The individual winner of the Spring 2000 edition of the Colonial Maths Challenge was Azeem Ansar of West Catholic, with 12 correct answers (out of 20).

Peter Baratta, William Clee, Joanne Darken, Dot French, Elena Koublanova, Margaret Hitczenko, David Santos helped with the competition.

You have 50 minutes to complete this exam. No consultation of any kind is allowed. Write your numerical answers on the answer-sheet provided. No partial credit is given.

1. Evaluate

$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}}.$$

2. If $3^{33}+3^{33}+3^{33}=3^x$, find x.

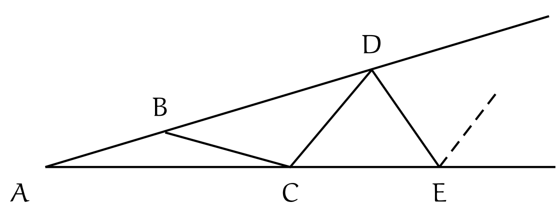
3. How many integers between 1 and 21 (excluding 1 and 21) do not have a common factor (i.e, are relatively prime) with 21?

4. A *palindrome* is an integer whose decimal expansion is symmetric. For example, 1, 11, 121, 344565443, are all palindromes. The sequence of palindromes, starting with 1 is written in ascending order

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, \ldots$$

Find the 1984-th positive palindrome.

5. A sequence of isosceles triangles is constructed starting with $AB = BC$, then $BC = CD$, and so on. If the angle $BAC = 17^\circ$, how many such triangles can be drawn?



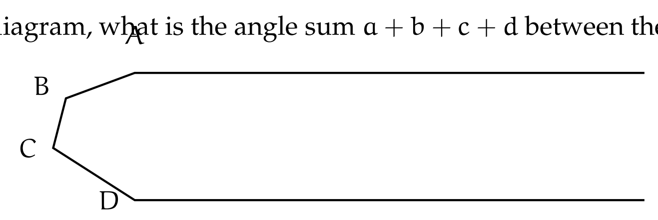
6. If $(x-1)^2+(y-2)^2+(z-3)^2+(w-4)^2=0$, what is $x+y+z+w$?

7. The sum of two numbers is 7 and their product 21. What is the sum of their reciprocals?

8. If $x^2+x-1=0$, find $x^4+2x^3+x^2$.

9. A car with 5 tires (four road tires and a spare) travelled 30 000 miles. If all five tires were used equally, how many miles' wear did each tire receive?

10. In the diagram, what is the angle sum $a+b+c+d$ between the two parallel lines?



11. In a river with a steady current, it takes a person 6 minutes to swim a certain distance upstream, but it takes him only 3 minutes to swim back. How many minutes would it take a piece of wood to float this same distance downstream?

12. If $(8x-5)^5 = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$, then $A + B + C + D + E + F =$

13. What is the angle between the hands of a clock at a quarter to five?

14. Leo the clockmaker has two antique clocks. One gains ten seconds every hour, while the other loses twenty seconds every hour. He set both clocks to show the correct time at 9 AM on 4 February 2001. On what date will they next show the correct times simultaneously?

15. A squeezable toothpaste tube is originally in the form of a cylinder 12 cm long, with diameter 4 cm. The short cylindrical nozzle has diameter 0.5 cm. What length of toothpaste can the tube produce?

16. Given that

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{1022}+\sqrt{1023}} + \frac{1}{\sqrt{1023}+\sqrt{1024}}$$

is an integer, find it.

17. Solve the system

$$\begin{aligned} x+y+u &= 4, \\ y+u+v &= -5, \\ u+v+x &= 0, \\ v+x+y &= -8. \end{aligned}$$

18. If eggs had cost x cents less per dozen, it would have cost 3 cents less for $x+3$ eggs than if they had cost x cents more per dozen. What is x?

19. A quiz has 25 questions with four points awarded for each correct answer and one point deducted for each incorrect answer, with zero for each question omitted. Anna scores 77 points. How many questions did she omit?

20. At a classroom party, the average age of b boys is g, and the average age of g girls is b. If the average age of everyone at the party (all boys and girls and their 42-year-old teacher) is $b+g$, what is the value of $b+g$?

1. Proceeding from the innermost fraction one easily sees that

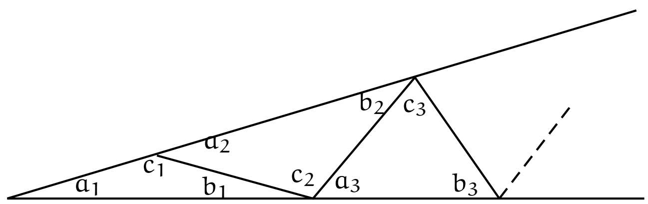
$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}} = \frac{1}{2-\frac{1}{2-\frac{2}{3}}} = \frac{1}{2-\frac{3}{4}} = \frac{4}{5}.$$

2. $3^{33} + 3^{33} + 3^{33} = 3 \cdot 3^{33} = 3^{34}$, whence $x = 34$.

3. Of the 21 numbers, one eliminates the seven which are multiples of 3 and the three which are multiples of 7. But one has included 21 twice, so the total number is $21 - 7 - 3 + 1 = 12$.

4. It is easy to see that there are 9 palindromes of 1 digit, 9 palindromes with two digits, 90 with three digits, 90 with 4 digits, 900 with 5 digits and 900 with 6 digits. The last palindrome with 6 digits, 999999, constitutes the $9 + 9 + 90 + 90 + 900 + 900 = 1998$th palindrome. Hence, the 1997th palindrome is 998899, the 1996th palindrome is 997799, the 1995th palindrome is 996699, the 1994th is 995599, etc., until we find the 1984th palindrome to be 985589.

5. Consider the diagram below.



Let $a_1 = x = 17°$. As the triangles formed are isosceles, $b_1 = a_1 = x$. This entails that $c_1 = 180 - a_1 - b_1$ and so

$$\begin{aligned} a_2 &= 180 - c_1 = 2x, \\ b_2 &= a_2 = 2x, \\ c_2 &= 180 - a_2 - b_2 = 180 - 4x, \\ a_3 &= 180 - b_1 - c_2 = 3x, \\ b_3 &= 3x, \\ &\vdots \\ a_n &= nx \\ b_n &= nx. \end{aligned}$$

In other words n-th triangle has two angles equal to nx. We can construct such triangles as long as $2nx < 180$. Since $x = 17°$, that means $34n < 180$. The biggest possible n is 5. We can construct 5 triangles.

6. Each summand is non-negative so if the sum is to equal 0, each summand must be zero. This entails $x = 1, y = 2, z = 3, w = 4$, whence $x + y + z + w = 10$.

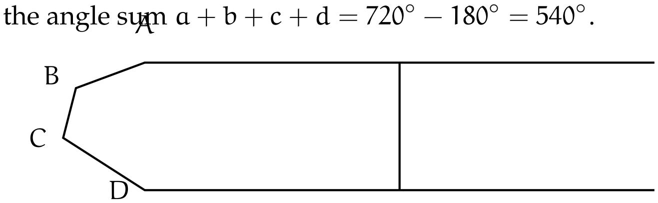
7. Let x, y be the numbers. One has $x + y = 7, xy = 21$ whence $\frac{1}{x} + \frac{1}{y} = \frac{y+x}{xy} = \frac{7}{21} = \frac{1}{3}$.

8. Observe that $x^2 + x = 1$. Hence $x^4 + x^3 = x^2$ and $x^3 + x^2 = x$. Thus

$$x^4 + 2x^3 + x^2 = x^4 + x^3 + x^3 + x^2 = x^2 + x = 1.$$

9. Travelling 30000 miles with 4 tires is as travelling 120000 miles on one tire. The average wear of each of the 5 tires is thus $120000 \div 5 = 24000$ miles.

10. Draw a straight line perpendicular to both parallel lines, as shown below. This closed figure forms a hexagon, with interior angle measure $4 \times 180° = 720°$. Subtract both right angles from the closed figure to obtain the angle sum $a + b + c + d = 720° - 180° = 540°$.



11. Let w be the speed of the swimmer and let c be the speed of the current. From $d = st$ where d is distance, s speed, and t time, one has $w - c = \frac{d}{6}$ and $w + c = \frac{d}{3}$. Whence $2c = (w + c) - (w - c) = \frac{d}{3} - \frac{d}{6} = \frac{d}{12}$. This yields $12 = \frac{d}{c}$. The piece of wood therefore needs 12 minutes to float downstream.

12. Letting $x = 1$ one gathers that

$$243 = 3^5 = (8(1) - 5)^5 = A + B + C + D + E + F.$$

13. Measure angles clockwise, with origin (0°), at 12:00. Each minute ran by the minute hands accounts for $\frac{360°}{60} = 6°$, and so, each, when jumping from hour to hour, the hands travel $5(6°) = 30°$. When the minute hand is on the 8, it has travelled $\frac{40}{60} = \frac{2}{3}$ of a circumference, that is

$$\frac{2}{3}360° = 240°$$

and the hour hand has moved $\frac{2}{3}$ of the way from 4 to 5, travelling

$$\left(4 + \frac{2}{3}\right)(30°) = 140°.$$

Hence, angle between the hands is $240° - 140° = 100°$.

14. The clocks will both show the right time again on 3 August 2001, the 180th day after 4 February 2001. The first clock will gain 1 minute every 6 hours, which is 1 hour every 360 hours, i.e., 1 hour every 15 days. The second clock will lose 2 hours every 15 days. After 15d days, the hour on the first clock is $9 + d \bmod 12$ and the hour on the second clock is $9 - 2d \bmod 12$. Since we want $9 + d \equiv 9 - 2d \bmod 12$, one must have $3d \equiv 0 \bmod 12$. Thus d must be a multiple of 4. Since one also wants the clocks to show the *exact hour* d must be a multiple of 12. The smallest such d is clearly $d = 12$. Thus the clocks agree on the correct time for the first time after $15d = 180$ days.

15. The volume of the toothpaste is $V = \pi r^2 h = \pi(2)^2(12) = 48\pi \text{ cm}^3$. After coming out of the nozzle, a cylinder of radius $= 0.25$ cm and height l cm is formed. Its volume is $\pi(.25)^2 l = 0.0625\pi l \text{ cm}^3$. Thus $l = 48 \div 0.0625 = 768$ cm.

16. Observe that $(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x}) = 1$ and so

$$\frac{1}{\sqrt{x+1} - \sqrt{x}} = \sqrt{x+1} + \sqrt{x}.$$

Thus

$$\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \cdots + \frac{1}{\sqrt{1022} + \sqrt{1023}} + \frac{1}{\sqrt{1023} + \sqrt{1024}}$$

becomes

$$(\sqrt{2} - \sqrt{1}) + (\sqrt{4} - \sqrt{3}) + \cdots + (\sqrt{1024} - \sqrt{1021}) = \sqrt{1024} - \sqrt{1} = 32 - 1 = 31.$$

17. Adding the equations $3(x+y+u+v)=-9$, and so $x+y+u+v=-3$. Thus

$$4+v=-3,$$
$$x-5=-3,$$
$$y+0=-3,$$
$$u-8=-3.$$

This yields $v=-7, x=2, y=-3, u=5$.

18. If eggs had cost x cents less per dozen, I would have saved $\frac{x}{12}$ cents per egg. If eggs had cost x cents more per dozen, I would have lost $\frac{x}{12}$ cents per egg. The difference between these two prices is $2(\frac{x}{12})$ per egg. Thus if I buy $x+3$ eggs and the total difference is 3 cents, I can write

$$(x+3)2\left(\frac{x}{12}\right)=3,$$

which is to say $x^2-3x-18=(x-3)(x+6)=0$. Since x has to be a positive number, $x=3$.

19. Anna answered 20 questions correctly (She could not answer less than 20, because then her score would have been less than $19\times 4=76<77$; She could not answer more than 20, because her score would have been at least $21\times 4-3=81.77$). To get exactly 77 points Anna had to answer exactly 3 questions wrong, which means she omitted 2 questions.

20. The total age of boys is bg. The total age of girls is bg. The total number of people at a classroom party is$b+g+1$. Thus the average age is

$$\frac{bg+bg+42}{b+g+1}=b+g,$$

whence $42=b^2+g^2+b+g$.

Since b and g are whole numbers (they represent the number of boys and girls respectively), the only numbers that satisfy the above equation are 5 and 8. Thus $b+g=8$.

1. (4 minutes, 4 marks) If $\frac{a}{b} = \frac{2}{3}$, find $900a^2 - 400b^2$.
Solution: From $\frac{a}{b} = \frac{2}{3}$, one gathers $3a - 2b = 0$. Thus
$$900a^2 - 400b^2 = 100(9a^2 - 4b^2) = 100(3a - 2b)(3a + 2b) = 0.$$

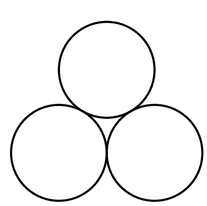
2. (4 minutes, 4 marks) A car with five tires (four road tires and a spare) travelled 30000 miles. All five tires were used equally. How many miles' wear did each tire received?
Solution: Travelling 30000 miles with 4 tires is as travelling 120000 miles on one tire. The average wear of each of the 5 tires is thus $120000 \div 5 = 24000$ miles.

3. (4 minutes, 6 marks) If a is 50 % larger than c, and b is 25 % larger than c, then a is what percent larger than b?
Solution: One has $a = 1.5c$ and $b = 1.25c$ Thus $a = \frac{1.5b}{1.25} = 1.2b$. Therefore a is 20% larger than b.

4. (6 minutes, 6 marks) Let
$$x = 6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{\vdots}}}}},$$
where there are an infinite number of fractions. Write x in the form $a + \sqrt{b}$, with a, b integers. Justify your answer. (Hint: Some animals, when cut, grow anew, the new "animal" resembling the old!)
Solution: One has $x = 6 + \frac{1}{x}$ or $x^2 - 6x - 1 = 0$. Solving for x one has $x = 3 \pm \sqrt{10}$. Since x must be positive, $x = 3 + \sqrt{10}$.

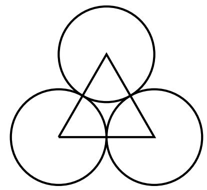
5. (7 minutes, 8 marks) Find the sum of all the integers from 1 to 1000 inclusive, which are not multiples of 3 or 5. You may use the formula $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
Solution: One computes the sum of all integers from 1 to 1000 and weeds out the sum of the multiples of 3 and the sum of the multiples of 5, but puts back the multiples of 15, which one has counted twice. The desired sum is
$$\begin{aligned}(1 + 2 + 3 + \cdots + 1000) - (3 + 6 + 9 + \cdots + 999) & \\ & -(5 + 10 + 15 + \cdots + 1000) \\ & +(15 + 30 + 45 + \cdots + 990) \\ = (1 + 2 + 3 + \cdots + 1000) - 3(1 + 2 + 3 + \cdots + 333) & \\ & -5(1 + 2 + 3 + \cdots + 200) \\ & +15(1 + 2 + 3 + \cdots + 66) \\ & = 500500 - 3 \cdot 55611 \\ & \quad -5 \cdot 20100 + 15 \cdot 2211 \\ & = 266332\end{aligned}$$

6. (7 minutes, 6 marks) Three circles of radius R are mutually tangential as shown in the figure below. What is the area of the region surrounded by the three circles?

