1. The curve $y=\frac{\ln x}{x}, x\ge 1$ is revolved about the *x*-axis. Is the volume of the resulting solid finite or infinite?

Solution. We apply the disks method and see that the question is about the convergence or divergence of the following improper integral. $\pi\int_1^{\infty}\frac{\ln^2 x}{x^2}dx$ **.** We will prove that the integral converges and find its value using integration by parts. Let $u=\ln^2 x, dv=\frac{1}{x^2}dx$. Then $du=2\ln x\frac{1}{x}dx,\ v=-\frac{1}{x}$, and $\pi\int_1^{\infty}\frac{\ln^2 x}{x^2}dx=-\pi\left.\frac{\ln^2 x}{x}\right|_1^{\infty}+2\pi\int_1^{\infty}\frac{\ln x}{x^2}dx$. The first term in this expression is equal to 0. Indeed, at the lower limit we have $\ln 1=0$ and at the upper limit we have to look at $\lim\limits_{x\to\infty}\frac{\ln^2 x}{x}$. The last limit is an indeterminate form $\frac{\infty}{\infty}$ and we can apply the L'Hospital's rule. $\lim\limits_{x\to\infty}\frac{\ln^2 x}{x}=\lim\limits_{x\to\infty}\frac{2\ln x\cdot\frac{1}{x}}{1}=\lim\limits_{x\to\infty}\frac{2\ln x}{x}=\lim\limits_{x\to\infty}\frac{2/x}{1}=0$. Therefore $\pi\int_1^{\infty}\frac{\ln^2 x}{x^2}dx=2\pi\int_1^{\infty}\frac{\ln x}{x^2}dx$. We apply integration by parts once again; this time we take $u=\ln x, dv=\frac{1}{x^2}dx$. Then $du=\frac{1}{x}dx, v=-\frac{1}{x}$, and $2\pi\int_1^{\infty}\frac{\ln x}{x^2}dx=-2\pi\left.\frac{\ln x}{x}\right|_1^{\infty}+2\pi\int_1^{\infty}\frac{1}{x^2}dx$. Like in the previous expression we see that the first term is 0 and it remains to evaluate $2\pi\int_1^{\infty}\frac{1}{x^2}dx=-2\pi\left.\frac{1}{x}\right|_1^{\infty}=2\pi$. So, the volume is finite and its value is 2π.

2. The same question about the area of the surface of revolution.

Solution. The area of the surface of revolution can be expressed as the following integral. $2\pi\int_1^{\infty}y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$. We will prove, using the comparison test that the integral diverges to ∞. To this end notice the simple inequality $1+\left(\frac{dy}{dx}\right)^2\ge 1$ whence $\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ge\sqrt{1}=1$ and $2\pi\int_1^{\infty}y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx\ge 2\pi\int_1^{\infty}y(x)dx=2\pi\int_1^{\infty}\frac{\ln x}{x}dx$. Let $u=\ln x$ then

$du = \frac{1}{x}dx$ and $2\pi\int_1^\infty \frac{\ln x}{x}dx = 2\pi\int_0^\infty u\,du = \pi u^2\Big|_0^\infty = \infty$. Therefore $2\pi\int_1^\infty y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \infty$ and the area of the surface of revolution is infinite.

3. Prove that the integral $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}dx$ converges and find its value.

Solution.

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}dx = \int_{-\infty}^{\infty} \frac{1}{(x+1)^2+1}dx = \int_{-\infty}^{\infty} \frac{1}{u^2+1}du \text{ (where } u = x+1) =$$

$$= \arctan u\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

(Recall that $\lim_{u\to\pm\infty} \arctan u = \pm\frac{\pi}{2}$)

4. Use the limit comparison test to show that the integral $\int_1^\infty \frac{x^3+3x+5}{x^4\ln x+\sqrt{x}}dx$ diverges.

Solution. We can write $\int_1^\infty \frac{x^3+3x+5}{x^4\ln x+\sqrt{x}}dx = \int_1^2 \frac{x^3+3x+5}{x^4\ln x+\sqrt{x}}dx + \int_2^\infty \frac{x^3+3x+5}{x^4\ln x+\sqrt{x}}dx$. The first term is a proper integral and its value is finite. To the second term we apply the limit comparison test. The leading term in the numerator is x^3 and in the denominator - $x^4\ln x$. Therefore the integral $\int_2^\infty \frac{x^3+3x+5}{x^4\ln x+\sqrt{x}}dx$ converges or diverges at the same time as the integral $\int_2^\infty \frac{x^3}{x^4\ln x}dx = \int_2^\infty \frac{1}{x\ln x}dx = \int_{\ln 2}^\infty \frac{1}{u}du = \ln u\Big|_{\ln 2}^\infty = \infty$, where $u = \ln x$. It proves that the integral $\int_1^\infty \frac{x^3+3x+5}{x^4\ln x+\sqrt{x}}dx$ diverges.

