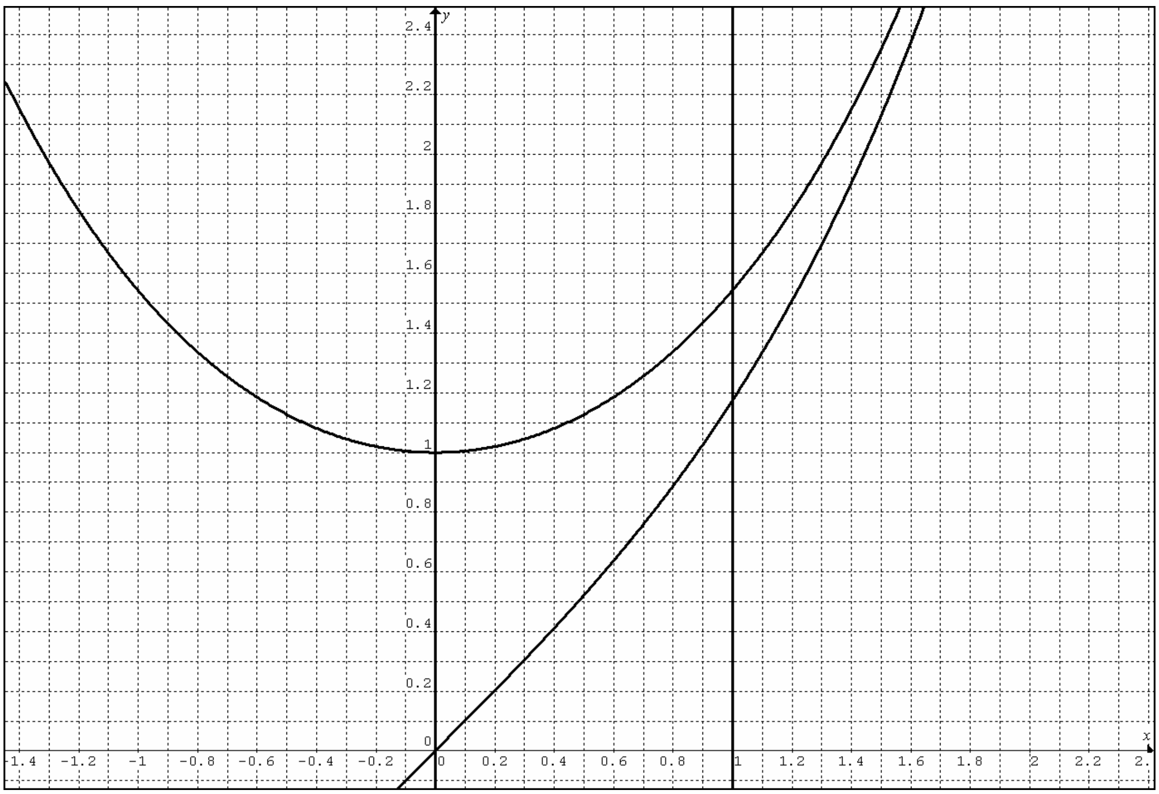
Calculus 2 172 Review 2

Consider the region between curves $y = \cosh x,\ y = \sinh x,\ x = 0,\ x = 1$.

1. Find the area of the region.

Solution. Recall that $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$, whence $\sinh x \le \cosh x$.



Therefore the area is given by the integral

$$A = \int_0^1 (\cosh x - \sinh x)dx = (\sinh x - \cosh x)\Big|_0^1 = \sinh 1 - \cosh 1 - \sinh 0 + \cosh 0 =$$

$$= \frac{e - e^{-1}}{2} - \frac{e + e^{-1}}{2} - 0 + 1 = 1 - \frac{1}{e} = \frac{e-1}{e} \approx 0.63.$$

2. Find the volume of the solid of revolution when the region is revolved about the *x*-axis.

Solution. We will compute the volume using the washers' method and the identity $\cosh^2 x - \sinh^2 x = 1$.

$$V_x = \pi \int_0^1 (\cosh^2 x - \sinh^2 x)dx = \pi \int_0^1 1dx = \pi.$$

3. Find the volume of the solid of revolution when the region is revolved about the *y*-axis.

Solution. We will use the cylindrical shells' method.