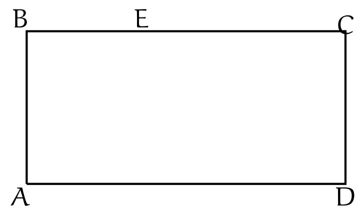


Solution: The area sought is the area of the equilateral triangle formed joining the centers of the three circles minus the three angular sectors. Since the equilateral triangle has side $2R$, it has area $R^2\sqrt{3}$. Since $60°$ is $1/6$

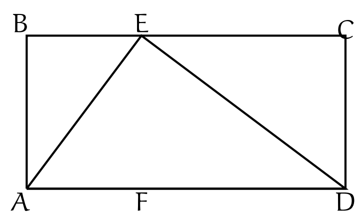
of the circumference, each of the angular sectors has area $\frac{\pi R^2}{6}$. Thus the area sought is $R^2\sqrt{3} - \frac{R^2\pi}{2}$.



7. (4 minutes, 4 marks) In the following diagram, $AE = 3, DE = 4$, and $AD = 5$. What is the area of the rectangle $ABCD$?



Solution: $\triangle AED$ is a 3-4-5 right triangle. Drop a perpendicular from E meeting AD at F. Then the area of $\triangle AED$ is both given by $\frac{1}{2}(3 \times 4) = 6$ and by $\frac{1}{2}(AD)(EF) = 2.5(EF)$. This entails that $EF = 2.4$ Since $EF = AB$, the area of the rectangle is given by $(AB)(AD) = 2.4 \times 5 = 12$.



8. (4 minutes, 6 marks) At a certain college 99% of the 100 students are female, but only 98% of the students living on campus are female. If some females live on campus, how many students live off campus?
Solution: Of the 100 students, only one is male. He is 2% of the on-campus population. Thus the whole on-campus population consists of 50 students, so there are 100-50=50 off-campus students.

9. (4 minutes, 6 marks) $DEFG$ is a square which is drawn on the outside of a regular pentagon $ABCDE$. How big is the angle EAF in degrees?
Solution: The sum of the internal angles of a pentagon is $3 \times 180° = 540°$. Each internal angle measures thus $540° \div 5 = 108°$. Thus $\angle AEF = 360° - \angle AED - \angle EDF = 360° - 108° - 90° = 162°$. As $\triangle AEF$ is isosceles, we gather that $\angle EAF = \frac{180° - 162°}{2} = 9°$.

10. (TIE BREAKER, 5 minutes) Solve for all real values of x

$$(x)(\sqrt[x]{x^3}) = \frac{x^x}{x}.$$

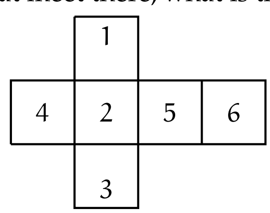
Solution: Clearly one solution is $x = 1$. Now $x^{1+3/x} = x^{x-1}$ entails $1 + \frac{3}{x} = x - 1$ or $x = -1, 3$. The real solution set is thus $\{-1, 1, 3\}$.

The individual winners of the Fall 2000 edition of the Colonial Maths Challenge were: Rich Anderson, senior, George Washington HS (first). Tie for second: Sasha Ovetsky, junior, Central HS (15) Maksim Rapoport, senior, George Washington. Tie for third: Bryan Ferguson, junior, George Washington (13) Feng-Yen Li, senior, Central Stephanie Moyerman, junior, Central Dmitry Pavolotsky, senior, Central HS. The Team Competition was won by Central HS.

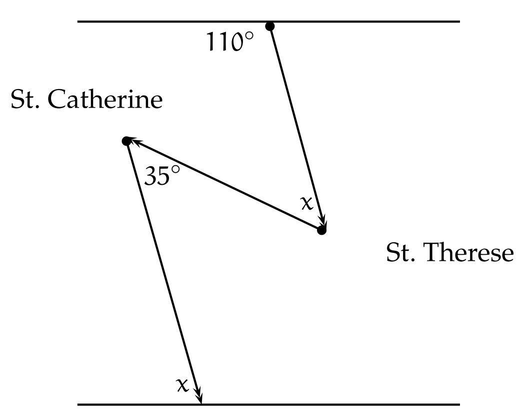
Peter Baratta, William Clee, Joanne Darken, Celina Evans, Dot French, Elena Koublanova, Margaret Hitczenko, David Santos helped with the competition.

Instructions: You have 1 hour to complete this exam. Scrap paper, graph paper, ruler, compass, and calculators are permitted. Write your answers on the answer sheet provided. No credit will be given for procedure. No partial credit whatsoever.

1. N is a prime number and $90 < N < 99$. What is 2N?
2. Let the binary operation $\otimes$ be defined as $a \otimes b = a^2 - b^2 + ab$. What is $(-3) \otimes (2)$?
3. Iblis entered an elevator in a tall building. She went up 4 floors, down 6 floors, up 8 floors and down 10 floors. She then found herself on the 23rd floor. In what floor did she enter the elevator?
4. A number of birds are resting on two branches of a tree, one branch vertically above the other. A bird on the lower branch says to the birds on the upper branch: “If one of you will come down here, we will have an equal number of birds on each branch. ” A bird on the upper branch replies, “If one of you will come up here, we will have twice as many up here as down there.” How many birds are on each branch?
5. If the figure shown is folded to form a cube, then three faces meet at every vertex. If for each vertex we take the product of the numbers on the three faces that meet there, what is the largest product we get?



6. At the annual Fourth of July picnic, the big event of the day is always the giant *Tug-of-War!* In the first round, four weightlifters (all of equal strength) tugged against five grandmothers (all of equal strength). The result was a tie! In the second round, Yogi Bear tugged against two grandmothers and one weightlifter. Again, the result was a tie! In the third and final round, Yogi Bear and two grandmothers tugged against four of the weightlifters. What was the outcome? Enter **Y** if Yogi and the grandmothers won, **W** if the weightlifters won, or **T** if it was a tie.
7. To traverse the St. Laurent river, first I arrived at Isle of St. Therese, and then I went to the Isle of St. Catherine, as shown in the diagram. Assume that the river banks are parallel. What is the angle x, in degrees?



8. Suppose the product

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{14^2}\right)\left(1-\frac{1}{15^2}\right)=\frac{a}{b},$$

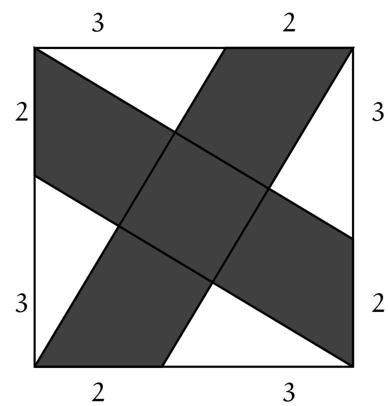
where a and b are positive integers with no prime factor in common. What is a^2+b^2?

9. If

$$\frac{4^5+4^5+4^5+4^5}{3^5+3^5+3^5}\cdot\frac{6^5+6^5+6^5+6^5+6^5+6^5}{2^5+2^5}=2^n,$$

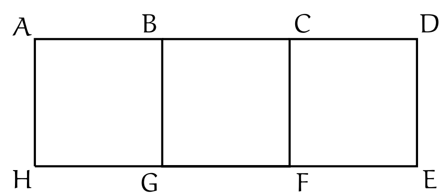
find n.

10. Find the area, in square units, of the cross-shaped region enclosed in the 5×5 square.



11. Four singers take part in a musical round of 4 equal lines, each finishing after singing the round through 3 times. The second singer begins when the first singer begins the second line, the third singer begins when the first singer begins the third line, the fourth singer begins when the first singer begins the fourth line. What is the fraction of the total singing time when all the singers are singing simultaneously?

12. Three identical squares are drawn side to side, as shown in the figure below. Compute the angle sum $\angle ACH+\angle ADH$ in degrees.



13. Find the smallest positive integer n such that $\frac{n}{2}$ is a (perfect) square, $\frac{n}{3}$ is a (perfect) cube, and $\frac{n}{5}$ is a (perfect)fifth power.

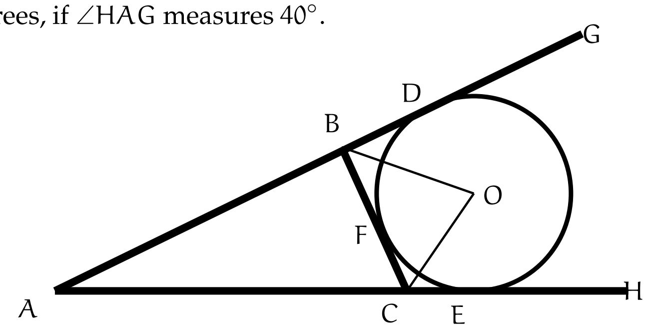
14. If

$$(x^2-x+1)^6(x^5-x+1)^7=a_{47}x^{47}+a_{46}x^{46}+a_{45}x^{45}+\cdots+a_3x^3+a_2x^2+a_1x+a_0,$$

find the value of the sum of the coefficients of every even power:

$$a_{46}+a_{44}+a_{42}\cdots+a_2+a_0.$$

15. The circle with centre at O is tangent to the line AG at D, to the line AH at E, and to the line at BC at F. Find $\angle BOC$ in degrees, if $\angle HAG$ measures $40°$.



16. Point P is interior to an equilateral triangle $\triangle ABC$ of side 3 units. The distance of P to AB is a, the distance of P to AC is $2a$ and the distance of P to CB is $3a$. Find the length of a.

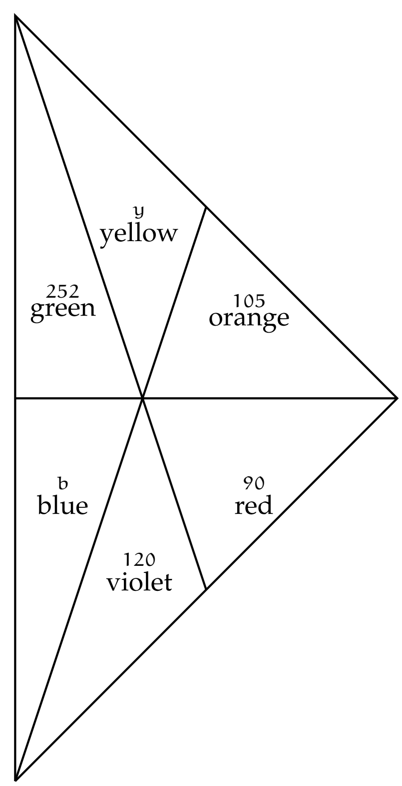
17. Five people are waiting in line. Starting with the second one, it is noticed that the age of a person added to three times the age of the person immediately in front is always equal to 110. What is the age of the first person in line? Assume that the ages are integral numbers.

18. Find all positive solutions of the system

$$xy = 2,$$
$$yz = 3,$$
$$zx = 4.$$

19. *One Price Shoes* sell all their shoes for the same price, which is an integral number of dollars. Anacleta and Sinforosa go shoe shopping to *One Price Shoes*. Anacleta has \$200 and she buys as many pairs of shoes as possible, and there remain \$32. Sinforosa has \$150 and she buys as many pairs of shoes as possible, and there remain \$24. What is the price of a pair of shoes?

20. A triangular banner is coloured as shown below, the area of each region given in cm^2. How many cm^2 of blue are there? How many cm^2 of yellow?



1. **194** $N = 97$ and $2N = 194$.

2. **−1** $(-3) \otimes (2) = (-3)^2 - 2^2 + (-3)(2) = 9 - 4 - 6 = -1$.

3. **27** If she entered at floor F, then $F + 4 - 6 + 8 - 10 = 23$, whence $F = 27$.

4. **7 and 5** Let x be the original number of birds in the upper branch and let y be the original numbers of birds in the lower branch. From the given data,

$$x - 1 = y + 1,$$
$$x + 1 = 2(y - 1).$$

This system solves to $(x, y) = (7, 5)$.

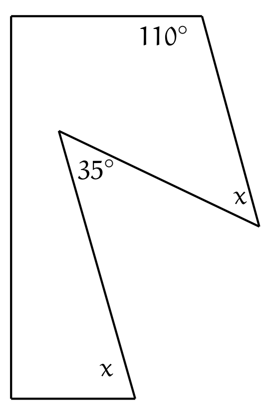
5. **90** Observe that 4 and 5 are on opposite sides, so they never meet. The largest product is $3 \times 5 \times 6 = 90$.

6. **Y** Denote by y the strength of Yogi Bear, by w the strength of a weightlifter and by g that of a grandmother. All these strengths are positive numbers. We are given that $4w = 5g$ and that $y = 2g + w$. Now

$$y + 2g = 2g + w + 2g = 4g + w = \frac{16}{5}w + w = \frac{21}{5}w > 4w,$$

and so Yogi and the grandmothers won.

7. **52.5° or 52°30′** Draw a line perpendicular to the shores to complete a hexagon as shown. The sum of



the interior angles of a hexagon is $180°(6 - 2) = 720°$. Thus

$$2x + 2 \cdot 90° + 110° + (360° - 35°) = 720°,$$

whence $x = 52.5°$.

8. **289** We use the fact that $1 - x^2 = (1 - x)(1 + x)$. First

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{14}\right)\left(1 - \frac{1}{15}\right) = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{13}{14} \cdot \frac{14}{15} = \frac{1}{15}.$$

Second

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{14}\right)\left(1+\frac{1}{15}\right)=\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{15}{14}\cdot\frac{16}{15}=\frac{16}{2}=8.$$

Hence

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{14^2}\right)\left(1-\frac{1}{15^2}\right)=\frac{8}{15},$$

and $a^2+b^2=289$.

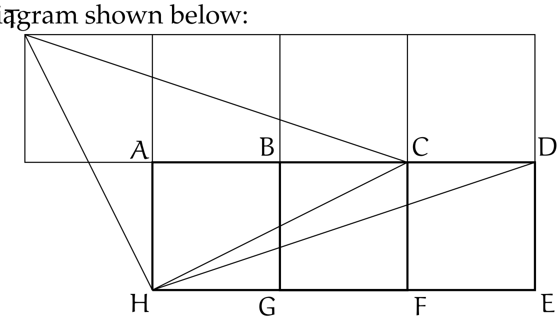
9. $\boxed{12}$ The product equals

$$\frac{4(4^5)}{3(3^5)}\cdot\frac{6(6^5)}{2(2^5)}=4(4^5)=2^{12}.$$

10. $\boxed{\frac{290}{17}}$ Let A be the area of one of the four smaller corner right triangles (the ones with hypotenuse 3). Then the area sought is $25-4A$. Now, each of these smaller triangles is similar to a large triangle (with legs 5 and 3). Since the hypotenuse of these triangles must be in proportion, $3k=\sqrt{34}$, for some constant k. Now, the areas keep the square proportionality, hence $Ak^2=\frac{15}{2}$. This gives $A=\frac{15}{2k^2}=\frac{135}{68}$. Hence $25-4A=25-\frac{135}{17}=\frac{290}{17}$.

11. $\boxed{\frac{3}{5}}$ Three lines have been sung before the fourth singer starts, and after that he sings 12 more lines. So the total span of lines is 15. They start singing simultaneously from line 4, and the first singer is the first to end, in line 12. Thus $12-4+1=9$ lines out of 15 are sung simultaneously and the fraction sought is $\frac{9}{15}=\frac{3}{5}$.