5. Use an appropriate test to find out whether the series $\sum_{n=1}^{\infty}\frac{n^2-5n+3}{n^3\ln^2 n-\sqrt[3]{n}}$ converges or diverges.

Solution. By the limit comparison test the series $\sum_{n=1}^{\infty}\frac{n^2-5n+3}{n^3\ln^2 n-\sqrt[3]{n}}$ behaves in the same way as the series $\sum_{n=2}^{\infty}\frac{n^2}{n^3\ln^2 n}=\sum_{n=1}^{\infty}\frac{1}{n\ln^2 n}$. The terms of the last series are positive and decreasing and therefore the series converges or diverges at the same time as the improper integral $\int_2^{\infty}\frac{1}{x\ln^2 x}dx=\int_{\ln 2}^{\infty}\frac{1}{u^2}du=-\frac{1}{u}\bigg|_{\ln 2}^{\infty}=\frac{1}{\ln 2}<\infty$. Therefore the series $\sum_{n=1}^{\infty}\frac{n^2-5n+3}{n^3\ln^2 n-\sqrt[3]{n}}$ converges.

6. Use the Ratio test to find out whether the series $\sum_{n=0}^{\infty}\frac{n!e^n}{(2n)!}$ converges or diverges.

Solution. Let $a_n=\frac{n!e^n}{(2n)!}$ be the nth term of our series. Then the ratio $\frac{a_{n+1}}{a_n}$ can be simplified as

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)!e^{n+1}}{[2(n+1)]!}\div\frac{n!e^n}{(2n)!}=\frac{(n+1)!}{n!}\cdot\frac{(2n)!}{[2(n+1)]!}\cdot\frac{e^{n+1}}{e^n}=$$

$$=(n+1)\frac{1}{(2n+1)(2n+2)}e=\frac{e}{2(2n+1)}\underset{n\to\infty}{\to}0.$$

By the ratio test the series converges.

7. Find the radius of convergence of the power series $\sum_{n=2}^{\infty}\frac{x^n}{n\ln n}$. What happens at the ends of the interval of convergence?

Solution. We will apply the ratio test to find the radius of convergence. Let $a_n=\frac{x^n}{n\ln n}$. Then $\frac{|a_{n+1}|}{|a_n|}=\frac{n\ln n}{(n+1)\ln(n+1)}|x|$. Next notice that $\lim_{n\to\infty}\frac{n\ln n}{(n+1)\ln(n+1)}=1$. Indeed, $\lim_{n\to\infty}\frac{n}{n+1}=1$. To find the limit $\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}$ we will apply the L'Hospital's rule, $\lim_{x\to\infty}\frac{\ln(u+1)}{\ln u}=\lim_{x\to\infty}\frac{1/(u+1)}{1/u}=\lim_{x\to\infty}\frac{u}{u+1}=1$. Therefore $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|x|$. By the ratio test the series converges absolutely if $|x|<1$ and diverges if $|x|>1$.

We have just proved that the radius of convergence of the series $\sum_{n=2}^{\infty}\frac{x^n}{n\ln n}$ is 1. To see what happens at the ends of the interval $[-1,1]$ let us first put $x=1$. Then our series becomes $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$. The terms of the series are positive and decreasing, hence we can apply the integral test. The series converges or diverges at the same time as the improper integral $\int_2^{\infty}\frac{1}{x\ln x}dx$ and we already have seen when solving Problem 4 that the last integral diverges; so our power series diverges to ∞ at the right end of the interval of convergence.

Now let us put $x=-1$. The series becomes $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$. This is an alternating series. Indeed, the sign of its terms alternates, the absolute values of its terms are decreasing, and its terms tend to 0 when $n\to\infty$. Therefore the series converges. We have just seen that the series of its absolute values diverges, hence the series converges conditionally.

8. Find the value of $\arcsin(0.499)$ with accuracy 10^{-7} using an appropriate Taylor series.

Solution. A "brute force" solution would be to use the Maclaurin series for $\arcsin$ at the point 0.499. The solution suggested below is much more effective.

As often in such problems we will try to state the problem in such a way that we can use Maclaurin series of known pattern. To this end notice that $\arcsin 0.5=\frac{\pi}{6}$.

