2}{xy} \implies xy = \frac{1}{3}.$$

Now

$$x^2 + y^2 = (x + y)^2 - 2xy = 2^2 - \frac{2}{3} = \frac{10}{3}.$$

6 Observe that there are $9 \times 10^{n-1}$ palindromes of n digits. The number 1003001 has seven digits. After writing 999999, the last palindrome with six digits, one has written

$$9 + 9 + 90 + 90 + 900 + 900 = 1998$$

palindromes. The 1999-th, 2000-th, 2001-st and 2002-nd are thus

$$1000001, 1001001, 1002001, 1003001,$$

and so 1003001 occupies the 2002-nd position.

7 Let s, s' be the respective speeds of messengers A and B. Starting from time 0, suppose it takes them t minutes to cross each other. To complete the trip, A spent $t + 16$ minutes and B spent $t + 9$ minutes. Since each travelled the same distance

$$s(t + 16) = s'(t + 9) \implies \frac{s}{s'} = \frac{t + 9}{t + 16}.$$

What A must complete now in 16 minutes is what B has completed in t minutes:

$$16s = s't \implies \frac{s}{s'} = \frac{t}{16}.$$

This gives

$$\frac{t + 9}{t + 16} = \frac{t}{16} \implies t = 12,$$

since $t \geq 0$. Observe that since what B must now complete in 9 minutes is what A completed in t minutes, we must also have

$$9s' = st \implies \frac{s}{s'} = \frac{9}{t} \implies \frac{9}{t} = \frac{t + 9}{t + 16} \implies t = 12,$$

as before. Thus $a = 12 + 16 = 28$ and $b = 12 + 9 = 21$. Finally, $a + b = 49$.

8 Let x be the initial amount of grass in 1 acre, let y be the amount that grows in 1 week in area of 1 acre, and let z be the amount of grass that is consumed by 1 bull in 1 week. The first condition gives

$$\frac{10}{3} \cdot x + \frac{10}{3} \cdot 4 \cdot y = 12 \cdot 4z \implies 5x + 20y = 72z.$$

The second condition gives

$$10x + 10(9y) = 21(9z) \implies 10x + 90y = 189z.$$

Now,

$$10x + 90y - 2(5x + 20y) = 189z - 2 \cdot 72z \implies 50y = 45z \implies y = \frac{9z}{10},$$

and

$$5x + 20\left(\frac{9z}{10}\right) = 72z \implies x = \frac{54z}{5}.$$

Let n be the number of bulls involved in the third condition. Then

$$24x + 24(18y) = n(18z) \implies 4x + 72y = 3nz.$$

Assembling the above we have

$$n = \frac{4x + 72y}{3z} = \frac{4\left(\frac{54z}{5}\right) + 72\left(\frac{9z}{10}\right)}{3z} \implies n = 36.$$

9 At each stage, the side of the triangle halves. Thus after two iterations, the perimeter of the shaded triangle is $\frac{P}{4}$.

10 We have

$$\begin{aligned}
\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1 &\iff \sqrt{x - 1 - 4\sqrt{x-1} + 4} + \sqrt{x - 1 - 6\sqrt{x-1} + 9} = 1 \\
&\iff \sqrt{(\sqrt{x-1} - 2)^2} + \sqrt{(\sqrt{x-1} - 3)^2} = 1 \\
&\iff \left|\sqrt{x-1} - 2\right| + \left|\sqrt{x-1} - 3\right| = 1
\end{aligned}$$

Put $a = \sqrt{x-1}$. Then

$$|a - 2| + |a - 3| = \begin{cases} 5 - 2a & \text{if } a \leq 2 \\ 1 & \text{if } 2 \leq a \leq 3 \\ 2a - 5 & \text{if } a \geq 3 \end{cases}$$

This means that

$$\left|\sqrt{x-1} - 2\right| + \left|\sqrt{x-1} - 3\right| = 1 \iff 2 \leq \sqrt{x-1} \leq 3 \iff 5 \leq x \leq 10.$$

Thus $t + s = 15$.

