In problems 1 - 8 find the antiderivatives.

1. $\int \frac{3x^5}{\sqrt[4]{2x^6+1}}dx$. We make the following substitution $u = 2x^6+1$. Then $du = 12x^5 dx$ and therefore $3x^5 dx = \frac{1}{4}du$. After the substitution we get the following integral $\frac{1}{4}\int \frac{1}{\sqrt[4]{u}}du$ and by Power Rule $\frac{1}{4}\int u^{-(1/4)}du = \frac{1}{4}\times\frac{4}{3}u^{3/4}+C = \frac{1}{3}u^{3/4}+C = \frac{1}{3}(2x^6+1)^{3/4}+C$.

2. $\int \tan^2 t \sec^4 t dt$. Because the power of secant is even it is convenient to write the integral as $\int \tan^2 t \sec^2 t \sec^2 t dt$ and, recalling that $\sec^2 t = \tan^2 t + 1$, as $\int \tan^2 t(\tan^2 t+1)\sec^2 t dt$. After performing the substitution $u = \tan t$ (recall that $du = \sec^2 t dt$) we have $\int u^2(u^2+1)du = \int (u^4+u^2)du = \frac{u^5}{5}+\frac{u^3}{3}+C = \frac{\tan^5 t}{5}+\frac{\tan^3 t}{3}+C$.

3. $\int \frac{dx}{x^2\sqrt{x^2-5}}$. We will use here a trigonometric substitution $x = \sqrt{5}\sec t$. Then $x^2-5 = 5\sec^2 t - 5 = 5\tan^2 t$ and $\sqrt{x^2-5} = \sqrt{5}\tan t$. Also $dx = \sqrt{5}\tan t \sec t dt$. We plug in these expressions and obtain the following integral $\int \frac{\sqrt{5}\tan t \sec t dt}{5\sec^2 t\sqrt{5}\tan t} = \frac{1}{5}\int \frac{1}{\sec t}dt = \frac{1}{5}\int \cos t dt = \frac{1}{5}\sin t + C$. Using the identity $\sin t = \frac{\tan t}{\sec t}$ and recalling that $\tan t = \frac{\sqrt{x^2-5}}{\sqrt{5}}$ and $\sec t = \frac{x}{\sqrt{5}}$ we can express our result as $\int \frac{dx}{x^2\sqrt{x^2-5}} = \frac{\sqrt{x^2-5}}{5x}+C$.

4. $\int \sqrt{2x^2+2x+3}dx$. First we will complete the square.
$2x^2+2x = 2(x^2+x) = 2[(x+\frac{1}{2})^2-\frac{1}{4}] = 2(x+\frac{1}{2})^2-\frac{1}{2}$. The integral becomes $\int \sqrt{2(x+\frac{1}{2})^2+\frac{5}{2}}dx$ which is a little bit more convenient to write as $\frac{1}{\sqrt{2}}\int \sqrt{4\left(x+\frac{1}{2}\right)^2+5}dx$. Let $u = x+\frac{1}{2}$ then $du = dx$ and the integral becomes $\frac{1}{\sqrt{2}}\int \sqrt{4u^2+5}du$. Now we perform a trigonometric substitution. $u = \frac{\sqrt{5}}{2}\tan t$.

Then $4u^2+5=5\tan^2 t+5=5\sec^2 t$ and $du=\frac{\sqrt{5}}{2}\sec^2 tdt$. Thus our integral becomes $\frac{5}{2\sqrt{2}}\int \sec^3 tdt=\frac{5}{4\sqrt{2}}\left(\sec t\tan t+\ln|\sec t+\tan t|\right)+C$. But $\tan t=\frac{2u}{\sqrt{5}}=\frac{2x+1}{\sqrt{5}}$ and $\sec t=\frac{\sqrt{4u^2+5}}{\sqrt{5}}=\frac{\sqrt{4x^2+4x+6}}{\sqrt{5}}$ whence the answer can be written as $\frac{1}{4\sqrt{2}}(2x+1)\sqrt{4x^2+4x+6}+\ln\left(\frac{2x+1+\sqrt{4x^2+4x+6}}{\sqrt{5}}\right)+C$. We can drop the sign of absolute value because the expression $2x+1+\sqrt{4x^2+4x+6}$ is always positive. Finally, because we can include $\ln\sqrt{5}$ in the constant of integration, we have $\int\sqrt{2x^2+2x+3}dx=\frac{1}{4\sqrt{2}}(2x+1)\sqrt{4x^2+4x+6}+\ln(2x+1+\sqrt{4x^2+4x+6})+C$.