$V = 2\pi\int_0^1 x(\cosh x - \sinh x)dx$. We will integrate by parts taking $u = x$

and $dv = (\cosh x - \sinh x)dx$. Then $du = dx$ and $v = \sinh x - \cosh x$ whence

$$V = 2\pi x(\sinh x - \cosh x)\Big|_0^1 - 2\pi\int_0^1 (\sinh x - \cosh x)dx =$$

$$= 2\pi(\sinh 1 - \cosh 1) + 2\pi\int_0^1 (\cosh x - \sinh x)dx.$$

The last integral we have already computed in Problem 1 and therefore

$$V_y = -2\pi\frac{1}{e} + 2\pi\frac{e-1}{e} = 2\pi\frac{e-2}{e} \approx 1.66$$

4. Find the coordinates of the geometric center of the region.

Solution. By the Pappus's Centroid Theorem we have

$x_c = \dfrac{V_y}{2\pi A} = \dfrac{e-2}{e-1} \approx 0.42$, and

$$y_c = \frac{V_x}{2\pi A} = \frac{e}{2(e-1)} \approx 0.80.$$

5. Find the area of the surface of revolution when the region is revolved about the *x*-axis.

Solution. The area in question consists of four parts.

(a) The area of the surface generated by the rotation of the segment of the *y*-axis from 0 to 1about the *x*-axis. This surface is a disk with the radius 1 and its area is equal to π .

(b) The area of the surface generated by the rotation of the segment of the vertical line $x = 1$ from $\sinh 1$ to $\cosh 1$ about the *x*-axis. This surface is an annulus with the radii $\sinh 1$ and $\cosh 1$ and its area is equal to $\pi(\cosh^2 1 - \sinh^2 1) = \pi$.

(c) The area of the surface generated by the rotation of the curve $y = \cosh x$ about the *x*-axis. This area is given by the formula

$$S_1 = 2\pi\int_0^1 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^1 \cosh x\sqrt{1+\sinh^2 x}\,dx =$$

$$= 2\pi\int_0^1 \cosh x\sqrt{\cosh^2 x}\,dx = 2\pi\int_0^1 \cosh^2 x\,dx = \pi\int_0^1 (\cosh 2x+1)\,dx =$$

$$= \frac{\pi}{2}\sinh 2x\Big|_0^1 + \pi = \pi\left(\frac{e^2-e^{-2}}{4}+1\right) = \pi\frac{e^4+4e^2-1}{4e^2}.$$

(d) The area of the surface generated by the rotation of the curve $y=\sinh x$ about the x-axis. This area is given by the formula

$S_2 = 2\pi\int_0^1 \sinh x\sqrt{1+\cosh^2 x}\,dx$. Let $u=\cosh x$, then $du=\sinh x\,dx$ and therefore

$$S_2 = 2\pi\int_1^{\cosh 1}\sqrt{1+u^2}\,du \overset{formula\,21}{=} 2\pi\left(\frac{u}{2}\sqrt{1+u^2}+\frac{1}{2}\ln(u+\sqrt{1+u^2})\right)\Bigg|_1^{\cosh 1} =$$

$$= \pi\left[\cosh 1\sqrt{1+\cosh^2 1}-\sqrt{2}-\ln(1+\sqrt{2})+\ln(\cosh 1+\sqrt{1+\cosh^2 1})\right].$$

Adding all four areas from (a) – (d) we get that the surface area in question is approximately 20.65

6. Find the area of the surface of revolution when the parametric curve $x=\cosh t,\ y=\sinh t,\ 0\le t\le 1$, is revolved about the x-axis and about the y-axis.