12. $\boxed{45^\circ}$ Construct the diagram shown below:



Observe that $\angle ADH=\angle ACT$ and that $\triangle THC$ is an isosceles right triangle. Hence $\angle ADH+\angle ACH=45^\circ$.

13. $\boxed{30233088000000}$ The integer must be of the form $n=2^a3^b5^c$, with a,b,c positive integers. The conditions imply that $a-1$ is even and that a is divisible by 15; that $b-1$ is divisible by 3 and that b is divisible by 10; and that $c-1$ is divisible by 5 and c divisible by 6. Clearly, the smallest positive integers satisfying those conditions are $a=15, b=10, c=6$. Hence the integer sought is $n=2^{15}3^{10}5^6=30233088000000$.

14. $\boxed{365}$ Put

$$p(x) = (x^2 - x + 1)^6(x^5 - x + 1)^7 = a_{47}x^{47} + a_{46}x^{46} + a_{45}x^{45} + \cdots + a_3x^3 + a_2x^2 + a_1x + a_0.$$

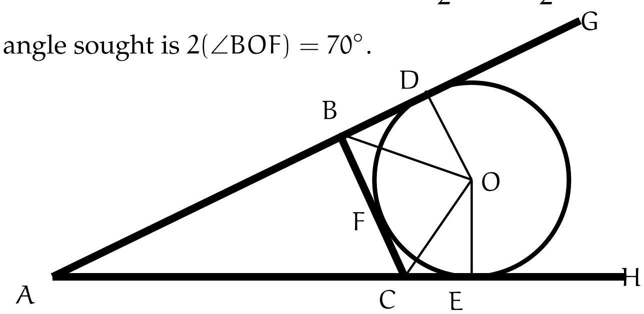
Then

$$a_{46} + a_{44} + \cdots + a_0 = \frac{p(1) + p(-1)}{2} = \frac{1 + 3^6}{2} = 365.$$

15. $\boxed{70^\circ}$ First observe that $\angle ADO$ and $\angle AEO$ are right angles. $\angle DOA = 90^\circ - 20^\circ = 70^\circ$. As BD and BF are tangent to the circle, OB bisects $\angle DOF$. Hence

$$\angle BOF = \frac{\angle DOA}{2} = \frac{70^\circ}{2} = 35^\circ.$$

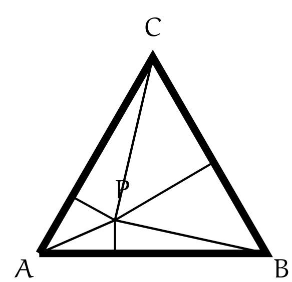
Therefore, the angle sought is $2(\angle BOF) = 70^\circ$.



16. $\boxed{a = \frac{\sqrt{3}}{4}}$ The area of $\triangle ABC =$ Area $\triangle APB+$ Area $\triangle APC+$ Area $\triangle CPA$. Now,

$$\frac{1}{2} \cdot 3 \cdot \sqrt{3^2 - \left(\frac{3}{2}\right)^2} = \frac{1}{2} \cdot 3 \cdot a + \frac{1}{2} \cdot 3 \cdot 3a + \frac{1}{2} \cdot 3 \cdot 2a,$$

which is to say $a = \frac{\sqrt{3}}{4}$.



17. $\boxed{28}$ Let x_k be the age of the k-th person. Then

$$x_2 + 3x_1 = 110,$$

$$x_3 + 3x_2 = 110,$$

$$x_4 + 3x_3 = 110,$$

$$x_5 + 3x_4 = 110.$$

These equations imply that each of x_2, x_3, x_4, x_5 is an integer leaving remainder 2 upon division by 3. Also

$$\begin{aligned} x_5 &= 110 - 3x_4 \\ &= 110 - 3(110 - 3x_3) \\ &= 110 - 3(110 - 3(110 - 3x_2)) \\ &= 110 - 3(110 - 3(110 - 3(110 - 3x_1))), \end{aligned}$$

from where it follows that

$$x_5 = 110(1 - 3 + 9 - 27) + 81x_1 = -2200 + 81x_1.$$

Hence,

$$x_1 = 27 + \frac{x_5 + 13}{81}.$$

Since x_1 is an integer, $x_5 + 13$ must be divisible by 81. The least positive x_5 for which this occurs is $x_5 = 68$. By successively solving the equations we gather that $x_5 = 68, x_4 = 14, x_3 = 32, x_2 = 26, x_1 = 28$.

18. $\boxed{z = \sqrt{6}, x = \frac{2\sqrt{6}}{3}, y = \frac{\sqrt{6}}{2}}$ Multiplying all the equations together, we obtain $x^2y^2z^2 = 24$, whence $xyz = 2\sqrt{6}$. Dividing xyz successively by each of the equations, we find,

$$z = \sqrt{6}, x = \frac{2\sqrt{6}}{3}, y = \frac{\sqrt{6}}{2}.$$

19. $\boxed{\$42}$ Let A be the number of pairs of shoes Anacleta buys and let S be the number of pairs of shoes Sinforosa buys. Let P be the price, in dollars, of each pair of shoes. Observe that $P > 32$. Then $200 = AP + 32, 150 = SP + 24$, whence

$$AP = 168 = 2^3 \cdot 3 \cdot 7, SP = 126 = 2 \cdot 3^2 \cdot 7.$$

The price of each pair of shoes is a common divisor of 168 and 126, but it must exceed 32. The only such divisor is 42.

20. $\boxed{y = 210, b = 168}$ Observe from the figure below that $\triangle ABE$ and $\triangle OBE$ share the same base BE. Thus the ratio of their areas is proportional to the ratio of their heights, which we will name h and h' respectively. Therefore

$$\frac{b + 372}{120} = \frac{h}{h'}.$$

Also, $\triangle AEC$ shares the same base EC with $\triangle OEC$ and their heights are also h and h', giving

$$\frac{h}{h'} = \frac{y + 195}{90}.$$

Therefore

$$\frac{b + 372}{120} = \frac{y + 195}{90}.$$

Again, $\triangle ACD$ and $\triangle AOD$ share the same base AD. The ratio of their areas is proportional to the ratio of their heights h_1 and h_2. This gives

$$\frac{y + 357}{252} = \frac{h_1}{h_2}.$$

Also, $\triangle CDB$ shares the same base DB with $\triangle ODB$ and their heights are also h_1 and h_2, giving

$$\frac{h_1}{h_2} = \frac{b + 210}{b}.$$

Therefore

$$\frac{y + 357}{252} = \frac{b + 210}{b}.$$

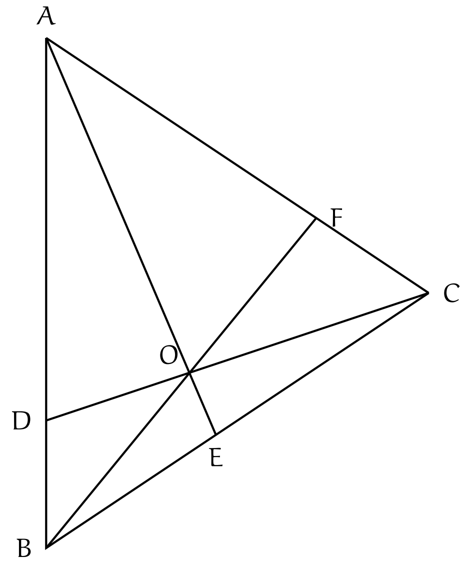
Upon solving the equations

$$\frac{b + 372}{120} = \frac{y + 195}{90},$$

and

$$\frac{y+357}{252} = \frac{b+210}{b}$$

we deduce that $y = 210$, $b = 168$.



1. (4 minutes, 4 marks) *Junk Pizzas* sell pizzas with diameters of 4 inches, 6 inches, 8 inches and 10 inches for \$2, \$4, \$8, and \$12 respectively. If the pizzas have all the same thickness, which size will give the most pizza per dollar?

Solution: **6 inches** A \$2 pizza gives $\frac{\pi 2^2}{2} = 2\pi$ square inches per dollar, a \$4 gives $\frac{\pi 3^2}{4} = 2.25\pi$ square inches per dollar, an \$8 gives $\frac{\pi 4^2}{8} = 2\pi$ square inches per dollar, and a \$12 gives $\frac{\pi 5^2}{12} \approx 2.08\pi$ square inches per dollar. The best deal is the \$4 pizza, with 6 inches of diameter.

2. (4 minutes, 5 marks) The average of six numbers is 4. A seventh number is added and the new average increases to 5. What was the seventh number?

Solution: **11** The sum of the original 6 numbers is $S = 6 \cdot 4 = 24$. If the 7th number is x, then

$$\frac{24 + x}{7} = 5,$$

whence $x = 11$.

3. (4 minutes, 6 marks) Assume that there is a positive real number x such that

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2,$$

where there is an infinite number of x's. What is the value of x?

Solution: **$x = \sqrt{2}$** We have $x^2 = 2$, and since x is positive, $x = \sqrt{2}$.

4. (7 minutes, 6 marks) The year is 2001. Dwayne's father gives Dwayne \$N dollars on the Nth day of the year, but he uses only \$1 and \$10 bills, using as many \$10 bills as possible. Thus, on February 1st he gives Dwayne \$32 using three \$10s and two \$1s. On February 1st Dwayne has $1 + 2 + 3 + \ldots + 32 = 528$ dollars saved, of these 138 are \$1s and 39 are \$10s. Dwayne will leave on vacation the day he has exactly as many \$10s as \$1s. When will Dwayne leave on vacation? You may use the formula

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}.$$

Solution: **8 April 2001** Let x, y be, respectively, the number of \$1s he uses and the number of \$10s he uses. On day number $10n + m$, $(n \geq 0, 0 \leq m \leq 9)$, we have

$$x = 45n + \frac{m(m+1)}{2}$$

and

$$y = 5n(n-1) + n(m+1).$$

If follows that

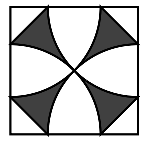
$$45n \leq x \leq 45(n+1)$$

and

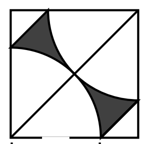
$$5n^2 - 4n \leq y \leq 5n^2 + 5n.$$

From this we deduce that if $n \leq 7$, $5n^2 + 5n < 45n$ and thus $x < y$, and if $n \geq 11$, then $5n^2 - 4n > 45(n+1)$, hence $x > y$. We must examine one by one the thirty values corresponding for $n \in \{8, 9, 10\}$. The only possibilities giving $x = y$ are $n = m = 8$ or $n = m - 1 = 8$. Dwayne will leave after saving up money for 98 days, on 8 April.

5. (7 minutes, 8 marks) The square has side 1. The shaded region below is formed by overlapping quarter-circles each centred at a corner of the square. Diagonally opposite quarter-circles are tangent. Find the area of the cross-shaped figure shaded.



Solution: $\boxed{2\sqrt{2}-1-\frac{\pi}{2}}$ The area of the figure is twice that of the figure shaded below.



Now,

$$(2r)^2 = 1^2 + 1^2,$$

whence $r = \frac{\sqrt{2}}{2}$. The figure above has as area that of the square minus two quarter circles and two isosceles right triangles. That is, its area is

$$1 - 2\cdot\frac{\pi}{4}\cdot\left(\frac{\sqrt{2}}{2}\right)^2 - 2\cdot\frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right)^2 = \sqrt{2}-\frac{1}{2}-\frac{\pi}{4}.$$

The required area is thus

$$2\sqrt{2}-1-\frac{\pi}{2}.$$

6. (6 minutes, 6 marks) Betty and Wilma can paint a building in two hours, Wilma and Pebbles can paint the same building in three hours and Betty and Pebbles can paint the very same building in 5 hours. How long would it take the three of them working together to paint the building?

Solution: $\boxed{\frac{60}{31}\text{ hours}}$ Let b, w, p be the fractional effort put by Betty, Wilma, and Pebbles, respectively, for one hour. We are given that

$$b + w = \frac{1}{2},$$

$$w + p = \frac{1}{3},$$

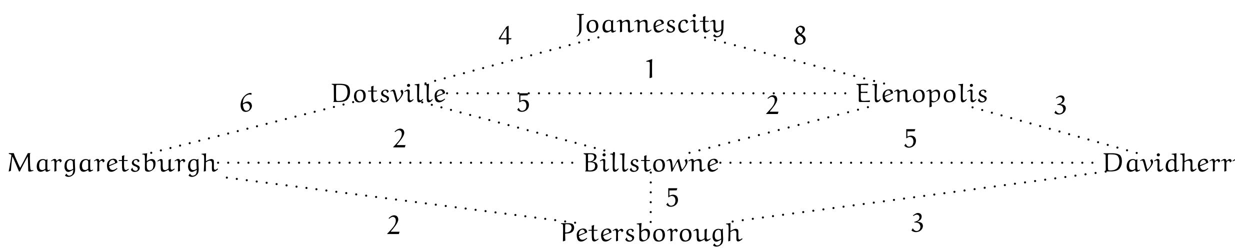
$$p + b = \frac{1}{5}.$$

Adding these equations,

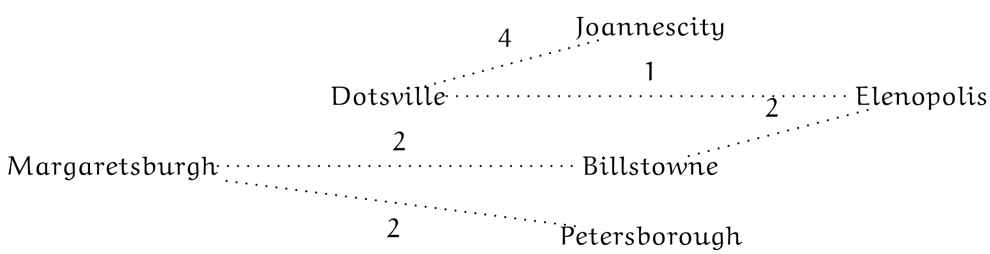
$$2(b + w + p) = \frac{31}{30},$$

hence $b + w + p = \frac{31}{60}$. In one hour they can do $\frac{31}{60}$ of the building, hence they can do the whole building in $\frac{60}{31}$ hours.

7. (6 minutes, 6 marks) The diagram is a sketch road map of roads linking Joannescity to Petersborough, passing through Dotsville, Elenopolis, Billstowne, Margaretsburgh, and Davidherri. The number on each section of road is the distance in miles from one town to another. How long is the shortest route from Joannescity to Petersborough?



Solution: **11 miles** Starting from Joannescity, go to the nearest town. The shortest path is shown below.



8. (4 minutes, 5 marks) List all the triangles with integral sides and perimeter 10. Lists that differ only in order are considered identical.

Solution: **$\{2,4,4\}$ and $\{3,3,4\}$** Let $a \leq b \leq c$ be the sides of the triangle. Then a, b, c, are integers satisfying

$$a + b + c = 10, a + b > c.$$

This entails

$$10 - c > c,$$

whence $c < 5$. Thus the only such lists are 2, 4, 4, and 3, 3, 4.

Tie Breaker I (4 minutes, 3 marks.)
Find the units digit of

$$7^{2000}.$$

Solution: **1** We have $7^4 = 2401 = 10N + 1$, with $N = 240$. Hence

$$7^{2000} = (7^4)^{500} = (10N + 1)^{400} = 10Q + 1,$$

for some integer Q, and so the units digit is 1.