11 Let b be the price of bread, u be the price of butter, and c the price of caviar. We are given that

$$\begin{aligned}
1.05b + 0u + 1.2c &= b + u + c + 0.6 \\
1.05b + 1.26u + 1.2 \times 0.85c &= b + u + c + 0.6
\end{aligned}$$

Subtracting the first equation from the second,

$$1.26u - 0.18c = 0 \implies c = 7u.$$

Replacing c by $7u$ in the first equation,

$$1.05b + 8.4b = b + u + 7u + 0.60 \implies b + 8u = 12.$$

The price sought is $b + u + c = b + 8u = 12$ dollars.

12 The sum of the roots of the equation $x^2 + ax + b = 0$ is $-a$ and their product b. If d and c are roots of $x^2 + ax + b = 0$, then we must have

$$c + d = -a, \qquad dc = b.$$

In the same manner, the sum of the roots of the equation $x^2 + cx + d = 0$ is $-c$ and their product d. If a and b are roots of $x^2 + cx + d = 0$, then we must have

$$a + b = -c, \qquad ab = d.$$

Now,

$$dc = b, \quad ab = d \implies abcd = bd \implies ac = 1.$$

Also,

$$a + c = -d, \quad a + c = -b \implies b = d.$$

Since $ab = d$, we deduce $a = 1$, and from $ac = 1$, we gather $c = 1$.

Finally, through addition,

$$a + c = -d, \quad a + c = -b \implies a + b + c + d = -a - c = -2.$$

13 Put the 6 in any of the 6 faces, leaving five faces. You have only one face to put the 1 (opposite of the 6), leaving 4 faces. Put the 4 in any of the 4 remaining faces, leaving 3 faces. You must put the 3 in the opposite face, leaving 2 faces. You can now put the 2 in any of the two remaining faces, and in the last face you put the 5. In total you have $6 \cdot 4 \cdot 2 = 48$ different dice.

14 We have

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \implies 4^2 = 2+2(ab+bc+ca) \implies ab+bc+ca = 7.$$

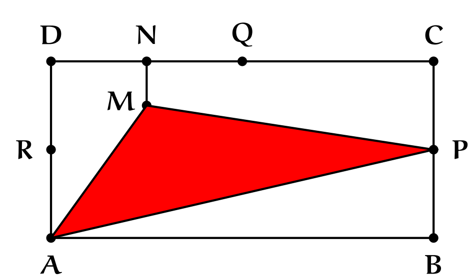
Hence

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \implies -4-3abc = 4(2-7) \implies abc = \frac{16}{3}.$$

15 Let N be the midpoint of the segment $[PQ]$. By similar triangles, $RD = 2MN$, which implies that $MN = \frac{DA}{4}$ and $DN = \frac{CD}{4}$. Observe that $a = (AD)(DC) = (AB)(BC)$ and that

$$\begin{aligned}[ABCD] &= [AMND] + [\triangle APM] + [CPMN] + [\triangle APB] \\ &= \left(\frac{DA+MN}{2}\right) DN + b + \left(\frac{CP+MN}{2}\right) NC + \frac{AB \times PB}{2} \\ &= \left(\frac{5DA}{8}\right)\frac{CD}{4} + b + \left(\frac{3CB}{8}\right)\frac{3CD}{4} + \frac{AB \times BC}{4} \\ &= \frac{5a}{32} + b + \frac{9a}{32} + \frac{a}{4} \\ &= \frac{22a}{32} + b\end{aligned}$$

Thus $\frac{a}{b} = \frac{16}{5}$.