Solution. (a) To find the surface area in case when the curve is revolved about the x-axis we use the formula

$$S_x = 2\pi\int_0^1 y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = 2\pi\int_0^1 \sinh t\sqrt{\sinh^2 t+\cosh^2 t}\,dt = 2\pi\int_0^1 \sinh t\sqrt{2\cosh^2 t-1}\,dt.$$

Taking $u=\cosh t$ we have $du=\sinh t\,dt$ and $S_x = 2\pi\int_1^{\cosh 1}\sqrt{2u^2-1}\,du$. Let $v=\sqrt{2}u$ then

$$S_x = \sqrt{2}\pi\int_{\sqrt{2}}^{\sqrt{2}\cosh 1}\sqrt{v^2-1}\,dv \overset{\text{formula 39}}{=} \frac{\sqrt{2}\pi}{2}\left(v\sqrt{v^2-1}-\ln(v+\sqrt{v^2-1})\right)\Bigg|_{\sqrt{2}}^{\sqrt{2}\cosh 1} =$$

$$= \pi(\cosh 1\sqrt{2\cosh^2 1-1}-1)+\frac{\sqrt{2}\pi}{2}\ln\frac{\sqrt{2}+1}{\sqrt{2}\cosh 1+\sqrt{2\cosh^2 1-1}} \approx 5.07$$

(b) If the curve is revolved about the y-axis we have

$$S_y = 2\pi\int_0^1 x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2\pi\int_0^1 \cosh t\sqrt{\sinh^2 t + \cosh^2 t}\,dt = 2\pi\int_0^1 \cosh t\sqrt{2\sinh^2 t + 1}\,dt.$$

Taking $u = \sinh t$ we have $du = \cosh t dt$ and $S_y = 2\pi\int_0^{\sinh 1}\sqrt{2u^2+1}\,du$. Let $v = \sqrt{2}u$ then

$$S_y = \sqrt{2}\pi\int_0^{\sqrt{2}\sinh 1}\sqrt{v^2+1}\,dv \overset{\text{formula 21}}{=} \frac{\sqrt{2}\pi}{2}\left[v\sqrt{v^2+1} + \ln(v + \sqrt{v^2+1})\right]\Big|_0^{\sqrt{2}\sinh 1} =$$

$$= \pi\sinh 1\sqrt{2\sinh^2 1 + 1} + \frac{\sqrt{2}\pi}{2}\ln(\sqrt{2}\sinh 1 + \sqrt{2\sinh^2 1 + 1}) \approx 10.00$$

7. Find the arc length of the curve $y = 3^x, \quad 0 \le x \le 1.$

Solution. According to the formula for arc length we have

$$L = \int_0^1\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1\sqrt{1 + (3^x\ln 3)^2}\,dx = \int_0^1\sqrt{1 + 9^x\ln^2 3}\,dx.$$

Let $u = \sqrt{1 + 9^x\ln^2 3}$. Then $u^2 = 1 + 9^x\ln^2 3$ whence

$2udu = 9^x(\ln 9)(\ln^2 3)dx = 2(\ln^3 3)9^x dx$ and, because $9^x = \frac{u^2-1}{\ln^2 3}$, we

have $dx = \frac{du}{(\ln^3 3)9^x} = \frac{du}{(\ln 3)(u^2-1)}$. Therefore

$$L = \frac{1}{\ln 3}\int_{\sqrt{1+\ln^2 3}}^{\sqrt{1+9\ln^2 3}}\frac{u^2du}{u^2-1} = \frac{1}{\ln 3}\int_{\sqrt{1+\ln^2 3}}^{\sqrt{1+9\ln^2 3}}\left(1 + \frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}\right)du =$$

$$= \frac{1}{\ln 3}\left[\sqrt{1+9\ln^2 3} - \sqrt{1+\ln^2 3} + \frac{1}{2}\ln\frac{\sqrt{1+9\ln^2 3}-1}{\sqrt{1+9\ln^2 3}+1} - \frac{1}{2}\ln\frac{\sqrt{1+\ln^2 3}-1}{\sqrt{1+\ln^2 3}+1}\right] \approx 2.25.$$