Tie Breaker II (6 minutes, 6 marks)
Find positive integers a, b, c, d such that

$$x^{80} + x^{79} + x^{78} + \cdots + x^3 + x^2 + x + 1 = (x^{2a} + x^a + 1)(x^{2b} + x^b + 1)(x^{2c} + x^c + 1)(x^{2d} + x^d + 1).$$

You may use the identity $T^3 - 1 = (T - 1)(T^2 + T + 1)$.

Solution: $\boxed{(x^{54} + x^{27} + 1)(x^{18} + x^9 + 1)(x^6 + x^3 + 1)(x^2 + x + 1)}$ Put $S = x^{80} + x^{79} + \cdots + x + 1$. Then

$$xS = x^{81} + x^{80} + \cdots + x = x^{81} + S - 1.$$

Solving for S,

$$x^{80} + x^{79} + \cdots + x + 1 = S = \frac{x^{81} - 1}{x - 1}.$$

Now,

$$\frac{x^{81} - 1}{x - 1} = \frac{x^{81} - 1}{x^{27} - 1} \cdot \frac{x^{27} - 1}{x^9 - 1} \cdot \frac{x^9 - 1}{x^3 - 1} \cdot \frac{x^3 - 1}{x - 1}.$$

Using the identity $T^3 - 1 = (T - 1)(T^2 + T + 1)$ we deduce

$$\frac{x^{81} - 1}{x - 1} = (x^{54} + x^{27} + 1)(x^{18} + x^9 + 1)(x^6 + x^3 + 1)(x^2 + x + 1).$$

The individual winners of the Spring 2002 edition of the Colonial Maths Challenge were: Sasha Ovetsky, senior, Central HS (first); Michael Segal, freshman Central HS (second); George Nesterenko, junior, Central HS (third). The Team Competition was won by Central HS.

Peter Baratta, William Clee, Joanne Darken, Dot French, Dan Jacobson, Elena Koublanova, Margaret Hitczenko, Mi Mi Lian, Clark Loveridge, David Santos, Benjamin Wong helped with the competition.

Instructions: You have 1 hour to complete this exam. Scrap paper, graph paper, ruler, compass, and calculators are permitted. Write your answers on the answer sheet provided. No credit will be given for procedure. No partial credit whatsoever.

1. 20 (**Forming Squares**) If any four different dots are chosen, how many squares can be formed from the design in Figure 1?

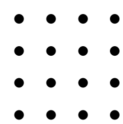


Figure 1: Forming Squares

Solution: There are 9 squares of side of length 1, 4 of side of length 2, 1 of side of length 3, 4 of side of length $\sqrt{2}$, and 2 of side $\sqrt{5}$. Thus the total number of squares is $9+4+1+4+2=20$.

2. 61 If

$$\frac{1}{1+\frac{1}{5}}=\frac{a}{b},$$

where the fraction $\frac{a}{b}$ is in least terms, find a^2+b^2.
Solution:

$$\frac{1}{1+\frac{1}{5}}=\frac{1}{\frac{6}{5}}=\frac{5}{6}=\frac{a}{b},$$

whence $a^2+b^2=5^2+6^2=61$.

3. 6 An isosceles triangle has one side measuring 3 units and another side measuring 6 units. What is the length of the third side?
Solution: Another side of length 3 is impossible, since the triangle inequality is not satisfied. Thus the third side has length 6.

4. 21 A binary operation ■ is defined by

$$a■b=\frac{a+b}{2}.$$

If

$$x■(x■21)=x,$$

find x.
Solution: We have

$$\begin{aligned} x■(x■21)=x &\iff \frac{x+\left(\frac{x+21}{2}\right)}{2}=x \\ &\iff x+\frac{x+21}{2}=2x \\ &\iff 2x+x+21=4x \\ &\iff x=21. \end{aligned}$$

5. 28 How many digits does $4^{16}5^{25}$ have?
Solution: There are 28 digits, since
$$4^{16}5^{25} = 2^{32}5^{25} = 2^7 2^{25}5^{25} = 128 \times 10^{25},$$
which is the 3 digits of 128 followed by 25 0's.

6. 10 In how many different ways can one change 50 cents using nickels, dimes or quarters?
Solution: We want the number of solutions of
$$5x + 10y + 25z = 50,$$
that is, of
$$x + 2y + 5z = 10,$$
with integer $0 \leq x \leq 10$, $0 \leq y \leq 5$, $0 \leq z \leq 2$. The table below exhausts all ten possibilities.

z	y	x
2	0	0
1	2	1
1	1	3
1	0	5
0	5	0
0	4	2
0	3	4
0	2	6
0	1	8
0	0	10

7. 400 A *palindrome* is an integer whose decimal expansion is symmetric and does not end in 0. How many palindromes of 5 digits are even?
Solution: A five digit even palindrome has the form $ABCBA$, where $A \in \{2, 4, 6, 8\}$, and $(B, C) \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}^2$. Thus there are 4 choices for the first digit, 10 for the second, and 10 for the third. Once these digits are chosen, the palindrome is completely determined. Therefore, there are $4 \times 10 \times 10 = 400$ even palindromes of 5 digits.

8. $\frac{3}{101}$ Find the exact value of the product
$$\left(1 - \frac{2}{5}\right)\left(1 - \frac{2}{7}\right)\left(1 - \frac{2}{9}\right)\cdots\left(1 - \frac{2}{99}\right)\left(1 - \frac{2}{101}\right).$$
Solution: We have
$$\begin{aligned}\left(1 - \frac{2}{5}\right)\left(1 - \frac{2}{7}\right)\left(1 - \frac{2}{9}\right)\cdots\left(1 - \frac{2}{99}\right)\left(1 - \frac{2}{101}\right) &= \frac{3}{5}\cdot\frac{5}{7}\cdot\frac{7}{9}\cdot\frac{9}{11}\cdots\frac{97}{99}\cdot\frac{99}{101} \\ &= \frac{3}{101}.\end{aligned}$$

9. 13 Let $t = 3^{2001}$. If
$$3^{2001} + 3^{2002} + 3^{2003} = at,$$
find a.
Solution: We have
$$3^{2001} + 3^{2002} + 3^{2003} = 3^{2001}(1 + 3 + 3^2) = (13)3^{2001},$$
whence $a = 13$.

10. 120 As a publicity stunt, a camel merchant has decided to pose the following problem: "If one gathers all of my camels into groups of 4, 5 or 6, there will be no remainder. But if one gathers them into groups of 7 camels, there will be 1 camel left in one group." The number of camels is the smallest positive integer satisfying these properties. How many camels are there?
Solution: The least common multiple of 4, 5 and 6 is 60, hence we want the smallest positive multiple of 60 leaving remainder 1 upon division by 7. This is easily seen to be 120.

11. 36 (**Area of Quadrilateral**) Figure 2 is composed of a grid of identical squares. The vertices of the quadrilateral $ABCD$ are corners of some of these squares. Find the *exact* area, in square units, of the quadrilateral $ABCD$.

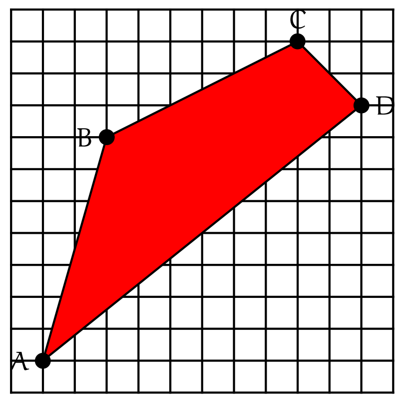
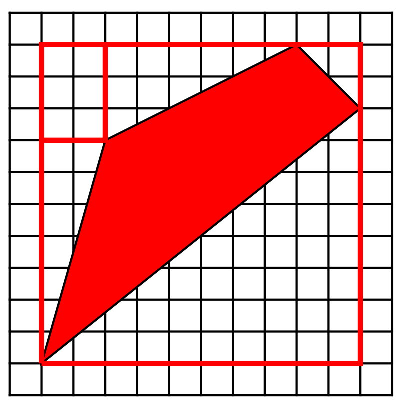


Figure 2: Area of Quadrilateral

Solution: Enclose the quadrilateral as shewn below.



The area sought is that of the enclosing square minus the area of the small rectangle and the four right-triangles formed, whence it is

$$10^2 - (3)(2) - \frac{1}{2}(3)(6) - \frac{1}{2}(2)(2) - \frac{1}{2}(8)(10) - \frac{1}{2}(7)(2) = 36.$$

12. -79 You are given that $x + y = -3$ and $xy = 8$. Find

$$x^4 + y^4.$$

Solution: We have

$$x^2 + y^2 = (x + y)^2 - 2xy = 9 - 16 = -7,$$

and

$$x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 = 49 - 128 = -79.$$

13. 28 **(Seven Circles)** Each of six circles of radii 1 is tangent to its neighbour and to a seventh larger circle, as in figure 3. The *exact* area that these six circles bound can be written in the form $a\sqrt{3} + b\pi$, where a and b are integers. Determine the value of $a^2 + b^2 + ab$.

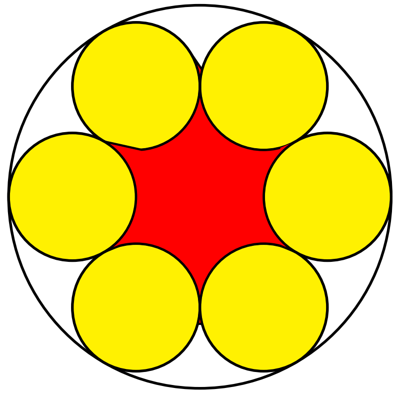
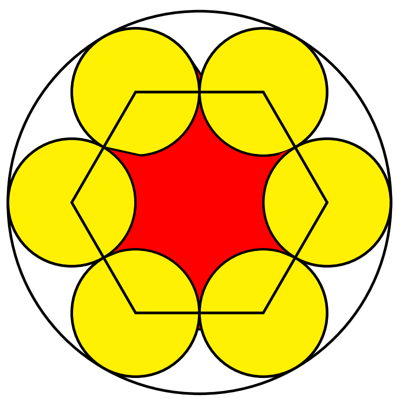


Figure 3: Seven Circles

Solution: The desired area is the area of the regular hexagon of side 2 with vertices at the centres of the smaller circles minus the six third-of-circles. The apothem of the hexagon is, by the Pythagorean Theorem $= \sqrt{2^2 - 1} = \sqrt{3}$. The area of the hexagon is thus $6(\frac{1}{2}\sqrt{3}(2)) = 6\sqrt{3}$. The area of the six thirds-of-circles is $6(\frac{1}{3}\pi) = 2\pi$. The area wanted is thus

$$6\sqrt{3} - 2\pi.$$

Hence $a = 6, b = -2$, from where it follows that $a^2 + b^2 + ab = 36 + 4 - 12 = 28$.



14. 13501 The integers from 1 to 1000 are written in succession. Find the sum of all the digits.
Solution: When writing the integers from 000 to 999 (with three digits), $3 \times 1000 = 3000$ digits are used. Each of the 10 digits is used an equal number of times, so each digit is used 300 times. The the sum of the digits in the interval 000 to 999 is thus

$$(0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)(300) = 13500.$$

Therefore, the sum of the digits when writing the integers from 000 to 1000 is $13500 + 1 = 13501$.

15. 96 How many 4-digit integers can be formed with the set of digits $\{0, 1, 2, 3, 4, 5\}$ such that no digit is repeated and the resulting integer is a multiple of 3?
Solution: The integers desired have the form $D_1D_2D_3D_4$ with $D_1 \neq 0$. Under the stipulated constraints, we must have

$$D_1 + D_2 + D_3 + D_4 \in \{6, 9, 12\}.$$

We thus consider three cases.

Case I: $D_1 + D_2 + D_3 + D_4 = 6$. Here we have $\{D_1, D_2, D_3, D_4\} = \{0, 1, 2, 3\}$, $D_1 \neq 0$. There are then 3 choices for D_1. After D_1 is chosen, D_2 can be chosen in 3 ways, D_3 in 2 ways, and D_1 in 1 way. There are thus $3 \times 3 \times 2 \times 1 = 3 \cdot 3! = 18$ integers satisfying case I.

Case II: $D_1 + D_2 + D_3 + D_4 = 9$. Here we have $\{D_1, D_2, D_3, D_4\} = \{0, 2, 3, 4\}, D_1 \neq 0$ or $\{D_1, D_2, D_3, D_4\} = \{0, 1, 3, 5\}, D_1 \neq 0$. Like before, there are $3 \cdot 3! = 18$ numbers in each possibility, thus we have $2 \times 18 = 36$ numbers in case II.

Case III: $D_1 + D_2 + D_3 + D_4 = 12$. Here we have $\{D_1, D_2, D_3, D_4\} = \{0, 3, 4, 5\}, D_1 \neq 0$ or $\{D_1, D_2, D_3, D_4\} = \{1, 2, 4, 5\}$. In the first possibility there are $3 \cdot 3! = 18$ numbers, and in the second there are $4! = 24$. Thus we have $18 + 24 = 42$ numbers in case III.

The desired number is finally $18 + 36 + 42 = 96$.

16. $3^{666}4$ Determine the largest number which is the product of positive integers whose sum is 2002.
Solution: We are given some positive integers $a_1, a_2, \ldots, a_n$ with $a_1 + a_2 + \cdots + a_n = 2002$. To maximise $a_1 a_2 \cdots a_n$, none of the a_k's can be 1. Let us shew that to maximise this product, we make as many possible $a_k = 3$ and at most two $a_j = 2$.

    Suppose that $a_j > 4$. Substituting a_j by the two terms $a_j - 3$ and 3 the sum is not changed, but the product increases since $a_j < 3(a_j - 3)$. Thus the a_k's must equal 2, 3 or 4. But $2 + 2 + 2 = 3 + 3$ and $2 \times 2 \times 2 < 3 \times 3$, thus if there are more than two 2's we may substitute them by 3's. As $2002 = 3(667) + 1 = 3(666) + 4$, the maximum product sought is $3^{666} \times 4$.

17. 117856 Let S be the set of all natural numbers whose digits are chosen from the set $\{1, 3, 5, 7\}$ such that no digits are repeated. Find the sum of the elements of S.
Solution: First observe that $1 + 7 = 3 + 5 = 8$. The numbers formed have either one, two, three or four digits. The sum of the numbers of 1 digit is clearly $1 + 7 + 3 + 5 = 16$.

    There are $4 \times 3 = 12$ numbers formed using 2 digits, and hence 6 pairs adding to 8 in the units and the tens. The sum of the 2 digits formed is $6((8)(10) + 8) = 6 \times 88 = 528$.

    There are $4 \times 3 \times 2 = 24$ numbers formed using 3 digits, and hence 12 pairs adding to 8 in the units, the tens, and the hundreds. The sum of the 3 digits formed is $12(8(100) + (8)(10) + 8) = 12 \times 888 = 10656$.

    There are $4 \times 3 \times 2 \cdot 1 = 24$ numbers formed using 4 digits, and hence 12 pairs adding to 8 in the units, the tens the hundreds, and the thousands. The sum of the 4 digits formed is $12(8(1000) + 8(100) + (8)(10) + 8) = 12 \times 8888 = 106656$.