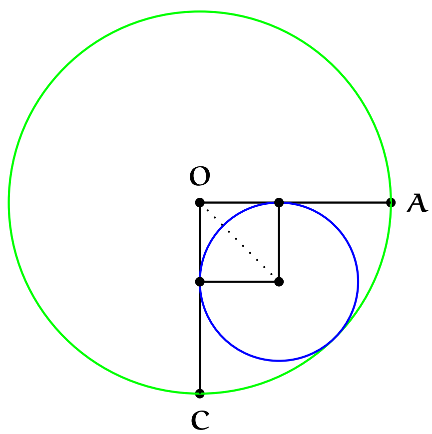


16 By joining the centers of both circles, observe that $R = a + 2r$, where r is the radius of the smaller circle and a is the length of the line segment outside the smaller circle, from the center of the larger circle. From the center of the smaller circle, draw two radii, parallel to the drawn radii of the larger circle, as shown below. Thus we obtain a square of side r, and by the Pythagorean Theorem

$$r^2 + r^2 = (a + r)^2 \implies a = (\sqrt{2} - 1)r,$$

since $a > 0$. Finally,

$$R = a + 2r = (\sqrt{2} - 1 + 2)r \implies r = \frac{R}{\sqrt{2} + 1} = (\sqrt{2} - 1)R.$$



17 We will show that it requires ten swings. If the Hydra has head A connected by not more than 10 necks, then by cutting off all the heads A is connected to, Hercules will separate head A from the rest of the Hydra. If the Hydra has head B connected to all heads except n (n & 9), then by first cutting off head B and then the n heads it was originally not connected to, Hercules will separate head B. If every head has at least 11 necks and, at the same time, is not connected to at least 10 other heads, then there must be at least 11+10+1 = 22 heads and at least $22 \times 11 > 100$ necks. On the other hand, heres an example of Hydra with 100 necks which cannot be killed with 9 swings: 20 heads in two groups of 10, where each head from one group is connected to all heads in the other group. Indeed, after at most 9 swings each group will contain heads connected within its group with all the heads which have not been cut off. Therefore, if at least one head has been cut off in each group, Hydra remains connected. Obviously, if all 9 swings have been used to cut off heads within one group, then Hydra will also remain connected.

18 There are none used when writing the numbers from 1 through 9.

When writing the numbers from 10 to 99, there are 9 zeroes used, when writing $\{10, 20, \ldots, 90\}$.

When writing a three-digit integer ABC (numbers in the 100-999 range), one can use either one or two zeroes. If ABC has exactly one zero, then it is either B or C. If one of B or C is 0, then there are 9 choices for the other and 9 for A. Thus of the numbers ABC there are $9 \cdot 9 \cdot 2 = 162$ that use exactly one 0. If ABC has exactly two 0's then B and C must be 0 and there are 9 choices for A. Those 9 numbers use $2 \cdot 9 = 18$ zeroes. Thus in this range we have used $162 + 18 = 180$ zeroes.

A number in the 1000-1999 range has the form 1ABC. When writing them, one may use exactly one, two, or three zeroes. If there is only exactly one zero, then exactly one of A, B, or C, is 0, and since there are 9 choices for each of the other two letters, one has used $9 \cdot 9 \cdot 3 = 243$ zeroes this way. If there are exactly two zeroes, then either A and B, or A and C, or B and C, are zero, and there are 9 for the remaining letter. Therefore there are $9 \cdot 3 = 27$ numbers with 2 zeroes and $27 \cdot 2 = 54$ zeroes are used. Also, there is exactly one number in the 1000-1999 range using 3 zeroes. Altogether in this range we have used $243 + 54 + 3 = 300$ zeroes in this range.

Finally, in the range 2000-2007, there is one number using 3 zeroes, and 7 numbers using 2 zeroes. Hence in this range we have used $3 + 2 \times 7 = 17$ zeroes.

In summary, we have used

$$9 + 180 + 300 + 17 = 506$$

zeroes.