5. $\int\cos(3x)x^2dx$. We have to integrate by parts twice. First time we take $u=x^2, dv=\cos(3x)dx$. Then $du=2xdx$ and $v=\frac{1}{3}\sin(3x)$. Plugging these expressions into the formula $\int udv=uv-\int vdu$ we get $\int\cos(3x)x^2dx=\frac{1}{3}x^2\sin(3x)-\frac{2}{3}\int x\sin(3x)dx$. To the integral in the right part we apply integration by parts once again taking $u=x$ and $dv=\sin(3x)dx$. Then $du=dx$ and $v=-\frac{1}{3}\cos(3x)dx$. Thus we have

$$\int\cos(3x)x^2dx=\frac{1}{3}x^2\sin(3x)-\frac{2}{3}\left(-\frac{1}{3}x\cos(3x)+\frac{1}{3}\int\cos(3x)dx\right)=$$

$$=\frac{1}{3}x^2\sin(3x)+\frac{2}{9}x\cos(3x)-\frac{2}{27}\sin(3x)+C$$

6. $\int\frac{x^2+x+1}{x^3(x+1)}dx$. We integrate a proper rational fraction so we can start with its decomposition into partial fractions.

$\frac{x^2+x+1}{x^3(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x+1}$. Multiplying both parts of this identity by the common denominator $x^3(x+1)$ we get the identity

$$x^2+x+1=Ax^2(x+1)+Bx(x+1)+C(x+1)+Dx^3 \quad (*)$$

Two of the coefficients in (*) can be found quite easily. If we plug in $x=0$ we get $C=1$, and if we plug in $x=-1$ we get $D=-1$. Next, let us compare coefficients by x^3 in both parts of (*). In the left part the coefficient is 0, in the right it is $A+D$.

Thus, $A=1$. Finally, to find B we compare coefficients by x^2 in both parts of (*). We get $1=A+B$ whence $B=0$. Now we can finish the integration.

$$\int \frac{x^2+x+1}{x^3(x+1)}dx=\int\left(\frac{1}{x}+\frac{1}{x^3}-\frac{1}{x+1}\right)dx=\ln|x|-\frac{1}{2x^2}-\ln|x+1|+C=\ln\left|\frac{x}{x+1}\right|-\frac{1}{2x^2}+C.$$

7. $\int \frac{3x^2-5x+6}{(x^2-1)(x^2+4)^2}dx$. Again we integrate a proper rational function. The corresponding decomposition into partial fractions is

$\frac{3x^2-5x+6}{(x^2-1)(x^2+4)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+4}+\frac{Ex+F}{\left(x^2+4\right)^2}$; After multiplying both parts by the common denominator we have

$$3x^2-5x+6=A(x+1)(x^2+4)^2+B(x-1)(x^2+4)^2+(Cx+D)(x^2-1)(x^2+4)+(Ex+F)(x^2-1).$$

If we plug in $x=1$ we get $4=50A$ whence $A=\frac{2}{25}$. Respectively, taking $x=-1$ gives us $14=-50B$ and $B=-\frac{7}{25}$. Comparing coefficients by x^5 provides the equation $0=A+B+C$ whence $C=\frac{1}{5}$. On the other hand if we compare coefficients by x^4 we get $0=A-B+D$ and $D=-\frac{9}{25}$. To find E we will compare coefficients by x, $-5=16A+16B-4C-E$, whence $E=5+16A+16B-4C=5+\frac{32}{25}-\frac{112}{25}-\frac{4}{5}=1$. Finally, comparing the constant terms we get

$6=16A-16B-4D-F$ whence $F=16A-16B-4D-6=\frac{32}{25}+\frac{112}{25}+\frac{36}{25}-6=\frac{30}{25}=\frac{6}{5}$.