    The desired sum is finally
$$16 + 528 + 10656 + 106656 = 117856.$$

18. 25 Fifty pounds of caviar are for sale at a store. Several customers are lined up to buy this caviar. Having sold the demanded portion to a customer, the cashier calculates the average weight of the portions already sold and tells the number of customers for which there is enough caviar left provided these customers each buy exactly the average pounds of caviar that she has declared. It turns out that for each of the first 10 customers the cashier is able to announce, after each of these customers has bought his caviar, that there is still enough caviar for the next 10 customers. How much caviar will there be left after the first 10 customers have made their purchases, provided the stated conditions are met?
Solution: If a_k is the amount of caviar sold to the k-th customer, then for each k, $1 \leq k \leq 10$ we are given that
$$50 - (a_1 + a_2 + \cdots + a_k) = 10\frac{a_1 + a_2 + \cdots + a_k}{k},$$
whence
$$\frac{50k}{10 + k} = a_1 + a_2 + \cdots + a_k.$$
If the stipulated conditions are met then
$$25 = \frac{50 \cdot 10}{10 + 10} = a_1 + a_2 + \cdots + a_{10}.$$
Hence there are $50 - 25 = 25$ pounds of caviar left.

Remark: Since this is valid for each k, $1 \leq k \leq 10$, subtracting the k-th equation from the $k-1$-th equation, we deduce that

$$a_k = \frac{50k}{10+k} - \frac{50(k-1)}{9+k} = \frac{500}{(10+k)(9+k)}.$$

As long the k-th customer buys exactly $\frac{500}{(10+k)(9+k)}$ pounds, the conditions of the problem will be met.

19. 2000 Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 3.6 \rfloor = 3$, $\lfloor -\pi \rfloor = -4$. Find the largest integer N verifying the following two conditions.

 - First condition: $\lfloor \frac{N}{3} \rfloor$ has three digits, each identical.
 - Second condition: There is a natural number n such that $\lfloor \frac{N}{3} \rfloor = 1 + 2 + 3 + \cdots n$.

 Solution: From the first condition

 $$\lfloor \frac{N}{3} \rfloor = 111k$$

 for some $k \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. From the second condition

 $$\frac{n(n+1)}{2} = 111k \iff n^2 + n - 222k = 0 \iff n = \frac{-1 + \sqrt{1 + 888k}}{2}.$$

 We test each $k \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and see that $1 + 888k$ is a square only when $k = 6$, whence $n = 36$ and

 $$1 + 2 + \cdots + 36 = 666.$$

 From the first condition

 $$666 \leq \frac{N}{3} < 667,$$

 whence

 $$1998 \leq N < 2001.$$

 The largest integer meeting both conditions is thus $N = 2000$.

20. 4913 and 5832 Find all the natural numbers of 4 digits that are equal to the cube of the sum of their digits.

Solution: Let $n = 1000a + 100b + 10c + d$ be the integer sought and let $s = a + b + c + d$ be the sum of its digits. From $1000 \leq n \leq 9999$ and $n = s^3$ it follows that $11 \leq s \leq 21$. Observe that

$$(s-1)s(s+1) = s^3 - s = 999a + 99b + 9c = 9(111a + 11b + c),$$

whence 9 divides one of the three consecutive integers $s-1$, s or $s+1$. Since $11 \leq s \leq 21$, we have three cases to consider: $s^3 - s = 16 \cdot 17 \cdot 18$, $s^3 - s = 17 \cdot 18 \cdot 18$ or $s^3 - s = 18 \cdot 19 \cdot 20$.

If $s^3 - s = 16 \cdot 17 \cdot 18$, then $111a + 11b + c = 544$ whence $a = 4, b = 9, c = 1, s = 17$ and so, $d = 3$. The number is $n = 4913$.

If $s^3 - s = 17 \cdot 18 \cdot 19$, then $111a + 11b + c = 646$ whence $a = 5, b = 8, c = 3, s = 18$ and so, $d = 2$. The number is $n = 5832$.

If $s^3 - s = 18 \cdot 19 \cdot 20$, then $111a + 11b + c = 760$ whence $a = 6, b = 8, c = 6, s = 19$ and no such digit d exists. The only two numbers are thus 4913 and 5832.

1. 2004 (4 minutes) Find the value of n in
$$(10^{2002}+25)^2-(10^{2002}-25)^2=10^n.$$
Solution: We have
$$\begin{aligned}(10^{2002}+25)^2-(10^{2002}-25)^2 &= 10^{4004}+2\cdot 25\cdot 10^{2002}+25^2-(10^{4004}-2\cdot 25\cdot 10^{2002}+25^2)\\ &= 4\cdot 25\cdot 10^{2002}\\ &= 100\cdot 10^{2002}=10^{2004},\end{aligned}$$
whence $n=2004$.

2. 13 (4 minutes) In a group of 100 camels, 46 eat wheat, 57 eat barley, and 10 eat neither. How many camels eat both wheat and barley?
Solution: Let A be the set of camels eating wheat, and $|A|$ its number, and B be the set of camels eating barley, and $|B|$ its number. Then
$$90=100-10=|A\cup B|=|A|+|B|-|A\cap B|=46+57-|A\cap B|=103-|A\cap B|,$$
whence $|A\cap B|=13$.

3. (6 minutes) Write positive integers in the empty boxes of the grid below so that the sum of any three consecutive boxes be always the same and the sum of all the boxes be 217.

A	B	C	D	E	F	G	H	I	J	K	L	M
			17				20					

Solution: Since the sum of any three consecutive boxes is the same, we deduce from

A	B	C	D	E	F	G	H	I	J	K	L	M
a	b	c	17				20					

that $a=17$. If the conditions are met we must have

A	B	C	D	E	F	G	H	I	J	K	L	M
17	b	c	17	b	c	17	20					

from where $b=20$. This gives an arragement of the form

A	B	C	D	E	F	G	H	I	J	K	L	M
17	20	c	17	20	c	17	20	c	17	20	c	17

from where
$$17\times 5+20\times 4+4c=217.$$
This gives $c=13$.

The required arrangement is finally

A	B	C	D	E	F	G	H	I	J	K	L	M
17	20	13	17	20	13	17	20	13	17	20	13	17

4. $\frac{2002}{2003}$ (5 minutes) Find
$$\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cdots\cfrac{1}{2-\cfrac{1}{2}}}}}},$$

where the digit 2 is repeated 2002 times.
Solution: Put

$$x_1 = 2, \quad x_n = 2 - \frac{1}{x_{n-1}}, \quad n \geq 2.$$

We see that

$$x_2 = 2 - \frac{1}{2} = \frac{3}{2},$$
$$x_3 = 2 - \frac{2}{3} = \frac{4}{3},$$

and by induction we can prove that

$$x_n = \frac{n+1}{n}.$$

Thus $x_{2002} = \dfrac{2003}{2002}$ and we want

$$\frac{1}{x_{2002}} = \frac{2002}{2003}.$$

5. $\dfrac{1}{2\pi}$ (8 minutes) (**Area of circular semi-segment**) In Figure 4 each of the chords divides the larger circle into two parts of areas in ratio $\dfrac{1}{3}$. The intersection of the chords forms a square concentric with the larger circle. Find the ratio of the area shaded to the area of the circle exscribed to the square (dashed circle). Solution:

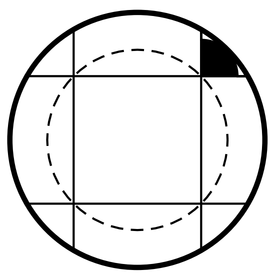
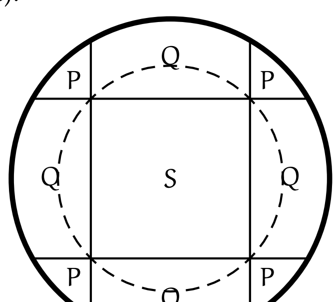


Figure 4: Area of circular semi-segment.

The outer circle of area A can be divided into 1 square of area S, four regions of area Q and four smaller regions of area P (the area shaded above).



Notice that any two parallel chords described divide the outer circle into three "bands", two "outer" bands of identical area $2P + Q$, and a "central" band of area $2Q + S$. Any one of the given chords divides the outer circle into two parts, one of area $\dfrac{A}{4}$ and the other of area $\dfrac{3A}{4}$. The area of the central band is $2Q + S = \dfrac{A}{2}$ is twice the area of one of the outer bands, each of area $2P + Q = \dfrac{A}{4}$. This gives $S = 4P$. If x is the side of the central square, then we have $P = \dfrac{S}{4} = \dfrac{x^2}{4}$. The area of the inner circle is thus $I = \pi\dfrac{x^2}{2}$ and thus finally

$$\frac{P}{I} = \frac{\frac{x^2}{4}}{\pi\frac{x^2}{2}} = \frac{1}{2\pi}.$$

6. 100π (5 minutes) (**Concentric Circles**) In figure 5, the two circles are concentric, AB is tangent to the inner circle, and $AB = 20$. Find the exact value of the area of the shaded annulus.

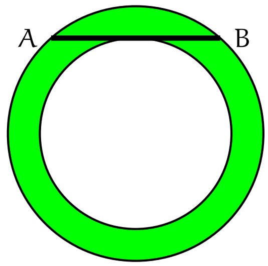


Figure 5: Concentric Circles

Solution: Let R and r be the radii of the outer and inner circle, respectively. The area sought is $\pi(R^2 - r^2)$. Now, by the Pythagorean Theorem, $R^2 - r^2 = (\frac{20}{2})^2 = 100$. Hence the desired area is 100π.

7. $2:1$ (4 minutes) (**Ratio of Areas**) In figure 6, each side of the rectangle is divided into three equal segments. The endpoints of the segments are joined to the center of the rectangle in order to form the banner shown. Find the ratio of the areas of the white portion to the shaded portion.

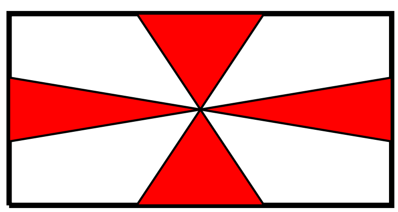


Figure 6: Ratio of Areas.

Solution: Without loss of generality, assume that the rectangle is a $(3a) \times (3b)$ rectangle of area $9ab$. Four isosceles triangles, two with base a and height $\frac{3b}{2}$, and two with base b and height $\frac{3a}{2}$ are formed. The area that these triangles comprise is thus

$$2\left(\frac{1}{2}a\frac{3b}{2}\right) + 2\left(\frac{1}{2}b\frac{3a}{2}\right) = 3ab.$$

The area of the white portion is thus $9ab - 3ab = 6ab$ and the desired ratio is

$$6ab : 3ab = 2 : 1.$$

8. 9901 (5 minutes) Given that

$$1,\ 000,\ 002,\ 000,\ 001$$

has a prime factor greater than 9000, find it.

Solution:

$$\begin{aligned} 1,000,002,000,001 &= 10^{12} + 2 \cdot 10^6 + 1 \\ &= (10^6 + 1)^2 \\ &= ((10^2)^3 + 1)^2 \\ &= (10^2 + 1)^2((10^2)^2 - 10^2 + 1)^2 \\ &= 101^2 9901^2, \end{aligned}$$

whence the prime sought is 9901.

The individual winners of the Spring 2003 edition of the Colonial Maths Challenge were: 1st Place: David Heinz - Central HS Wayne Whitfield - Central HS 2nd Place: George Nesterenko - Central HS Jacky Yuen - George Washington 3rd Place: Lynn Haimowitz - Central Jarret Leiberman - George Washington. The Team Competition was won by Central HS.

Baratta, William Clee, Joanne Darken, Dot French, Dan Jacobson, Margaret Hitczenko, Wimayra Luy, Clark Loveridge, David Santos, Yun Yoo helped with the competition.

Instructions: You have 1 hour to complete this exam. Scrap paper, graph paper, ruler, compass, and calculators are permitted. Write your answers on the answer sheet provided. No credit will be given for procedure. No partial credit whatsoever.

1. Given that
$$\frac{10+10^2}{\frac{1}{10}+\frac{1}{100}}$$
is an integer, find it.

   1000 Solution: $= \dfrac{10+10^2}{\frac{1}{10}+\frac{1}{100}} = \dfrac{10^3+10^4}{10+1} = \dfrac{11000}{11} = 1000.$

2. If today is Thursday, what day will it be 100 days from now?

   Saturday Solution: Since $100 = 7 \times 14 + 2$, 98 days from today will be a Thursday, and 100 days from today will be a Saturday.

3. What is the closest (integer) perfect square to 2003?

   2025 Observe that $\sqrt{2003} > 44$. We see that $44^2 = 1936$ and $45^2 = 2025$. Thus the closest square is 2025.

4. In the circle with center O, $\angle COD = 2\angle ABC$ and $\widehat{AB} = 150°$. See figure 7. What is the measure of $\widehat{CD}$ in degrees?

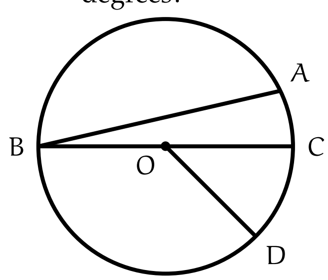


Figure 7: Problem 4.

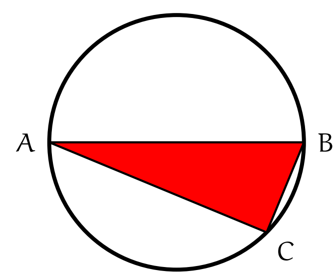


Figure 8: Problem 5.

   30° Solution: $\widehat{AC} = 30°$ and hence $\angle ABC = \frac{1}{2}\widehat{AC} = 15°$. Thus $\widehat{CD} = \angle COD = 2\angle ABC = 30°$.

5. The circle in figure 8 has radius 1. Side AB of $\triangle ABC$ is a diameter of the circle, and $AC = \sqrt{3}$. Find the area inside the circle but outside the triangle. Leave your answer in the form $a\pi + b\sqrt{3}$, where a and b are rational numbers.

   $\pi - \frac{\sqrt{3}}{2}$ Solution: $\triangle ABC$ is a right triangle at C since this $\angle C$ is inscribed in a semicircle. By the Pythagorean Theorem $BC = \sqrt{4-3} = 1$. Hence the area of $\triangle ABC$ is $\frac{\sqrt{3}}{2}$, and the area sought is $\pi - \frac{\sqrt{3}}{2}$.

6. Find all positive primes of the form $n^3 - 8$, where n is a positive integer.

   19 Solution: Since $n^3 - 8 = (n-2)(n^2 + 2n + 2)$ and since $n - 2 < n^2 + 2n + 4$, we must have $n - 2 = 1 \implies n = 3$ and $n^3 - 8 = 19$.

7. If k is a positive integer, the notation $k!$ (read "k factorial") indicates the product of the integers between 1 and k, e.g., $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. Find the last two digits of the integer

$$1! + 2! + 3! + \cdots + 100!.$$

   13 For $k \geq 10$, $k!$ ends in two or more 0's. Thus the last two digits of the desired sum are the last two digits of

$$1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! = 409113,$$

   thus the desired sum ends in 13.

8. p and q are positive integers such that

$$\frac{7}{10} < \frac{p}{q} < \frac{11}{15}.$$

   What is the least value of q?

   7 Solution: If all a, b, c, d are positive, then

$$\frac{a}{b} < \frac{c}{d} \implies \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

   Hence

$$\frac{7}{10} < \frac{11}{15} \implies \frac{7}{10} < \frac{18}{25} < \frac{11}{15} \implies \frac{7}{10} < \frac{25}{35} < \frac{18}{25} < \frac{11}{15}.$$

   Since $\frac{25}{35} = \frac{5}{7}$, we have $q \leq 7$. Could it be smaller? Observe that $\frac{5}{6} > \frac{11}{15}$ and that $\frac{4}{6} < \frac{7}{10}$. Thus by considering the cases with denominators $q = 1, 2, 3, 4, 5, 6$, we see that no such fraction lies in the desired interval. The smallest denominator is thus 7.