19 Suppose n has k digits and that $x = 10n + u$. Then $x^* = 10^k u + n$. Now

$$7x^* = 2x \iff 7(10^k u + n) = 2(10n + u) \iff (7 \cdot 10^k - 2)u = (10 \times 2 - 7)n = 13n.$$

But $7 \cdot 10^k - 2 = 6\underbrace{9\ldots9}_{k-1 \text{ nines}}8$. Observe that the smallest value of k for which 13 divides $7 \cdot 10^k - 2$ is $k = 5$. Hence $699998u = 13n$, which gives $n = 53846u$. Taking $u = 1$ we find $n = 53846$ and $x = 5384610 + 1 = 538461$. This satisfies the equation.

20 Consider the following categorization:

1. with (x, y) in the first quadrant
2. $(0, 0)$, the origin,
3. $(x, 0)$ with $1 \leq x \leq 99$.

Thus the number of lattice points sought is four times the number in (1), plus 1, plus four times the number in (3).

Clearly, the number of lattice points in (3) is 99.

The number of (1) is the number of strictly positive solutions to $x + y < 100$. Let $z = 100 - x - y$, the discrepancy of $x + y$ from 100. Then we are counting the number of strictly positive solutions to $x + y + z = 100$. To count these, write 100 as a sum of 100 ones:

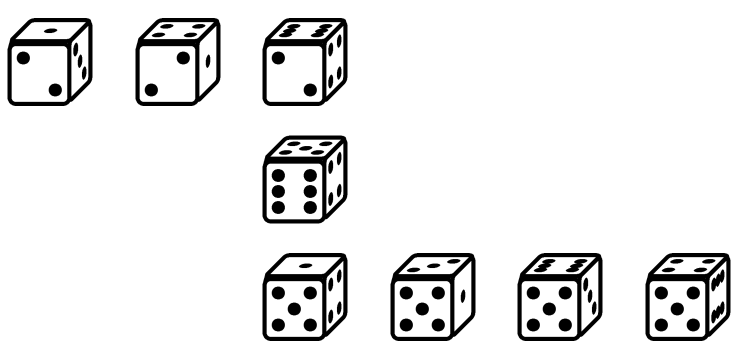
$$\underbrace{1 + 1 + \cdots + 1}_{100 \text{ ones}}.$$

Observe that there are 99 plus signs. Of these, we must choose two, because the equation $x + y + z = 100$ has two. Thus there are $\binom{99}{2} = 4851$ such points.

The required number of points is thus

$$4 \cdot 4851 + 4 \cdot 99 + 1 = 19801.$$

21 The trajectory is shown below.



22 Put

1	a	b	c	d	e	f	g	2

Then

$$1 + a + b = 10, \quad a + b + c = 10 \implies c = 1.$$

Similarly,

$$f + g + 2 = 10, \quad e + f + g = 10 \implies e = 2$$

So we have

1	a	b	1	d	2	g	h	2

This gives

$$1 + d + 2 = 10 \implies d = 7$$

and

1	a	b	1	7	2	g	h	2

Hence

$$b + 1 + 7 = 10 \implies b = 2, \quad 7 + 2 + g = 10 \implies g = 1.$$

Now we have

1	a	2	1	7	2	1	h	2

Finally,

$$1 + a + 2 = 10 \implies a = 7, \quad 1 + 2 + h = 10, h = 7,$$

giving

1	7	2	1	7	2	1	7	2

23 We have

$$\begin{aligned}\left(\sqrt{5+\sqrt{12}} - \sqrt{5-\sqrt{12}}\right)^2 &= \left(\sqrt{5+\sqrt{12}}\right)^2 - 2\left(\sqrt{5+\sqrt{12}}\right)\left(\sqrt{5-\sqrt{12}}\right) + \left(\sqrt{5-\sqrt{12}}\right)^2 \\ &= 5 + \sqrt{12} - 2\left(\sqrt{25-12}\right) + 5 - \sqrt{12} \\ &= 10 - 2\sqrt{13}.\end{aligned}$$

24 Let x be the distance along the horizontal path and y the distance on the hill. Then the duration of the run is

$$\frac{x}{4} + \frac{y}{6} + \frac{y}{3} + \frac{x}{4} = 5.$$

hours. From this equation we obtain that the total distance of the run was $2(x+y) = 20$ miles.

