Important theorems of analysis, topology and algebra

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Analysis, topology	Algebra
R has nested interval completeness: given an increasing (even non-decreasing) sequence of real numbers, all of whose members are less than or equal to all of the members of a decreasing (even non-increasing) sequence, if the terms of the lesser sequence approach the corresponding terms of the greater sequence arbitrarily closely, there exists a common limit to the two sequences. **R** (or **R**^n, or a complete, separable, perfect metric space) is uncountable. Baire Category Theorem: a countable intersection of dense open subsets of **R** (or **R**^n, or a complete, separable, perfect metric space) is non-empty; equivalently, **R** (or **R**^n, or a complete, separable, perfect metric space) is not a countable union of closed sets with empty interiors. Intermediate Value Theorem: if f: $[a,b] \to$ **R** is continuous and y is between $f(a)$ and $f(b)$, then there exists x between a and b such that $f(x) = y$. Urysohn's Lemma: two disjoint closed sets in **R** (or **R**^n, or a complete, separable, perfect metric space) can be separated by a continuous real function. Tietze's Extension Theorem: a continuous function defined on a closed subset of **R** (or **R**^n, or a complete, separable, perfect metric space) can be extended to a continuous function on the whole space.	Z is a Euclidean domain; therefore every ideal of Z is principal; therefore the Fundamental Theorem of Arithmetic: every natural number has a unique factorization as a product of primes. Every odd degree polynomial over **R** has a zero in **R**. Every (countable) field has an algebraic closure. Every (countable) ordered field has a real closure.
Heine-Borel Theorem: [0,1] (any closed, bounded interval) is compact; every closed, bounded subset of **R** (or **R**^n, or a complete, separable, perfect, totally bounded metric space) is compact (every open cover has a finite subcover). Every continuous function on [0,1] (any closed, bounded interval) is uniformly continuous; is bounded; has a maximum and a minimum; attains a maximum and a minimum; (Weierstrass) can be approximated arbitrarily closely (in the sup norm) by a polynomial; is Riemann integrable. Rolle's Theorem: if f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, then there exists x between a and b such that $f'(x) = 0$; equivalently, the Mean Value Theorem: if f is continuous on $[a,b]$ and differentiable on (a,b), then there exists x between a and b such that $f'(x) = (f(b) - f(a))/(b - a)$. Mean Value Theorem for Integrals: if f is continuous on $[a,b]$, there exists x between a and b such that the integral of f from a to b equals $f(x)(b - a)$. Fundamental Theorems of Calculus: 1) if f is continuous on $[a,b]$, and $F(x)$ = the integral of f from a to x, then $F'(x) = f(x)$; 2) if $F'(x) = f(x)$, then the integral of f from a to $b = F(b) - F(a)$. Local existence of solutions of finite systems of ordinary differential equations. Hahn-Banach Theorem. Fundamental Theorem of Algebra: every polynomial over **C** has a zero in **C**.	Artin-Schreier Theorem: every (countable) formally real field can be ordered; has a real closure; has a unique real closure. Every (countable) field has a unique algebraic closure. Every (countable) commutative, associative ring with unit has a radical ideal; has a prime ideal.
Open mapping theorem. Closed graph theorem.	

Least upper bound (respectively, greatest lower bound) principle: **R** is sequentially complete; that is, every bounded, increasing (respectively, decreasing) sequence of real numbers has a least upper bound (respectively, greatest lower bound); every bounded sequence of real numbers has a least upper bound and a greatest lower bound. Bolzano-Weierstrass Theorem: every bounded sequence of real numbers (or points of **R**^*n*, or of a complete, separable, perfect, totally bounded metric space) has a convergent subsequence. Every Cauchy sequence of real numbers (or points of **R**^*n*, or of a complete, separable, perfect metric space) converges. Every sequence of points in a compact metric space has a convergent subsequence. Ascoli Lemma: every bounded, equicontinuous sequence of real functions on a bounded interval has a uniformly convergent subsequence.	Every (countable) vector space has a basis. Every (countable) field has a transcendence base. Every (countable) commutative, associative ring with unit has a maximal ideal. Every (countable) abelian group has a torsion subgroup; has a unique divisible closure.
Souslin's Theorem: a set is Borel if and only if it is analytic and co-analytic. Lusin Separation Theorem: two disjoint analytic sets can be separated by a Borel set. Every uncountable closed (or analytic) subset of **R** (or **R**^*n*, or a complete, separable, perfect metric space) has a non-empty perfect subset.	
Cantor-Bendixson Theorem: every closed subset of **R** (or **R**^*n*, or a complete, separable, perfect metric space) is the union of a countable set and a perfect set.	Every (countable) abelian group is the direct sum of a divisible group and a reduced group.

The classification of most of the theorems in this table comes from Stephen G. Simpson's Subsystems of Second-Order Arithmetic. The six rows of the table represent increasing levels of axiomatic power required to prove the theorems on that row. The theorems that I added to the table may be placed on the correct row, but I am not completely certain. I do not make any claims about the proper order to study and prove these theorems; however, the first four rows of theorems in the table are certainly the most useful. Numerous important theorems have been omitted, but they are either so elementary that they would be placed on the first row of the table, or they are consequences of the basic theorems listed elsewhere in the table.