9. Find the number of ordered pairs (x, y) of integers (positive, negative or zero) satisfying the equation

$$x + y + xy = 120.$$

   Here (x, y) is different from (y, x) if $x \neq y$.

   6 Solution: Observe that

$$x + y + xy + 1 = 121 \implies (x+1)(y+1) = 11^2.$$

   Thus $x + 1 = \pm 1, \pm 11$ or ± 121. Thus there are six solutions

$$(0, 120), (120, 0), (-2, -122), (-122, -2), (10, 10), (-12, -12).$$

10. An urn has 900 chips, numbered 100 through 999. Chips are drawn at random and without replacement from the urn, and the sum of their digits is noted. What is the smallest number of chips that must be drawn in order to guarantee that at least three of these digital sums be equal?

    53 Solution: There are 27 different sums. The sums 1 and 27 only appear once (in 100 and 999), each of the other 25 sums appears thrice. Thus if $27 + 25 + 1 = 53$ are drawn, at least 3 chips will have the same sum.

11. The quadratic equation $ax^2 + bx - 3 = 0$ has -1 as one of its roots. If a and b are positive primes, what is $a^2 + b^2$?

29 Solution: We have

$$a(-1)^2 + b(-1) - 3 = 0 \implies a - b = 3.$$

The difference between odd primes is always even, so one of the primes must be 2. Thus $a = 5$ and $b = 2$. Hence $a^2 + b^2 = 29$.

12. A circle is inscribed in right $\triangle ABC$, with a right angle at C. The circle is tangent to hypotenuse $\overline{AB}$ at P where $AP = 20$ and $BP = 6$. Find the radius of the circle.

4 Solution: Tangents to a circle from the same point have equal lengths. Thus $AC = 20 + r$ and $CB = 6 + r$. Equality of areas reveals that

$$\frac{1}{2}(20 + r)(6 + r) = r^2 + 2\left(\frac{1}{2}(20)(r)\right) + 2\left(\frac{1}{2}(6)(r)\right) \implies 120 - 26r - r^2 = 0 \implies (30 + r)(4 - r) = 0.$$

Since $r > 0$ we must have $r = 4$.

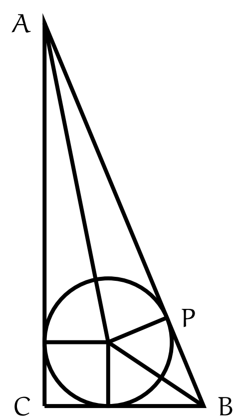


Figure 9: Problem 12.

13. The vertices of a triangle have coordinates $(-1, 2)$, $(1, 8)$, and $(9, 5)$. There is a point (x, y) inside the triangle with the following property: if line segments are drawn from each vertex to its opposite side and passing through (x, y), the triangle is divided into six regions of equal area. What is the point (x, y)?

$(3, 5)$ Solution: The point (x, y) must be the centroid of the triangle, that is, (x, y) is the point of intersection of the medians of the triangle. The centroid of the triangle with vertices at (a_1, b_1), (a_2, b_2), and (a_3, b_3) is located at $((a_1 + a_2 + a_3)/3, (b_1 + b_2 + b_3)/3)$, so the point required is $(3, 5)$.

14. Little Dwayne has 100 cards where the integers from 1 through 100 are written. He also has an unlimited supply of cards with the signs $+$ and $=$. How many true equalities can he make, if he uses each card no more than once?

33 The shortest equality under the stated conditions must involve 3 numbers, and hence a maximum of

33 equalities can be achieved. The 33 equalities below show that this maximum can be achieved.

$1+75=76$	$23+64=87$	$45+53=98$
$3+74=77$	$25+63=88$	$47+52=99$
$5+73=78$	$27+62=89$	$49+51=100$
$7+72=79$	$29+61=90$	$24+26=50$
$9+71=80$	$31+60=91$	$20+28=48$
$11+70=81$	$33+59=92$	$16+30=46$
$13+69=82$	$35+58=93$	$12+32=44$
$15+68=83$	$37+57=94$	$8+34=42$
$17+67=84$	$39+56=95$	$2+38=40$
$19+66=85$	$41+55=96$	$4+6=10$
$21+65=86$	$43+54=97$	$14+22=36$

15. In figure 10 $\triangle DEC$ has area 4, $\triangle AEB$ has area 10 and $AB \| CD$. Find the area of the trapezoid $ABCD$.

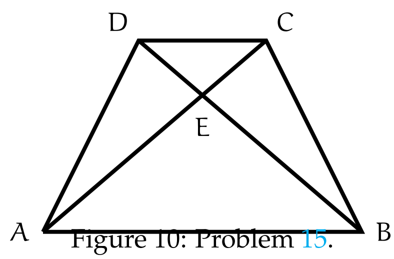


Figure 10: Problem 15.

$14+4\sqrt{10}$ Solution: By considering $\triangle ADB$ and $\triangle ACB$ we deduce that $\triangle AED$ and $\triangle BEC$ have the same area, call it x. The area of the trapezoid is $14+2x$. By considering similar triangles

$$\frac{x}{4}=\frac{10}{x} \implies x=2\sqrt{10},$$

whence the area of the trapezoid is $14+4\sqrt{10}$.

16. Find x if

$$\sqrt{13}+3=2x+\cfrac{2}{x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\ddots}}}}$$

3 Solution: We have

$$\frac{\sqrt{13}+3}{2} = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ddots}}}} \implies \frac{\sqrt{13}+3}{2} = x + \frac{2}{\sqrt{13}+3}$$

$$\implies x = \frac{\sqrt{13}+3}{2} - \frac{2}{\sqrt{13}+3}$$

$$\implies x = \frac{6(\sqrt{13}+3)}{2(\sqrt{13}+3)}$$

$$\implies x = 3.$$

17. How many paths consisting of a sequence of horizontal and/or vertical line segments, each segment connecting a pair of adjacent letters in figure 11 spell BIPOLAR?

```
            B
          B I B
        B I P I B
      B I P O P I B
    B I P O L O P I B
  B I P O L A L O P I B
B I P O L A R A L O P I B
```

Figure 11: Problem 17.

127 Solution: Split the diagram, as in figure 12. Since every required path must use the R, we count paths starting from R and reaching up to a B. Since there are six more rows that we can travel to, and since at each stage we can go either up or left, we have $2^6 = 64$ paths. The other half of the figure will provide 64 more paths. Since the middle column is shared by both halves, we have a total of $64 + 64 - 1 = 127$ paths.

```
            B
          B I
        B I P
      B I P O
    B I P O L
  B I P O L A
B I P O L A R
```

Figure 12: Problem 17.

18. A survey shows that 90% of high-schoolers in Philadelphia like at least one of the following activities: going to the movies, playing sports, or reading. It is known that 45% like the movies, 48% like sports, and 35% like reading. Also, it is known that 12% like both the movies and reading, 20% like only the movies, and 15% only reading. What percent of high-schoolers like all three activities?

5%

Solution: We make the Venn diagram in as in figure 13. From it we gather the following system of equations

$$\begin{array}{rcl} x + y + z + 20 &=& 45 \\ x + z + t + u &=& 48 \\ x + y + t + 15 &=& 35 \\ x + y &=& 12 \\ x + y + z + t + u + 15 + 20 &=& 90 \end{array}$$

The solution of this system is seen to be $x = 5$, $y = 7$, $z = 13$, $t = 8$, $u = 22$. Thus the percent wanted is 5%.

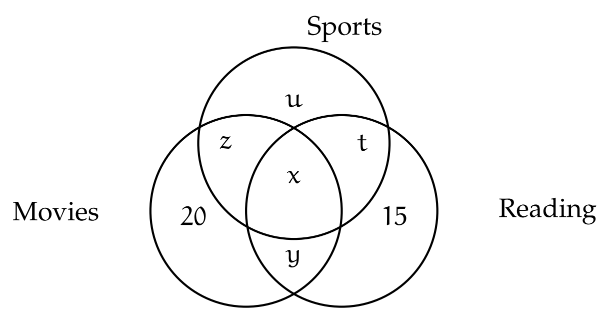


Figure 13: Problem 18.

19. The numbers

$$1, 2, 3, \cdots, 2003$$

are written on a blackboard, in increasing order. Then the first, the fourth, the seventh, etc. are erased, leaving the numbers

$$2, 3, 5, 6, 8, 9, 11, 12, 14, \cdots$$

on the board. This process is repeated, leaving the numbers

$$3, 5, 8, 9, \cdots.$$

The process continues until one number remains on the board and is finally erased. What is the last number to be erased?

1598

Let J_n be the first number remaining after n erasures, so $J_0 = 1$, $J_1 = 2$, $J_3 = 3$, $J_4 = 5$, etc. We prove by induction that

$$J_{n+1} = \frac{3}{2} J_n \text{ if } J_n \text{ is even,}$$

and

$$J_{n+1} = \frac{3}{2}(J_n + 1) - 1 \text{ if } J_n \text{ is odd.}$$

Assume first that $J_n = 2N$. Consider the number $3N$. There are initially N smaller numbers $\equiv 1 \mod 3$. So after the first erasure, it will lie in $2N$-th place. Hence, it will lie in first place after $n + 1$ erasures. Assume now that $J_n = 2N + 1$. Consider $3N + 2$. There are initially $N + 1$ smaller numbers $\equiv 1 \mod 3$. So after the first erasure, it will lie in $2N + 1$-st place. Hence, it will lie in first place after $n + 1$ erasures. That completes the induction. We may now calculate successively the members of the sequence: $1, 2, 3, 5, 8, 12, 18, 27, 41, 62, 93, 140, 210, 315, 473, 710, 1065, 1598, 2397$. Hence 1598 is the last surviving number from $1, 2, ..., 2003$.

20. $\mathcal{A}$ is a set of one hundred distinct natural numbers such that any triplet a, b, c of $\mathcal{A}$ (repetitions are allowed in a triplet) gives a non-obtuse triangle whose sides measure a, b, and c. Let $S(\mathcal{A})$ be the sum of the perimeters obtained by adding all the triplets in $\mathcal{A}$. Find the smallest value of $S(\mathcal{A})$. Note: we count repetitions in the sum $S(\mathcal{A})$, thus all permutations of a triplet (a, b, c) appear in $S(\mathcal{A})$.

868500000 Solution: Let m be the largest member of the set and let n be its smallest member. Then $m \geq n + 99$ since there are 100 members in the set. If the triangle with sides n, n, m is non-obtuse then $m^2 \leq 2n^2$ from where

$$(n+99)^2 \leq 2n^2 \iff n^2 - 198n - 99^2 \geq 0 \iff n \geq 99(1+\sqrt{2}) \iff n \geq 240.$$

If $n < 240$ the stated condition is not met since $m^2 \geq (n+99)^2 \geq 2n^2$ and the triangle with sides of length n, n, m is not obtuse. Thus the set

$$\mathcal{A} = \{240, 241, 242, \ldots, 339\}$$

achieves the required minimum. There are $100^3 = 1000000$ triangles that can be formed with length in $\mathcal{A}$ and so 3000000 sides to be added. Of these $3000000/100 = 30000$ are 240, 30000 are 241, etc. Thus the value required is

$$30000(240 + 241 + \cdots + 339) = (30000)\left(\frac{100(240+339)}{2}\right) = 868500000.$$

1. (3 minutes) What is the difference between the sum of all odd positive integers up to 2003 and the sum of all the even positive integers up to 2003?

   $\boxed{1002}$ We need

   $$(1+3+\cdots+2003)-(2+4+\cdots+2002) = (1-2)+(3-4)+\cdots+(2001-2002)+2003 = -1001+2003 = 1002.$$

2. (4 minutes) Evaluate the sum

   $$\left(\frac{1\cdot 2\cdot 4+2\cdot 4\cdot 8+3\cdot 6\cdot 12+\cdots+100\cdot 200\cdot 400}{1\cdot 3\cdot 9+2\cdot 6\cdot 18+3\cdot 9\cdot 27+\cdots+100\cdot 300\cdot 900}\right)^{1/3}$$

   $\boxed{\frac{2}{3}}$ Solution: Let

   $$a = 1\cdot 2\cdot 4+2\cdot 4\cdot 8+3\cdot 6\cdot 12+\cdots+100\cdot 200\cdot 400 = (1^3+2^3+\cdots+100^3)(1\cdot 2\cdot 4)$$

   and

   $$b = 1\cdot 3\cdot 9+2\cdot 6\cdot 18+3\cdot 9\cdot 27+\cdots+100\cdot 300\cdot 900 = (1^3+2^3+\cdots+100^3)(1\cdot 3\cdot 9).$$

   Then

   $$\left(\frac{a}{b}\right)^{1/3} = \left(\frac{(1^3+2^3+\cdots+100^3)(1\cdot 2\cdot 4)}{1^3+2^3+\cdots+100^3)(1\cdot 3\cdot 9)}\right)^{1/3} = \left(\frac{8}{27}\right)^{1/3} = \frac{2}{3}.$$

3. (6 minutes) Figure 14 is symmetric and is made up of three squares each of side 1. What is the radius of the smallest circle that encloses the figure?

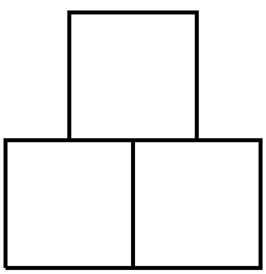


Figure 14: Problem 3.

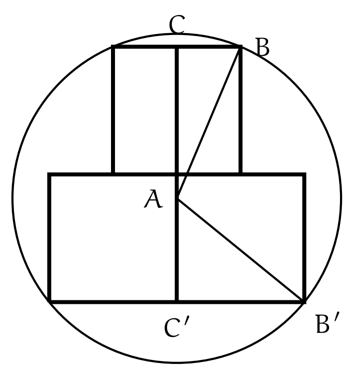


Figure 15: Problem 3.

   $\boxed{\frac{5\sqrt{17}}{16}}$ Solution: Let $r = AB = AB'$ be the radius sought, as in figure 15. From $\triangle ABC$ we deduce

   $$r^2 = \left(\frac{1}{2}\right)^2 + (2 - C'A)^2.$$

   From $\triangle AB'C'$ we deduce

   $$r^2 = 1^2 + (C'A)^2.$$

   Equating both equalities, we deduce $C'A = \dfrac{13}{16}$ and putting this back into either of the above equalities we deduce $r = \dfrac{5\sqrt{17}}{16}$

4. (3 minutes) Factor 111111 (six 1's) into primes.

$\boxed{(3)(7)(11)(13)(37)}$ Solution:

$$\begin{aligned} 111111 &= (111)(1001) \\ &= (3)(37)(11)(91) \\ &= (3)(37)(11)(7)(13) \\ &= (3)(7)(11)(13)(37) \end{aligned}$$

5. (4 minutes) Let $r \neq -2003$ be a real number. When $y = x^2 + rx + 2003$ and $y = x^2 - 2003x + r$ are graphed on the xy-plane, they intersect at a point (x_0, y_0). Find the value of x_0.