25 Let O be the center of the circle. Observe that

$$[ABCA'B'C'] = 2[\triangle AOB] + 2[\triangle BOC] + 2[\triangle COA'].$$

Now, observe that

$$[\triangle AOB] = [\triangle B'OA], \quad [\triangle BOC] = [\triangle COB'], \quad [\triangle COA'] = [\triangle AOC].$$

Hence

$$[ABCA'B'C'] = 2([\triangle B'OA] + [\triangle COB'] + [\triangle AOC]) = 2[\triangle ACB'] = 2.$$

26 Recall that $x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$. Hence $(k+1)^3 - 1 = k(k^2+3k+3)$ and $(k+1)^3 + 1 = (k+2)(k^2+k+1)$.

Thus

$$\left(\frac{2^3-1}{2^3+1}\right)\left(\frac{3^3-1}{3^3+1}\right)\left(\frac{4^3-1}{4^3+1}\right)\cdots\left(\frac{100^3-1}{100^3+1}\right)$$

equals

$$\left(\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdots\frac{98}{100}\cdot\frac{99}{101}\right)\left(\frac{3^2+3+1}{2^2-2+1}\cdot\frac{4^2+4+1}{3^2+3+1}\cdot\frac{5^2+5+1}{4^2+4+1}\cdots\frac{100^2+100+1}{99^2+99+1}\right).$$

This last simplifies to

$$\left(\frac{2}{100\cdot 101}\right)\left(\frac{100^2+100+1}{3}\right) = \frac{3367}{5050}.$$

27 Let P, and P' be the perpendicular projections of the center of the top circle on $[AB]$ and $[AC]$ respectively. Let Q, and Q' be the perpendicular projections of the center of the bottom circle on $[AC]$ and $[BC]$ respectively. Observe that

$$AP = AP', \qquad CQ = CQ', \qquad AP' + 2R + CQ = 5.$$

Put $AP = AP' = x$ and $CQ = CQ' = y$. By considering the appropriate right triangles (recall that a radius is perpendicular to a tangent of a circle), we obtain

$$\frac{x}{R} = \cot\frac{\widehat{A}}{2}, \qquad \frac{y}{R} = \cot\frac{\widehat{C}}{2}.$$

Hence

$$x + 2R + y = 5 \implies R\cot\frac{\widehat{A}}{2} + 2R + R\cot\frac{\widehat{C}}{2} = 5 \implies R = \frac{5}{\cot\frac{\widehat{A}}{2} + \cot\frac{\widehat{C}}{2} + 2}.$$

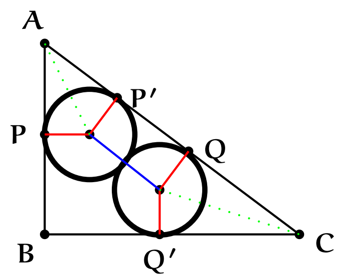
Furthermore,

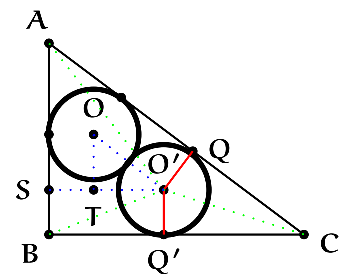
$$\cot\frac{\widehat{A}}{2} = \frac{\cos\frac{\widehat{A}}{2}}{\sin\frac{\widehat{A}}{2}} = \frac{\sqrt{\frac{1}{2}+\frac{\cos A}{2}}}{\sqrt{\frac{1}{2}-\frac{\cos A}{2}}} = \frac{\sqrt{\frac{1}{2}+\frac{2}{5}}}{\sqrt{\frac{1}{2}-\frac{2}{5}}} = 3.$$