Next we will use the formula $\arcsin a-\arcsin b=\arcsin\left(a\sqrt{1-b^2}-b\sqrt{1-a^2}\right)$. We can easily prove this formula if we notice that

$$\sin(\arcsin a-\arcsin b)=\sin(\arcsin a)\cos(\arcsin b)-\cos(\arcsin a)\sin(\arcsin b)$$

and recall that $\sin(\arcsin x) = x$ and $\cos(\arcsin x) = \sqrt{1-x^2}$. In our case we get $\frac{\pi}{6} - \arcsin(0.499) = \arcsin\left(\frac{1}{2}\sqrt{1-0.499^2} - 0.499\sqrt{1-\left(\frac{1}{2}\right)^2}\right) \approx \arcsin 0.0011543159$. Now we will use the Maclaurin series $\arcsin x = x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2^n n!(2n+1)} x^{2n+1}$.

If we compute the sum of the first two terms of the series above for $x = 0.0011543159$ we will get $\arcsin(.499) \approx \frac{\pi}{6} - 0.0011543159 - \frac{1}{2\cdot 3}(.0011543159)^3 \approx .5224444594$. All the digits are correct.

Remark. You will not be required to do it on the test but it is not difficult to estimate the error of our approximation. Indeed,

$$|Error| \le \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2^n n!(2n+1)} (0.0011543159)^{2n+1} \le \sum_{n=2}^{\infty} (0.0011543159)^{2n+1} =$$

$$= (0.0011543159)^5 \sum_{n=0}^{\infty} [(0.0011543159)^2]^n = \frac{(0.0011543159)^5}{1-(0.0011543159)^2} \approx .2049386957 \times 10^{-14}$$

9. Write the Maclaurin polynomial of degree four for the function $f(x) = \cos(\sin x)$.

Solution. We will look at two ways to solve the problem. The first one is straightforward. We use the formula

$$M_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 .$$

Successive differentiations provide

$f'(x) = -\sin(\sin(x))\cos(x),$

$f''(x) = -\cos(\sin(x))\cos^2(x) + \sin(\sin(x))\sin(x),$

$f'''(x) = \sin(\sin(x))\cos^3(x) + 3\cos(\sin(x))\cos(x)\sin(x) + \sin(\sin(x))\cos(x),$

$f^{(4)}(x) = \cos(\sin(x))\cos^4(x) - 6\sin(\sin(x))\cos^2(x)\sin(x) - 3\cos(\sin(x))\sin^2(x) +$
$4\cos(\sin(x))\cos^2(x) - \sin(\sin(x))\sin(x).$

Plugging in $x = 0$ we get $f(0) = 1,\ f'(0) = 0,\ f''(0) = -1,\ f'''(0) = 0,\ f^{(4)}(0) = 5$ and

$$M_4(x) = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 .$$

The second way makes use of standard Maclaurin series for $\sin x$ and $\cos x$. We know that $\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \dots$ Therefore $\cos(\sin x) = 1 - \frac{\sin^2 x}{2} + \frac{\sin^4 x}{24} - \dots$ Next notice that $\sin x = x - \frac{x^3}{3!} + \dots = x(1 - \frac{x^2}{6} + \dots)$ whence $\sin^2 x = (x - \frac{x^3}{3!} + \dots)^2 = x^2(1 - \frac{x^2}{6} + \dots)^2$ and $\sin^4 x = (x - \frac{x^3}{3!} + \dots)^4 = x^4(1 - \frac{x^2}{6} + \dots)^4$. Finally notice that the function $f(x)$ is even ($f(-x) = f(x)$) and therefore its Maclaurin series contains only even powers of *x*. Combining all the terms with exponent of *x* not greater than 4 we get $\cos(\sin x) = 1 - \frac{x^2}{2} - x^2 \cdot (-2 \cdot 1 \cdot \frac{x^2}{6})/2 + \frac{x^4}{24} + \dots = 1 - x^2 + \frac{5}{24}x^4 + \dots$, where the ellipsis indicates the terms with powers of *x* greater or equal than five.

10. Estimate the value of $\int_0^{\pi/4} \frac{\sin x}{x} dx$ with accuracy 0.000001 using an appropriate Taylor series.

Solution. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Therefore $\frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$ and

$$\int_0^{\pi/4} \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^{\pi/4} x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n+1}}{2n+1}\Bigg|_0^{\pi/4} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)(2n+1)!}.$$

Next we have to decide how many terms of the last series we want to use for approximation. The series is an alternating one so when we stop the error we make is not greater than the absolute value of the next term. We can compute that for $n = 5$ the absolute value of the fifth term is $.6543351347 \times 10^{-6}$ which is enough for the accuracy we need. Therefore we use the approximation

$$\int_0^{\pi/4} \frac{\sin x}{x} dx \approx \frac{\pi}{4} - \frac{\pi^3}{1152} + \frac{\pi^5}{614400} - \frac{\pi^7}{578027520} \approx .758976.$$