$\boxed{\dfrac{r-2003}{r+2003}}$ Solution: We have

$$x_0^2 + x_0 r + 2003 = x_0^2 - 2003x_0 + r \implies x_0(r+2003) = r - 2003 \implies x_0 = \frac{r-2003}{r+2003}.$$

6. (4 minutes) A brick in the form of a regular polygon is removed from a wall. It is observed that if the brick suffered a rotation of either $40°$ or $60°$ about its center it would fit back again into its original place. What is the smallest number of sides of this polygon?

$\boxed{\text{18 sides}}$ Solution: Let n be the required minimum of sides. Each exterior angle of the figure measures $\dfrac{360°}{n}$. This exterior angle must be small enough so that when we turn the figure either $40°$ or $60°$, we obtain the same shape. This will be accomplished when $\dfrac{360°}{n}$ divides the greatest common divisor of $40°$ and $60°$, which is $20°$. Since $\dfrac{360°}{n} = 20°$ has the solution $n = 18$ this is the sought minimum.

7. (3 minutes) An urn contains 28 blue marbles, 20 red marbles, 12 white marbles, 10 yellow marbles, and 8 magenta marbles. How many marbles must be drawn from the urn in order to assure that there will be 15 marbles of the same color?

$\boxed{59}$ Solution: If all the magenta, all the yellow, all the white, 14 of the red and 14 of the blue marbles are drawn, then in among these $8 + 10 + 12 + 14 + 14 = 58$ there are no 15 marbles of the same color. Thus we need 59 marbles in order to insure that there will be 15 marbles of the same color.

8. (4 minutes) Little Dwayne is back! He writes consecutive positive integers starting with 1 on a triangle, each time adding two numbers to a new row, as in figure 16.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & 3 & 4 & & \\ & 5 & 6 & 7 & 8 & 9 & \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Figure 16: Problem 8.

If he stops when he writes the number 2003, in what row does he stop?

$\boxed{\text{45-th}}$

Solution: The last number of the n-th row is n^2. Since $44^2 = 1936 < 2003 < 2025 = 45^2$, 2003 lies in the 45-th row.

9. (6 minutes) $\triangle ABC$ in figure 17 is equilateral, of side 2. Its vertex B is on the side AX of the square AXYZ of side 4. If the triangle rotates clockwise in such a way that each of its sides successively touches a side of the square traversing the whole square until the triangle returns to its original position (and original orientation), what is the length of the path travelled by vertex C? Write your answer in the form $a\pi$ where a is a rational number.

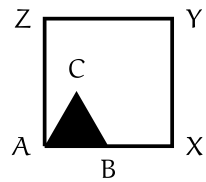


Figure 17: Problem 9.

$\boxed{\frac{40\pi}{3}}$ Solution: From figure 18 we need 8 iterations to turn $\triangle ABC$ into $\triangle CAB$. Thus we need 24 iterations to return $\triangle ABC$ to its original position and orientation. In 8 of the 24 iterations vertex C travels a third of the circumference, that is $\frac{4\pi}{3}$ radians per iteration. In 8 of the 24 iterations C is the pivot and so travels nothing. On the remaining 8 iterations vertex C travels one twelfth of the circle, that is, $\frac{\pi}{3}$ radians per iteration. Thus vertex C has travelled $8\left(\frac{4\pi}{3}\right) + 8\left(\frac{\pi}{3}\right) = \frac{40\pi}{3}$ radians.

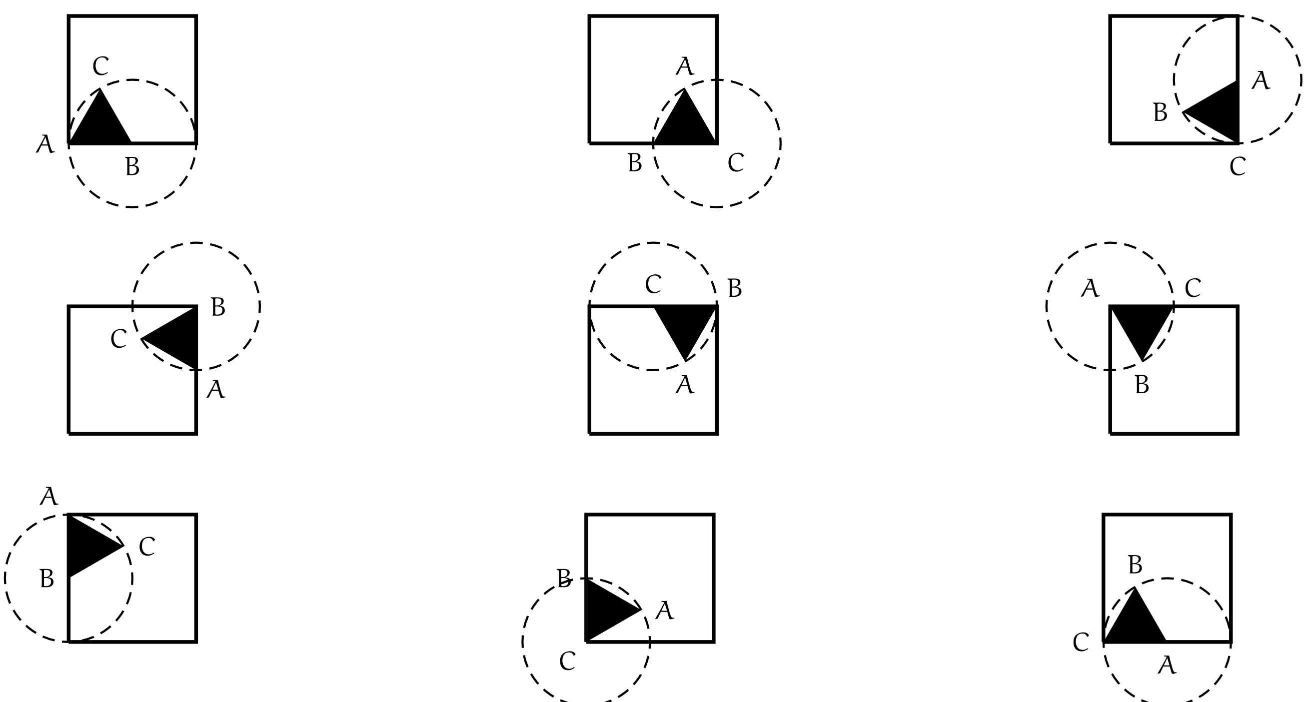


Figure 18: Problem 9.

The 2006 Colonial Math Contest, held on February 9, 2006, was a great success. Forty students on 9 teams participated and although we were expecting 22 more students, we still had an exciting and challenging contest.

The Individual Contest in the morning was close. The winner was Ian Cassel from Lansdale Catholic HS. There was a 4-way tie for 2nd place (just one point behind 1st place): Xiufen Chen from Lincoln HS, Steve Kroukowski, David Martz and Charles Yocum all from Lansdale Catholic HS.

The Team Contest in the afternoon consisted of 12 problems. Each problem was scored and team scores were posted before the next problem started, so teams could see the standings at all times. Lincoln HS jumped out to an early lead on the first 5 problems, but a late surge by the Lansdale Catholic 1 team, won the event. Members of the winning team were Ian Cassel, David Martz, Brianne Riviello and Jenna Wasylenko.

Members of the winning team and first and second place winners of the individual contest received copies of MAPLE 10 software (provided by Maplesoft) as prizes. Contest problems and solutions will be posted on the Math Dept website at http://faculty.ccp.edu/dept/math/index.html.

And now the thank yous: huge thanks to David Santos for preparing the problems and providing the solutions; to Mark Saks and Eleonora Chertok for feedback on the problems; to Chris Lewis and the Admissions Office for helping organize the contest and, of course, for lunch; to the AV staff for excellent support; to faculty Margaret Hitczenko, Robert Smith, Mohamed Teymour, John Jernigan, Elena Koublanova, Wimayra Luy and David Santos and to students Kyle Hofler and Joseph Heard for helping run the contest; a special thanks to Joanne Darken and Maplesoft for the prizes; and to the HS faculty Jackie Burton (Lincoln HS), Roger Cazeau (Blair Christian Academy), Jason File (MaST Community Charter),Casey Huckel (Franklintowne HS), Christine McCrane (Lansdale Catholic HS) and Hung Phan (Paul Robeson HS) for bringing and encouraging your students and for your help with the team contest. Kudos to all.

1 What is the greatest number of Mondays that can occur in 45 consecutive days?

2 A $4 \times 4 \times 4$ wooden cube is painted red and then cut into sixty-four $1 \times 1 \times 1$ cubes. How many of the $1 \times 1 \times 1$ cubes have exactly two of their sides painted red?

3 What is

$$(2 + 4 + \cdots + 2006) - (1 + 3 + \cdots + 2005),$$

that is, what is the difference between the sum of all the odd positive integers up to 2006 and the sum of all the even positive integers up to (and including) 2006?

4 How many isosceles triangles are there with perimeter 11 and each side of integral length?

5 Consider the two infinite repeating decimals

$$a = 0.\overline{12345} = 0.1234512345\ldots; \qquad b = 0.\overline{98765} = 0.9876598765\ldots.$$

If the sum $a + b$ is written as an improper fraction in lowest terms $\frac{p}{q}$, find $p^2 + q^2$.

6 Two circles of radius 2 and centers O and P are tangent to one another as shown in figure 19. If AD and BD are tangents, find the length of BD.

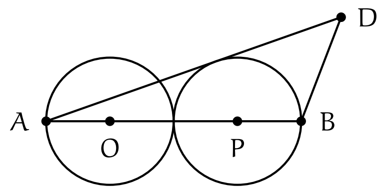


Figure 19: Problem 6.

7 In figure 20 $\triangle ABC, \triangle FDC, \triangle GEC$ are isosceles. Also $\overline{AB} = 3\overline{AC}$. The perimeter of $\triangle ABC$ is 84. D is the midpoint of BC; E is the midpoint of DC; F is the midpoint of AC and G is the midpoint of FC. Find the perimeter of the shaded quadrilateral DEGF.

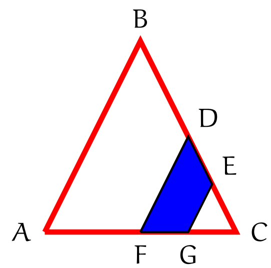


Figure 20: Problem 7.

8 In $\triangle ABC$, E and F are on $\overline{AB}$, with E between A and F and satisfying

$$AE : EF : FB = 1 : 2 : 3.$$

Points G and D are on CB with G between C and D and satisfying

$$CG : GD : DB = 4 : 3 : 2.$$

If FG intersects ED at H, find the ratio $DH : HE$.

9 Reduce to lowest terms: $\dfrac{116690151}{427863887}$.

10 The product of three consecutive even integers is $87 * * * * * 8$, where the asterisks represent five missing digits. What is the sum of the integers?

11 How many (unordered) triplets of real numbers $\{x, y, z\}$ are there such that

$$x(x + y + z) = 26; \qquad y(x + y + z) = 27; \qquad z(x + y + z) = 28?$$

12 If a, b, c are positive integers such that $a^2 + b - c = 100$ and $a + b^2 - c = 124$, find $a + b + c$.

13 Let $\frac{p}{q}$ be the unique rational number in reduced form, such that $\frac{59}{80} < \frac{p}{q} < \frac{45}{61}$ and $0 < q < 200$. Find the sum $p + q$.

14 A die consists of a cube which has a different color on each of 6 faces. How many distinguishably different kinds of dice can be made?

15 How many integers in the set $\{100, 101, 102, \ldots, 198, 199\}$ of 100 consecutive integers **are not** the sum of four consecutive integers?

16 Two squares are inscribed in a semicircle as in figure 21. If the area of the smaller square is a, what is the area of the larger square?

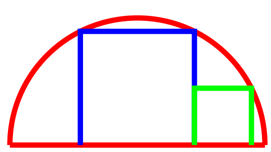


Figure 21: Problem 16.

17 Consider the sequence

$$1, \quad 1,2, \quad 1,2,3, \quad 1,2,3,4, \quad 1,2,3,4,5, \quad \ldots.$$

Find its 2005-th term. You may use the formula

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

18 Consider 2005 hexagonal "domino" pieces with the numbers $1,2,3,4,5,6$ written on the edges in clockwise fashion, as in figure 22. A "chain" is formed so that domino rules are observed. This means that two edges from different pieces are joined only when the numbers on the edges agree. The numbers on the edges joining two different domino pieces are now deleted and the remaining numbers are now added. What is the maximum value of this sum?

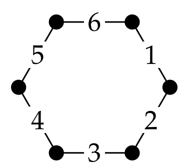


Figure 22: Problem 18.

19 Find the greatest common divisor of

$$a=\underbrace{1\ldots1}_{\text{forty ones}} \quad \text{and} \quad b=\underbrace{1\ldots1}_{\text{twelve ones}}.$$

20 The rectangle in figure 23 is decomposed into nine squares. If the shaded square has area 1, what is the area of the rectangle?

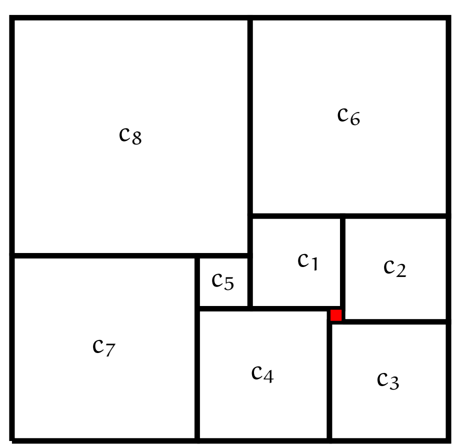


Figure 23: Problem 20

STUDENT:	**SCHOOL:**
(Optional) Age:	**(Optional) Sex:**
1.	2.
3.	4.
5.	6.
7.	8.
9.	10.
11.	12.
13.	14.
15.	16.
17.	18.
19.	20.

21 3 minutes The positive non-squares are ordered in increasing order:

$$2, \quad 3, \quad 5, \quad 6, \quad 7, \quad 8, \quad 10, \ldots.$$

Find the 100-th term.

22 4 minutes The odd positive integers are written in a triangular array as follows

$$\begin{array}{cccccc} & & & & & 1 \\ & & & & 3 & 5 \\ & & & 7 & 9 & 11 \\ & & 13 & 15 & 17 & 19 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

where the n-th row has n numbers. What is the sum of the 20-th row?

23 4 minutes Find the largest integer k such that 5^k divides

$$3 \times 10! + 12 \times 5! + 4 \times 7!.$$

Here $n! = 1 \times 2 \times 3 \times \cdots \times n$.

24 6 minutes What is the sum of the positive integers less than ten million having exactly 77 divisors?

25 4 minutes An equilateral triangle and a regular hexagon have equal perimeters. If T is the area of the triangle and H is the area of the hexagon, find the ratio $\dfrac{T}{H}$.

26 3 minutes If $\sqrt{m^3 + m^3 + m^3 + m^3 + m^3} = 25$, find m.

27 5 minutes Find the sum

$$\frac{1}{2} + \left(\frac{1}{3} + \frac{2}{3}\right) + \left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right) + \cdots + \left(\frac{1}{100} + \frac{2}{100} + \cdots + \frac{99}{100}\right)$$

of all positive proper fractions (not necessarily in lowest terms) whose numerators are positive integers and whose denominators are positive integers less than or equal to 100.