We got that

$$\int \frac{3x^2-5x+6}{(x^2-1)(x^2+4)^2}dx=\frac{2}{25}\int\frac{1}{x-1}dx-\frac{7}{25}\int\frac{1}{x+1}dx+\frac{1}{5}\int\frac{x}{x^2+4}dx-\frac{9}{25}\int\frac{1}{x^2+4}dx+\int\frac{x}{(x^2+4)^2}dx+\frac{6}{5}\int\frac{1}{(x^2+4)^2}dx$$

The computation of the first two integrals is immediate, $\int\frac{x}{x^2+4}dx=\frac{1}{2}\ln(x^2+4)$ because the numerator equals to ½ of the derivative of the denominator, and $\int\frac{1}{x^2+4}dx=\frac{1}{2}\arctan\left(\frac{x}{2}\right)$ according to the formula $\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\arctan\left(\frac{x}{a}\right)$, $a>0$.

To compute the integral $\int \frac{x}{(x^2+4)^2}dx$ we make the substitution $u = x^2+4$. It results in $\int \frac{x}{(x^2+4)^2}dx = \frac{1}{2}\int \frac{1}{u^2}du = -\frac{1}{2u} = -\frac{1}{2(x^2+4)}$. Finally, to compute the integral $\int \frac{1}{(x^2+4)^2}dx$ we use the reduction formula $\int \frac{1}{(x^2+a^2)^n}dx = \frac{1}{a^2}\left[\frac{1}{2n-2}\cdot\frac{x}{(x^2+a^2)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{1}{(x^2+a^2)^{n-1}}\right]$. Applying this formula in case when $n=2$ and $a=2$ we get $\int \frac{1}{(x^2+4)^2}dx = \frac{1}{8}\frac{x}{x^2+4} + \frac{1}{8}\int \frac{1}{x^2+4}dx = \frac{1}{8}\frac{x}{x^2+4} + \frac{1}{16}\arctan\left(\frac{x}{2}\right)$. Plugging these expressions into the formula above and combining like terms we obtain the answer

$$\int \frac{3x^2-5x+6}{(x^2-1)(x^2+4)^2}dx = \frac{2}{25}\ln|x-1| - \frac{7}{25}\ln|x+1| + \frac{1}{10}\ln(x^2+4) - \frac{21}{200}\arctan\left(\frac{x}{2}\right) - \frac{1}{2(x^2+4)} + \frac{3}{20}\frac{x}{x^2+4} + C$$

8. $\int \frac{\sin x}{\sin x - \cos x}dx$. We will compute the integral with the help of a rationalizing substitution. First we will divide both the numerator and the denominator by $\cos x$. Then $\int \frac{\sin x}{\sin x - \cos x}dx = \int \frac{\tan x}{\tan x - 1}dx$. Next we perform the substitution $u = \tan x$. Then $x = \arctan u$, $dx = \frac{1}{1+u^2}du$, and $\int \frac{\sin x}{\sin x - \cos x}dx = \int \frac{u}{(u-1)(u^2+1)}du$. The decomposition into partials provides $\frac{u}{(u-1)(u^2+1)} = \frac{A}{u-1} + \frac{Bu+C}{u^2+1}$ whence $u = A(u^2+1) + (Bu+C)(u-1)$. Plugging in $u=1$ we obtain $A = \frac{1}{2}$. Comparing coefficients by u^2 provides $A+B=0$ whence $B = -\frac{1}{2}$. Finally, comparing the constant terms we get $A-C=0$ whence $C = \frac{1}{2}$. Therefore

$$\int \frac{u}{(u-1)(u^2+1)}du = \frac{1}{2}\left[\int \frac{1}{u-1}du - \int \frac{u}{u^2+1}du + \int \frac{1}{u^2+1}du\right] = \frac{1}{2}\ln|u-1| - \frac{1}{4}\ln(u^2+1) + \frac{1}{2}\arctan u$$

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Recall that $u = \tan x$ whence $\tan u = x$ and $u^2+1 = \tan^2 x + 1 = \sec^2 x$. Now, we can write the answer as

$$\int \frac{\sin x}{\sin x - \cos x}dx = \frac{1}{2}\ln|\tan x - 1| - \frac{1}{2}\ln|\sec x| + \frac{x}{2} + C.$$