Similarly,

$$\cot\frac{\widehat{C}}{2} = \frac{\cos\frac{\widehat{C}}{2}}{\sin\frac{\widehat{C}}{2}} = \frac{\sqrt{\frac{1}{2}+\frac{\cos C}{2}}}{\sqrt{\frac{1}{2}-\frac{\cos C}{2}}} = \frac{\sqrt{\frac{1}{2}+\frac{3}{10}}}{\sqrt{\frac{1}{2}-\frac{3}{10}}} = 2.$$

Finally $R = \dfrac{5}{2+3+2} = \dfrac{5}{7}$.





Aliter: Let the circles have centers O and O' as in the figure. Let S be the perpendicular projection of O' onto $[AB]$, and let T be the perpendicular projection of O onto $[SO']$. We have

$$6 = [\triangle ABC] = [\triangle O'BC] + [\triangle O'AC] + [\triangle O'AB].$$

Because the circle with centre O' is tangent to sides $[BC]$ and $[AC]$, we get

$$[\triangle O'AC] = \frac{5}{2}R, \qquad [\triangle O'BC] = 2R.$$

We now need to find $[\triangle O'AB] = \frac{1}{2}(AB)(SO')$. Observe that $\triangle ABC \sim \triangle OTO'$. Thus

$$\frac{O'T}{BC} = \frac{OO'}{AC} \implies \frac{O'T}{4} = \frac{2R}{5} \implies O'T = \frac{8R}{5}.$$

Thus

$$SO' = ST + TO' = R + \frac{8R}{5} = \frac{13R}{5},$$

and

$$[\triangle O'AB] = \frac{1}{2}(AB)(SO') = \frac{3}{2}\cdot\frac{13R}{5} = \frac{39R}{10}.$$

Finally,

$$6 = \frac{39R}{10} + \frac{5R}{2} + 2R \implies R = \frac{5}{7}.$$

28 Observe that we need $x > y$. Since $x^2 - y^2 = 81 \iff (x+y)(x-y) = 81$, we examine the positive divisors of 81. We need

$$x + y = 81, x - y = 1, \qquad x + y = 27, x - y = 3, \qquad x + y = 9, x - y = 9.$$

Hence, by inspection, the following solutions lie on the first quadrant,

$$(41, 40), (15, 12),$$

and the solution $(9, 0)$ lies on the x-axis. Thus on each quadrant there are two solutions, and a solution each on the positive and the negative portion of the x-axis, giving a total of

$$4 \cdot 2 + 2 = 10$$

solutions.

29 Let y stand for the yellow portion, r for the red portion, and b for the blue portion, as in the diagram shown below. We are interested in $r + b$. Now,

$$r + b + 2y = \frac{\pi R^2}{4}, \qquad y + b = \frac{\pi (R/2)^2}{2} = \frac{\pi R^2}{8}.$$

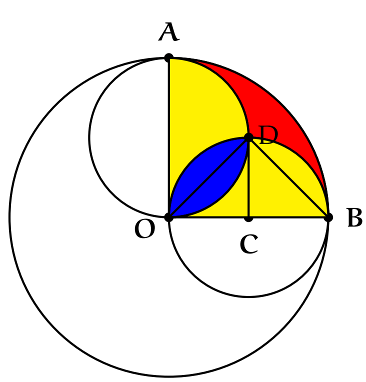
The last equality gives

$$2y = \frac{\pi R^2}{4} - 2b,$$

and substituting this into the first equation,

$$r + b = \frac{\pi R^2}{4} - 2y = 2b,$$

whence $r = b$.