28 3 minutes How many zeros are there at the end of the product

$$15^6 \cdot 28^5 \cdot 55^7 \qquad ?$$

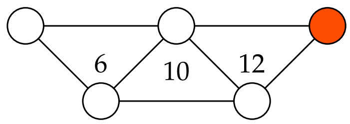
29 4 minutes As a sales gimmick, a merchant began measuring with a yardstick that was x inches too long. If this reduced his profit from 50% of his cost to 20% of his cost, find x.

30 5 minutes A seven-digit base ten number

$$N = 6666abc,$$

where a, b, and c represent digits, is a perfect square. Find $a^2 + b^2 + c^2$.

31 3 minutes Fill in the numbers from 1 up to 5 in the five little circles, in such a way that in each triangle the given number is the sum of the numbers on the vertices. Which number appears in the shaded circle?



32 4 minutes The mass of 200 kg cucumbers consists of 99% of water. The cucumbers are drying out due to the sun, till the mass consists of 98% of water. Determine the weight of the cucumbers now.

33 4 minutes John, Bob and Tom are brothers of Jill and Ingrid. Ingrid is three years younger than Jill. The age of Bob equals the average of the ages of John and Jill. John and Tom together, and also Bob and Ingrid together, are 1 year younger than twice the age of Jill. John and Ingrid together are 1 year older than twice the age of Jill. Who is the eldest child?

34 5 minutes The lengths of the sides of a right triangle are respectively $x - y$, x, and $x + y$, where $0 < y < x$. Find the ratio $\frac{x}{y}$.

35 5 minutes A positive integer, written in decimal notation, has 4 digits. The first two digits are the same and the last two digits are the same. The number is a perfect square. What is the sum of the digits of this number?

1 If one puts Monday at the very beginning of the period one obtains $[\![\frac{45}{7}]\!] = 7$ Mondays.

2 For every edge, there are 2 such cubes. Since there are 12 edges, there required total is $12 \times 2 = 24$..

3 The desired quantity is

$$(2-1)+(4-3)+\cdots+(2006-2005) = 1003.$$

4 If the sides of the triangle measure a, a, b, then $2a + b = 11$, from where b must be odd. The following triangle inequalities must also be fulfilled

$$2a > b, a + b > a.$$

Thus one has the following possibilities:

$$b = 1, a = 5; \quad b = 3, a = 4; \quad b = 5, a = 3,$$

so there are three such triangles.

5 Observe that $a + b = 1.\overline{1} = \frac{10}{9} = \frac{p}{q}$, hence $p^2 + q^2 = 181$

6 Let line AD be tangent to circle P at E. Then $DE = DB$ and

$$AE^2 = AP^2 - PE^2 = 36 - 4 = 32 \implies AE = 4\sqrt{2}.$$

Also

$$AD^2 = BD^2 - AB^2 \implies (AE + BD)^2 = 64 + BD^2 \implies (4\sqrt{2} + BD)^2 = 64 + BD^2,$$

which in turn

$$\implies 32 + 8BD\sqrt{2} + BD^2 = 64 + BD^2 \implies BD = 2\sqrt{2}.$$

7 Let $x = AC$. Then $x + 3x + 3x = 84 \implies x = 12$. Hence $FC = \frac{AC}{2} = 6$ and $FG = 3$. Also $DC = \frac{BC}{2} = 18$. From this the desired perimeter is 39.

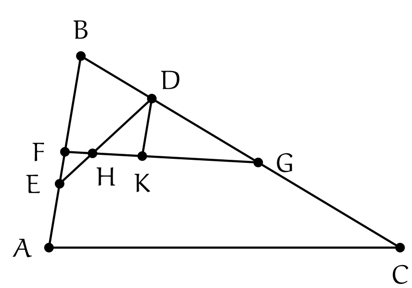
8 Choose K on line FG such that $DK \| EF$. As $\triangle GDK \sim \triangle GBF$, it follows that

$$\frac{DK}{BF} = \frac{DG}{BG} \implies DK = \frac{9y}{5}.$$

As $\triangle HFE \sim \triangle HKD$, it follows that

$$\frac{DH}{HE} = \frac{DK}{EF} = \frac{DK}{2y} = \frac{3}{2} \cdot \frac{DK}{3y} = \frac{3}{2} \cdot \frac{3x}{5x} = \frac{9}{10}.$$

The required ratio is thus $\frac{9}{10}$.



9 $\frac{116690151}{427863887} = \frac{3 \cdot 38896717}{11 \cdot 38896717} = \frac{3}{11}$.

10 Let the integers be $n-2, n, n+2$. Forcedly, they must end in $2, 4, 6$, since their product ends in 8. The product of the integers is $n^3 - 4n \approx n^3$. Now $n^3 \approx 87000008 \implies n \approx 443$. Inspection gives $n = 444$, and the sum is $442 + 444 + 446 = 1332$.

11 Adding the equations $(x + y + z)^2 = 91 \implies x + y + z = \pm 9$. Thus there are two triplets.

12 Subtracting both equations,

$$a - a^2 + b^2 - b = 24 \implies (b - a)(b + a - 1) = 24,$$

and one of the factors must be odd and the other even. One tries either $3 \cdot 8$ or $1 \cdot 24$. The first factorization is rejected, since it gives $b = 6$, $a = 3$, $c = -85$. The second factorization yields $a = 12, b = 13, c = 57$ and so $a + b + c = 82$.

13 Since we are given that this is the unique fraction, we don't have to prove uniqueness (which is a lot harder to prove). Observe that if a, b, c, d are positive integers then

$$\frac{a}{b} < \frac{c}{d} \implies \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

Hence

$$\frac{59}{80} < \frac{104}{141} < \frac{45}{61},$$

and $p + q = 245$.

14 Choose a specific colour for the upper face of the cube, say A. Then we have five choices for colouring the lower face of the cube, say with colour B. Rotate the cube so that some colour C is facing us. Now the remaining sides are fixed with respect to these three. We can distribute the three colours in $3 \times 2 \times 1 = 6$ ways, giving $5 \times 6 = 30$ possibilities.

15 If $n - 1, n, n + 1, n + 2$ are four consecutive integers, then their sum is $4n + 2$, that is, the integers sought do not leave remainder 2 upon division by 4. Since three quarters of the integers leave some other remainder, the answer is $\frac{3}{4} \cdot 100 = 75$.

16 Rotate the figure as in figure 24. The larger square is then decomposed into 4 of the smaller square. The area sought is $4a$.

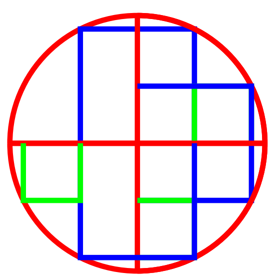


Figure 24: Problem 16.

17 The n-th group has n members and sum $\frac{n(n+1)}{2}$. Upon writing the last term of the n-th group, $\frac{n(n+1)}{2}$ terms have been written. Now $\frac{n^2}{2} \approx 2005 \implies n \approx 63$, that is, we should look around the 63rd group in order to find the 2005-th term. Now after the 63rd group has been written, $\frac{63(64)}{2} = 2016$ terms have been written. Hence the 2005-th term is the 52 in the group

$$1, 2, 3, \ldots, 62, 63.$$

Originally this problem asked for the sum of the first 2005 terms, but it was then considered too difficult. Let's see how to find this sum. We have to sum

$$\frac{1(1+1)}{2} + \frac{2(2+1)}{2} + \frac{3(3+1)}{2} + \cdots + \frac{62(62+1)}{2} + (1 + 2 + \cdots + 52).$$

This is

$$\left(\sum_{k=1}^{62} \frac{k^2+k}{2}\right) + 1378.$$

Since

$$(1^2+2^2+\cdots+n^2) = \frac{n(n+1)(2n+1)}{6} \qquad \text{and} \qquad 1+2\cdots+n = \frac{n(n+1)}{2},$$

the desired sum is

$$\frac{1}{2}\left(1^2+2^2+\cdots+62^2\right) + \frac{1}{2}(1+2+\cdots+62) + (1+2+\cdots+52) = \frac{1}{2}\cdot\frac{62\cdot 63\cdot 125}{6} + \frac{1}{2}\cdot\frac{62\cdot 63}{2} + \frac{52\cdot 53}{2} = 43042.$$

18 In order to obtain the maximal sum, we must have a chain where every piece is glued to the smallest possible of sides using the smallest numbers. The chain then must have 2003 pieces which are joined on two sides and the two end pieces are joined at only one side. If the 1 and the 2 were shared, the first and second piece must be joined at 1 and the second and third at 2. But this is impossible because then the first and the third piece would be joined at 3 and 6, violating domino rules. Hence only 3's and 1's are shared. The middle 2003 contribute $2+4+5+6=17$ each, one of the end pieces contributes $2+3+4+5+6=20$, and the other end piece contributes $1+2+4+5+6=18$ hence the maximum sum is $(2003)(17)+1(20)+1(18)=34089$.

19 Let $a_n = \underbrace{1\ldots1}_{n \text{ ones}}$. Then using the Euclidean Algorithm,

$$\begin{aligned}
\gcd(a_{40}, a_{12}) &= \gcd(a_{40}-a_{12}, a_{12}) \\
&= \gcd(a_{28}\cdot 10^{12}, a_{12}) \\
&= \gcd(a_{28}, a_{12}) \\
&= \gcd(a_{28}-a_{12}, a_{12}) \\
&= \gcd(a_{16}\cdot 10^{12}, a_{12}) \\
&= \gcd(a_{16}, a_{12}) \\
&= \gcd(a_{16}-a_{12}, a_{12}) \\
&= \gcd(a_{4}\cdot 10^{12}, a_{12}) \\
&= \gcd(a_{4}, a_{12}) \\
&= a_4,
\end{aligned}$$

whence the gcd sought is 1111.

20 Suppose c_1 has side c. Then we see that c_2 has side $c+1$, c_3 has side $c+2$, and c_4 has side $c+3$. From this it follows that c_5 has side 4. Continuing this process, we deduce that c_6 has side $2c+1$. The height of the initial rectangle is thus

$$(2c+1)+(c+1)+(c+2) = 4c+4.$$

Square c_7 has side

$$c+3+4 = c+7.$$

The width of the original rectangle is thus

$$(c+7)+(c+3)+(c+2) = 3c+12.$$

Square c_8 has side of length

$$c+7+4 = c+11.$$

Finally, two opposite sides of the rectangle have dimensions $4c+4$ and

$$(c+7)+(c+11) = 2c+18.$$

Since they must be equal we deduce $4c+4 = 2c+18 \implies c=7$. Conclusion: the original rectangle has area $33\times 32 = 1056$.

21 Since there are 10 squares between 1 and 110 inclusive 110 is the 100-th term.

22 Upon writing the last element of the $(n-1)-$th row,

$$1+2+\cdots+n-1=\frac{(n-1)n}{2}$$

numbers have been written. Thus the first element of the n-th row is the $\frac{(n-1)n}{2}+1$-st odd integer, namely $2(\frac{(n-1)n}{2}+1)-1=n^2-n+1$. This means that the sum of 20-th row is

$$381+383+\cdots+\cdots+419=\frac{20(381+419)}{2}=8000.$$

23 We have

$$\begin{aligned} 3\times 10!+12\times 5!+4\times 7! &= 6!(3\times 7\times 8\times 9\times 10+2+4\times 7) \\ &= 6!(3\times 7\times 8\times 9\times 10+30) \\ &= 30\cdot 6!(7\times 8\times 9+1) \\ &= 30\cdot 6!\cdot 505, \end{aligned}$$

whence $k=3$.

24 First observe that $2^{10}\approx 10^3$, and so $2^{30}\approx 10^6$, which implies that $2^{34}\approx 10^7$. This means that any exponent present of any prime must be ≤ 34. If n has the prime factorization

$$n=p_1^{a_1}\cdots p_k^{a_k}$$

with primes $p_1<p_2<\cdots<p_k$ and $a_i\geq 1$, then the number of divisors of n is

$$(a_1+1)\cdots(a_k+1).$$

If

$$(a_1+1)\cdots(a_k+1)=77,$$

then one must have $k\leq 2$, and then either $a_1=6, a_2=10$, and one number is $2^6 3^{10}=3779136$, or $a_1=10, a_2=6$, and another number is $2^{10}3^6=746496$. Observe that $2^{10}5^6=16000000>10^7$. The desired sum is thus $3779136+746496=4525632$.

25 Let $P=6a=3b$, where a is the length of a side of the hexagon and b is the length of a side of the triangle. This entails that $2a=b$ and $4a^2=b^2$. The area of the hexagon is $6\left(\frac{\sqrt{3}}{4}a^2\right)$ and the area of the triangle is $\frac{\sqrt{3}}{4}b^2$. Thus

$$\frac{\frac{\sqrt{3}}{4}b^2}{6\left(\frac{\sqrt{3}}{4}a^2\right)}=\frac{b^2}{6a^2}=\frac{4a^2}{6a^2}=\frac{2}{3}.$$

26 We have $\sqrt{5m^3}=25 \implies 5m^3=5^4 \implies m^3=5^3 \implies m=5$.

27 Recall from Gauß trick that

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

Hence the k-th sum, $k\geq 2$, in parentheses is

$$\left(\frac{1}{k}+\frac{2}{k}+\cdots+\frac{k-1}{k}\right)=\frac{(k-1)k}{2k}=\frac{k-1}{2}.$$

Thus the desired sum is

$$\frac{1}{2}\cdot(1+2+\cdots+99)=2475.$$

28 The product equals

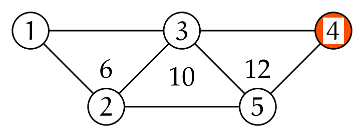
$$3^6 5^6 7^5 2^{10} 5^7 11^7=3^6 5^3 7^5 10^{10} 11^7,$$

which ends in 10 zeros.

29 One has
$$1.5 = 1.2(1 + \frac{x}{36}) \implies x = 9.$$

30 We have $\sqrt{6666000} \approx 2581.9 > 2581$. Now $2583^2 = 6671889$ and $2582^2 = 6666724$, and so the only answer is 6666724. Thus $7^2 + 2^2 + 4^2 = 69$.

31 The figure below shows the correct arrangement.



32 Originally, we have 198 kg of water, and 2 kg of cucumber mass. After evaporation, the 2 kg of cucumber mass make 2% of the current weight w and thus $.02w = 2 \implies w = 100$ kg.

33 Let j, b, t, l, i be the ages of John, Bob, Tom, Jill, and Ingrid, respectively. We are given
$$i + 3 = l; \quad b = \frac{j+l}{2}; \quad j + t = b + i = 2l - 1; \qquad j + i = 2l + 1.$$
We will express all other ages in terms of one, let us say, Jill's age. We have $i = l - 3$, so Ingrid is younger than Jill. Also, $b = 2l - 1 - i = l + 2$, so Bob is older than Jill. Continuing $j = 2l + 1 - i = l + 4$, so John is older than Bob. Now $t = 2l - 1 - j = l - 5$, so Tom is younger than Ingrid. Therefore, in increasing order of age we have Tom, Ingrid, Jill, Bob, and John.

34 We have that
$$(x-y)^2 + x^2 = (x+y)^2 \implies x^2 = 4xy \implies x = 4y,$$
whence $\frac{x}{y} = 4$.

35 The number is of the form $aabb$, where $a \neq 0$ and b are digits. Now, $aabb = 11(a0b)$ clearly divisible by 11. But since the number is a square, the number $a0b$ must be divisible by 11. This means that $a - 0 + b$ must be divisible by 11, and so the number could be either 4477 or 7744. The sum of the digits is thus 22.