We must now find the area of the blue portion. To do this, observe that $\triangle OCD$ is rectangle at C. Half of the area of the blue portion can be obtained by subtracting from the area of the sector OCD the area of $\triangle OCD$:

$$\frac{b}{2} = \frac{1}{4} \cdot \pi \left(\frac{R}{2}\right)^2 - \frac{1}{2} \cdot \frac{R}{2} \cdot \frac{R}{2} \implies b = \frac{R^2(\pi - 2)}{8}.$$

Thus the area sought is

$$b + r = 2b = \frac{(\pi - 2)R^2}{4}.$$

30 Writing in base 26: $2007 = 2 \cdot 26^2 + 25 \cdot 26 + 5$. Thus the label is BYE.

31 Since

$$x^2 - 2x - 2 = (x - a)(x - b),$$

we have $a + b = 2$ and $ab = -2$. It follows that

$$a^2 + b^2 = (a + b)^2 - 2ab = 2^2 - 2(-2) = 8.$$

Now

$$a^2 = 2a + 2, \quad b^2 = 2b + 2.$$

Therefore

$$a^3 = 2a^2 + 2a, \quad b^3 = 2b^2 + 2b,$$

whence

$$a^3 + b^3 = 2(a^2 + b^2) + 2(a + b) = 2(8) + 2(2) = 20.$$

32 For a number to be divisible by 8, the number formed by its last three digits must the divisible by 8. The largest number that can be formed with all ten digits is 9876543210, but this is not divisible by 8. A permutation of the last three digits to 9876543120 yields a number divisible by 8, and the largest such.

33 Let E' be the perpendicular projection of E on the segment $[AB]$. By the Pythagorean Theorem

$$AB = \sqrt{AD^2 + BD^2} = \sqrt{16^2 + 12^2} = \sqrt{4^2} \cdot \sqrt{4^2 + 3^2} = 20.$$

Since

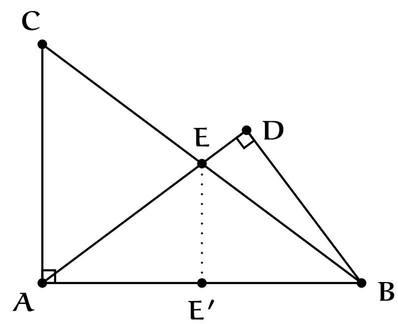
$$\frac{BD}{AD} = \frac{3}{4} = \frac{AC}{AB},$$

$\triangle ABC \sim \triangle DAB$. Thus $\widehat{BAE} = \widehat{ABE}$ and so $AE = EB$. It follows that $\triangle ABE$ is isosceles. Since EE' is the height of an isosceles triangle of base 20, $AE' = E'B = 10$. On the other hand $\triangle AE'E \sim \triangle BAC$ and thus

$$\frac{EE'}{AC} = \frac{AE'}{AB} = \frac{1}{2} \implies EE' = \frac{15}{2}.$$

Finally,

$$[\triangle AEB] = \frac{1}{2} \cdot AB \cdot EE' = \frac{1}{2} \cdot 20 \cdot \frac{15}{2} = 75.$$



34 We have

$$(x)\left(\sqrt[x]{x^3}\right) = \frac{x^x}{x} \iff x \cdot x^{3/x} = x^{x-1} \iff x^{1+3/x} = x^{x-1}.$$

Either the base is 1, so $x = 1$ is a solution, or both exponents are equal, giving

$$1 + \frac{3}{x} = x - 1 \iff x + 3 = x^2 - x \iff x^2 - 2x - 3 = 0 \iff (x-3)(x+1) = 0,$$

so $x = 3$ or $x = -1$. The solution set is $\{-1, 1, 3\}$ and the product of the positive solutions is $3(1) = 3$.

35 Clearly, $1^2 + 1 + 2 = 4 = 2^2$. Assume $n > 1$. Then $2n + 1 > n + 2$ and

$$n^2 < n^2 + n + 2 < n^2 + 2n + 1 = (n+1)^2,$$

that is, $n^2 + n + 2$ is between two consecutive squares and hence not a square.